ECE 619 Spring 2013
Midterm Exam

March 12 at 8:10AM-9:25AM

The exam consists of 5 questions each of several parts. The exam is closed book. All of the questions can be correctly answered in a reasonable amount of time and space. If you need an excessive amount of time or computations to answer a problem, then you are doing something wrong. Show your work **clearly**.

1. **SET COVERING (20 points):** Consider a problem where you wish to identify events that have occurred based on a set of alerts or alarms. You receive two alarms {a2 a3}. The relationships between events and the alarms are given below. Events e2 and e3 occur with equal likelihood and are more than twice as likely to occur as e1 e4 and e5. Formulate an optimization problem to find the minimal set of events that will most likely explain the alarms. Identify clearly all the constraints and the objective function and any other assumptions.

e1-> {a1 a4 }
e2-> {a1 a2}
e3-> {a1 a3}
e4-> {a3 a4}
e5-> {a2}

- Let x_i be whether an event has occurred
- Let c_i be the likelihood of event i

then $\underline{c}^T = [1 \; \frac{1}{2} \; \frac{1}{2} \; 1 \; 1]$

- Since alarms a_2 & a_3 received let

$\underline{b}^T = [0 \; 1 \; 1 \; 0]$

- The relationship between events & alarms is given by

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- $\min \; c^T \underline{x} \; > \; A\underline{x} \geq \underline{b} \qquad \underline{x} \geq 0 \qquad \underline{x} \leq 1$

binary constraint unneeded.

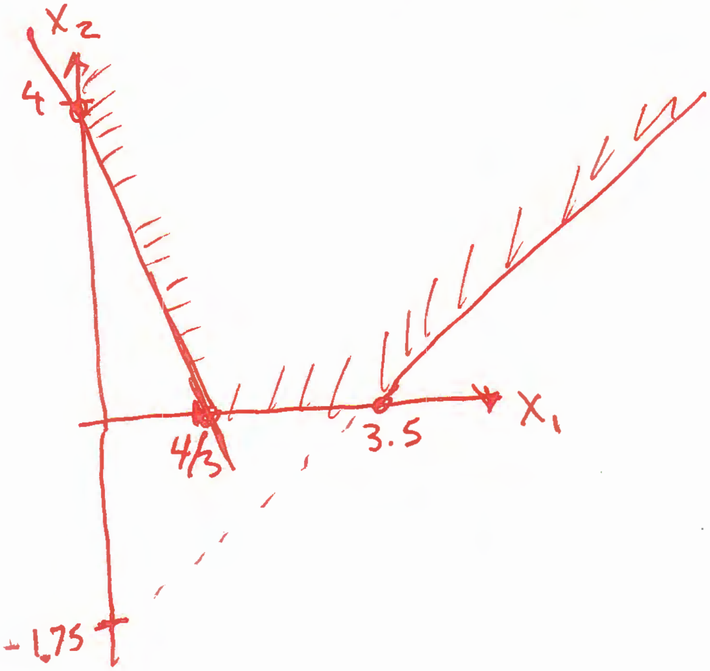
2. **LINEAR PROGRAM (25 points):** Given the linear programming problem below.

$$\min\ 5x_1 + 2x_2$$
such that
$$3x_1 + x_2 \geq 4$$
$$x_1 - 2x_2 \leq 3.5$$
$$x_1, x_2 \geq 0$$

a) Solve this problem graphically. Identify the solution and **all** the extreme points.

b) Now formulate this problem in standard form and select the extreme point, which is optimal. For this point:

- identify the basic variables, x_B, non-basic variables, x_N, and the basis **B**.



Standard form

$\max\ z = -5x_1 - 2x_2$

$\underline{c}^T = [-5\ -2\ 0\ 0]$

subject to

$A\underline{x} = \underline{b}$

$\underline{x} = [x_1\ x_2\ x_3\ x_4]^T$

$\underline{x} \geq 0$

$\underline{b} = [-4\ 3.5]$

$$A = \begin{bmatrix} -3 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

By inspection optimal point is $[4/3\ 0]$

with slack variables $x_3 = 0$ $x_4 = 2.17$

Thus $\underline{x}_B = [x_1\ x_4]^T$; $\underline{x}_N = [x_2\ x_3]^T$

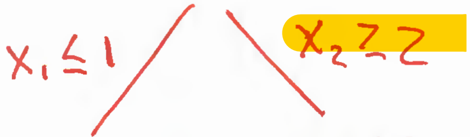
and $B = \begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix}$

3. **INTEGER PROGRAM (10 points):** Given the linear programming problem from problem 2, repeated below, suppose that x_1 is now restricted to be integer. Starting with the solution found in question (2a), solve using the branch-and-bound algorithm. (Note the solution at each branch should be obvious and a solution is found quickly for this problem so if you need to do a large number of branches you are making a mistake).

$$\begin{aligned} &\min\ 5x_1 + 2x_2 \\ &\text{such that} \\ &3x_1 + x_2 \geq 4 \\ &x_1 - 2x_2 \leq 3.5 \\ &x_1, x_2 \geq 0 \qquad x_1 \text{ integer} \end{aligned}$$

Solve previous with a branch for $x_1 \leq 1$ and a branch for $x_2 \geq 2$

Should be X1



by inspection

$z = 7$

$\underline{x} = [1\ 1\ 0\ \cancel{0}]^T$

4.5

$z = 10$

$\underline{x} = [2\ 0\ 2\ 1.5]^T$

<u>Done</u>

4. DUALITY (25 points): Consider the constrained optimization problem below

$$\min 2x_1^2 + 3x_2^2$$

such that

$$3x_1 + 2x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

a) Find the dual of this non-linear program.

b) Repeat if x_1 and x_2 are binary. Plot this dual function.

a) Form the Lagrangean

$$\mathcal{L}(\underline{x}, \lambda) = 2x_1^2 + 3x_2^2 + \lambda(4 - 3x_1 - 2x_2)$$

Minimize over X by applying $\partial \mathcal{L}/\partial x_i = 0$; substitute

$$\partial \mathcal{L}(\cdot)/\partial x_1 = 4x_1 + \lambda(-3) = 0 \Rightarrow x_1 = \frac{3}{4}\lambda$$

$$\partial \mathcal{L}(\cdot)/\partial x_2 = 6x_2 + \lambda(-2) = 0 \Rightarrow x_2 = \frac{1}{3}\lambda$$

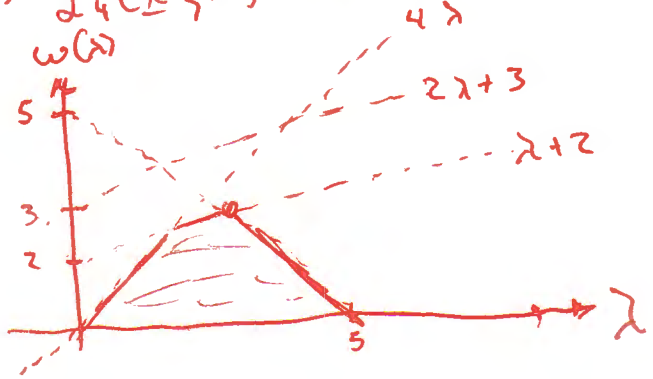
Now substitute

$$w(\lambda) = 2\left(\frac{3}{4}\lambda\right)^2 + 3\left(\frac{1}{3}\lambda\right)^2 + \lambda\left(4 - \frac{9}{4}\lambda - \frac{2}{3}\lambda\right)$$

$$= 4\lambda - \frac{35}{24}\lambda^2 \quad ; \quad \lambda \geq 0$$

b) if x_1 and x_2 binary then enumerate

$$\left.\begin{array}{ll} (0,0) \Rightarrow & \mathcal{L}_1(\underline{x}, \lambda) = 4\lambda \\ (0,1) \Rightarrow & \mathcal{L}_2(\underline{x}, \lambda) = 2\lambda + 3 \\ (1,0) \Rightarrow & \mathcal{L}_3(\underline{x}, \lambda) = \lambda + 2 \\ (1,1) \Rightarrow & \mathcal{L}_4(\underline{x}, \lambda) = 5 - \lambda \end{array}\right\} \quad w(\lambda) < \min(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$$



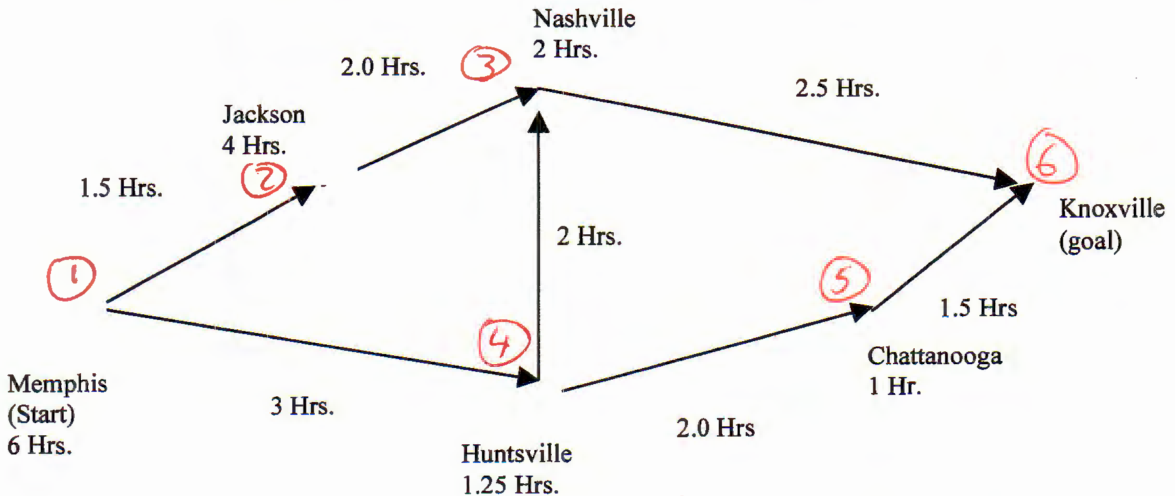
$$w^*(\lambda^*) = 3.5$$

$$\lambda^* = 1.5$$

5. **GRAPH SEARCH (20 points)**: Consider the graph below with possible routes to Knoxville leaving from Memphis. The paths are directional as indicated by the arrows. The times shown next to the arcs show the actual driving times. The numbers just below the cities show the estimated remaining time to reach Knoxville (the goal). Then:

   a) Formulate a linear program for this problem (but do not solve).
   b) Find the shortest path using Dijkstra's search. Show the steps clearly.
   c) Find the shortest path using A* search. Show the steps clearly. Does the estimate of the remaining distance to the goal satisfy the A* requirement for finding an optimal solution?

   Note, the numbers here aren't necessarily accurate values so don't get hung up on your knowledge of these cities.



a) Use labels above to define paths x_{ij}

so $\underline{x} = [x_{12}\ x_{23}\ x_{14}\ x_{36}\ \overset{x_{43}}{\vee}\ x_{45}\ x_{56}]$

then $\underline{c}^T = [1.5\ 2.0\ 3.0\ 2.5\ 2.0\ 2.0\ 1.5]$

and there are equalities for each of the six nodes

$A\underline{x} = \underline{b}$ with $\underline{b}^T = [1\ 0\ 0\ 0\ 0\ -1]$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$\min \ \underline{c}^T \underline{x}$

$\ni A\underline{x} = \underline{b}$

$\underline{x} \geq 0$

Extra page for work.

Dijkstra search build up the table

	①	②	③	④	⑤	⑥
①	0	1.5	~~0~~	3	—	—
①②	0	1.5	3.5	3	—	—
①②④	0	1.5	3.5	3	5	—
①②③④	0	1.5	3.5	3	5	6
①②③④⑤	0	1.5	3.5	3	5	6

Path is
1-2-3-6

c) A* algorith modifies above with estimate

	①	②	③	④	⑤	⑥
①	(0,6)	(1.5,5.5)	—	(3,4.8)	—	—
①④	(0,6)	(1.5,5.5)	(5,7)	(3,4.25)	(5,6)	—
①②④	(0,6)	(1.5,5.5)	(3.5,5.5)	(3,4.25)	(5,6)	—
①②④⑤	(0,6)	(1.5,5.5)	(3.5,5.5)	(3,4.25)	(5,6)	(6,6)

- No need to look at node ⑤ as it cannot be better than solution found
- Heuristic sastifies A* criteria