The exam consists of 5 questions each of several parts. The exam is closed book. All of the questions can be correctly answered in a reasonable amount of time and space. If you need an excessive amount of time or computations to answer a problem, then you are doing something wrong. Still, there is an extra page attached to the exam if you do need extra space. Show your work clearly.

1. **GENERAL DUALITY (25 points):** Consider the constrained optimization problem below

\[
\begin{align*}
& \text{min } 4x_1^2 + x_2^2 \\
& \text{such that} \\
& 3x_1 + 2x_2 \geq 4 \\
& 0 \leq x_1 \leq 2 \\
& 0 \leq x_2 \leq 1
\end{align*}
\]

a) Find the dual of this non-linear program.
b) Repeat if \(x_1\) and \(x_2\) are restricted to integers.

\[
\begin{align*}
\min L(x,\lambda) &= 4x_1^2 + x_2^2 + \lambda_1 (4 - 3x_1 - 2x_2) \\
& \quad + \lambda_2 (x_1 - 2) + \lambda_3 (x_2 - 1) \\
& \quad \lambda_i \geq 0 \quad \forall i
\end{align*}
\]

then minimize over \(\mathbf{x}\)

\[
\begin{align*}
\frac{\partial L(.)}{\partial x_1} &= 8x_1 + \lambda_1 (-3 - 2) + \lambda_2 = 0 \\
\frac{\partial L(.)}{\partial x_2} &= 2x_2 + \lambda_1 (-2) + \lambda_3 = 0
\end{align*}
\]

then \(x_1 = \frac{3\lambda_1 - \lambda_2}{8} \quad \text{and} \quad x_2 = \frac{2\lambda_1 - \lambda_3}{2}\)

substitute to find \(w(\lambda)\)

\[
w(\lambda) = 4 \left( \frac{3\lambda_1 - \lambda_2}{8} \right)^2 + \left( \frac{2\lambda_1 - \lambda_3}{2} \right)^2 + \lambda_1 \left( 4 - \frac{3}{8}(3\lambda_1 - \lambda_2) \right) \]

\[
+ \lambda_2 \left( \frac{3\lambda_1 - \lambda_2}{8} \right) + \lambda_3 \left( \frac{2\lambda_1 - \lambda_3}{2} - 1 \right)
\]

(see work page)
2. **SET COVERING (10 points):** Consider a problem where you wish to identify events that have occurred based on a set of alerts or alarms. You are to assume that given a set of alarms that there at most two events, which might have occurred. Formulate an optimization problem to find the minimal set of events that can explain the received alarms but do not attempt to solve. Identify clearly all the constraints and the objective function and any other assumptions. The relationships between events and the alarms are given below. The received alarms are \( \{a_1, a_3\} \).

\[
C^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\
A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[b = \begin{bmatrix} 1 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}
\]

\[X \text{ corresponds to the events}
\]

\[b \text{ corresponds to received alarm}
\]

\[\text{them}
\]

\[
\begin{align*}
\text{min} & \quad C^T X \\
\text{with} & \quad A^T X \geq b \\
& \quad \sum_i x_i \leq 2 \\
& \quad x_i \geq 0
\end{align*}
\]

\(x_i\) binary not needed

\(\exists\) or can add this to \(A\)
3. **LINEAR PROGRAM (25 points):** Given the linear programming problem below.

\[
\begin{align*}
\text{min} & \quad 2x_1 + 5x_2 \\
\text{such that} & \quad x_1 + 3x_2 \geq 4 \\
& \quad -2x_1 + x_2 \leq 3.25 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

a) Solve this problem graphically. Identify the solution and all the extreme points.

b) Now formulate this problem in standard form and select the extreme point, which is optimal. For this point, identify the basic variables, \( x_B \), non-basic variables, \( x_N \), and the basis \( B \).

**Extreme points** \((4, 0), (0, 4/3), (0, 3.25)\)

\( x_{\text{optimal}} = (0, 4/3) \) \( z^* = 6 \ 2/3 \)

**b) Standard form**

\[
\begin{align*}
\text{max} & \quad -2x_1 - 5x_2 \\
\text{with} & \quad x_1 + 3x_2 - x_3 = 4 \\
& \quad -2x_1 + x_2 + x_4 = 3.25 \\
& \quad x_i \geq 0
\end{align*}
\]

At optimal \((0, 4/3, 0, 1.92)\)

\[
\begin{align*}
X_B &= \begin{bmatrix} x_2, x_4 \end{bmatrix} \\
X_N &= \begin{bmatrix} x_1, x_3 \end{bmatrix} \\
B &= \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}
\end{align*}
\]
4. INTEGER PROGRAM (15 points): Given the linear programming problem from problem 3, repeated below, suppose that \( x_2 \) is now restricted to be integer. Starting with the solution found in question (3a), solve using the branch-and-bound algorithm. (Note the solution at each branch should be obvious and a solution is found quickly for this problem so if you need to do a large number of branches you are making a mistake).

\[
\begin{align*}
\text{min} & \quad 2x_1 + 5x_2 \\
\text{such that} & \quad x_1 + 3x_2 \geq 4 \\
& \quad -2x_1 + x_2 \leq 3.25 \\
& \quad x_1, x_2 \geq 0 \\
& \quad x_2 \text{ integer}
\end{align*}
\]

\[
\begin{align*}
X^* &= (1, 1) \\
Z^* &= 7 \\
\text{Integer Optimal}
\end{align*}
\]

(see previous graph)

**Integer but better solution found.**
C) Add constraint to part (a) of

\[ 1.5x_{12} + 2x_{23} + 3x_{44} + 2.5x_{36} + 2x_{43} + 2x_{45} + 1.5x_{56} \leq 6 \]

we relax this constraint and iterate on the \( \lambda \) for this constraint.

(i) \( \lambda = 0 \) (from above)

Distance = 345
Time = 6.5 hours

(ii) Constraint violation is 0.5. The next \( \lambda \) depends on the update parameter. Choose 5. Then \( \lambda^{(2)} = 2.5 \)

With \( \lambda = 2.5 \) then add in the cost of the time at each path. This will not be enough to change solution again violation is 0.5. The next \( \lambda^{(2)} = 5 \).
\[ \lambda_1^2 \left( \frac{36}{64} + 1 - \frac{1}{8} - 2 \right) = -1 \frac{9}{16} \]
\[ \lambda_1 \lambda_2 \left( -24/64 + \frac{3}{8} + \frac{3}{8} \right) = \frac{3}{8} \]
\[ \lambda_2 \left( \frac{16}{64} - \frac{1}{8} \right) = \frac{1}{8} \]
\[ \lambda_3 \lambda_2 - 1 + 1 = 0 \]
\[ \lambda_3 \left( \frac{1}{4} - \frac{1}{2} \right) = -\frac{1}{4} \]
\[ \lambda_1 (4) = 4 \]
\[ \lambda_2 (-2) = -2 \]
\[ \lambda_3 (-1) = -1 \]

\[ \begin{array}{c|c|c}
\lambda_1 & \lambda_2 & \omega(\lambda) \\
0 & 0 & 4\lambda_1 - 2\lambda_2 - \lambda_3 \\
1 & 0 & 4 + 1\lambda_1 - \lambda_2 - \lambda_3 \\
1 & 1 & 5 - \lambda_1 - \lambda_2 \\
2 & 0 & 16 - 2\lambda_1 - \lambda_3 \\
2 & 1 & 17 - 4\lambda_1 \\
0 & 1 & 1 + 2\lambda_1 - 2\lambda_2 \\
\end{array} \]

or for \( \lambda_2 = \lambda_3 = 0 \)

\[ \begin{array}{c|c|c}
\lambda_1 & \lambda_2 & \omega(\lambda) \\
0 & 0 & 4\lambda \\
1 & 0 & 4 + \lambda \\
1 & 1 & 5 - \lambda \\
2 & 0 & 16 - 2\lambda \\
2 & 1 & 17 - 4\lambda \\
0 & 1 & 1 + 2\lambda \\
\end{array} \]