Consider the system of non-linear simultaneous equations $g(x) = \mathbf{0}$ where $g : \mathbb{R}^2 \rightarrow$ defined by:

$$\forall x \in \mathbb{R}^2, g(x) = \begin{bmatrix} (x_1)^2 - 0.1x_1x_2 \\ (x_2)^2 - 0.1x_1x_2 \end{bmatrix}.$$

(i) Write out explicitly the Newton–Raphson update to solve $g(x) = \mathbf{0}$. Invert the Jacobian matrix explicitly using the formula for the inverse of a 2×2 matrix.

(ii) Is there a solution to $g(x) = \mathbf{0}$?

(iii) Starting at $x^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, calculate the first three iterates.

(iv) Now replace the exact Jacobian by the approximation $\tilde{J}(x)$ obtained by neglecting the off-diagonal terms of the Jacobian. Write down the new update equations.

(v) Starting at $x^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, calculate the first five iterates based on the approximate Jacobian $\tilde{J}(x)$.

(vi) Which method takes less effort overall to achieve a solution satisfying the condition $\|g(x^{(\nu)})\| < 0.3$: using the exact Jacobian or using the approximate Jacobian? (Hint: The answer is: "it all depends.")

7.4 Consider a plate capacitor, illustrated in Figure 7.12. (We will consider a similar arrangement in the sizing of interconnects in integrated circuits case study in Section 15.5.) The capacitor consists of two conductors separated by a non-conducting dielectric. The upper conductor is of length L and width w and has thickness T. The upper conductor is separated from the lower conductor by a dielectric of thickness d and dielectric constant ε.