Summary of crown reduction

Definition: A crown is an order pair \((I, H)\) of subsets of vertices from a graph \(G\) that satisfy the following criteria.
1. \(I \neq \emptyset\) is an independent set of \(G\)
2. \(H = N(I)\)
3. There exists a matching \(M\) on the edges connecting \(I\) and \(H\) such that all elements of \(H\) are matched.

\(H\) is called the head of the crown and the width of the crown is \(|H|\).

Theorem 1: If \(G\) is a graph with a crown \((I, H)\) then there is a vertex cover of \(G\) with minimum size that contains all the vertices in \(H\) and none of the vertices in \(I\).

Proof: Since there is a matching \(M\) of the edges between \(I\) and \(H\), any vertex cover must contain at least one vertex from each matched edge. Thus the matching will require at least \(|H|\) vertices in the vertex cover. This minimum number can be realized by selecting \(H\) to be in the vertex cover. It is further noted that vertices from \(H\) can be used to cover edges that do not connect \(I\) and \(H\) while this is not true for vertices in \(I\). Thus, including the vertices from \(H\) does not increase, and may decrease, the size of the vertex cover as compared to including vertices from \(I\). Therefore, there is a minimum size vertex cover that contains all the vertices in \(H\) and none of the vertices in \(I\).

Algorithm: Given a graph \(G\) the following algorithm can be used to find a crown.

1. Find a maximal matching \(M_1\) of the graph and identify the set of all unmatched vertices as the set \(O\) of outsiders.
2. Find a maximum auxiliary matching \(M_2\) of the edges between \(O\) and \(N(O)\).
3. Let \(I_0\) be vertices in \(O\) that are unmatched by \(M_2\).
4. Repeat steps 4a and 4b until \(n = N\) so that \(I_{N-1} = I_N\).
   4a. Let \(H_n = N(I_n)\).
   4b. Let \(I_{n+1} = I_n \cup N_{M_2}(H_n)\).

The crown is now the ordered pair \((I, H)\) where \(I = I_N\) and \(H = H_N\).

Theorem 2: The algorithm produces a crown as long as the set \(I_0\) of unmatched outsiders in not empty.

Proof: First, since \(M_1\) is a maximal matching, the set \(O\), and consequently its subset \(I\), are both independent. Second, because of the definition of \(H\), it is clear that \(H = N(I_{N-1})\) and since \(I = I_N = I_{N-1}\) we know that \(H = N(I)\).

The third condition for a crown is proven by contradiction. Suppose there were an element of \(h \in H\) that were unmatched by \(M_2\). Then the construction of \(H\) would produce and augmented (alternating) path of odd length. For \(h\) to be in \(H\) there must have been an unmatched vertex in \(O\) that begins the path. Then the repeated step 4a would always produce an edge that is not in the matching while the next step 4b would produce an edge that is part of the matching, this process repeats until the vertex \(h\) is reached. The
resulting path begins and ends with unmatched vertices and alternates between matched and unmatched edges. Such a path cannot exist if \( M_2 \) is in fact a maximum matching because we could increase the size of the matching by swapping the matched and unmatched edges along the path. Therefore every element of \( H \) must be matched by \( M_2 \). The actual matching used in the crown is the matching \( M_2 \) restricted to edges between \( H \) and \( I \).

**Theorem 3:** If the matching \( M_1 \) is of size less than or equal to \( K \) then the graph \( G \) has at most \( 4K \) vertices that are not in the crown.

**Proof:** Since the size of the matching \( M_1 \) is less than or equal to \( K \), it contains at most \( 2K \) vertices. This implies there are at most \( 2K \) vertices in \( O \) that are matched by \( M_2 \). All the remaining vertices are included in \( I_0 \) and are therefore in \( I \). Thus the largest number of vertices in \( G \) that are not included in \( I \) and \( H \) is \( 4K \).

**Example:** For our example consider the following graph.

Step 1: pick \( M_1 = \{(f, g), (h, i)\} \) as the maximal matching. This implies that \( O = \{a, b, c, d, e\} \).

Step 2: pick \( M_2 = \{(a, f), (b, g), (d, i), (e, h)\} \) which is a maximum matching since the edges between \( O \) and \( N(O) \) form a bipartite graph and \( |N(O)| = 4 \).
Step 3: $I_0 = \{c\}$.

Step 4a: $H_0 = N(I_0) = \{f\}$.

Step 4b: $I_1 = I_0 \cup N_{\mathcal{M}_2}(H_0) = \{a, c\}$.

Step 4a: $H_1 = N(I_1) = \{f, g\}$.

Step 4b: $I_2 = I_1 \cup N_{\mathcal{M}_2}(H_1) = \{a, b, c\}$.

Further repetition of steps 4a and 4b will not add new vertices, so we are done $I = \{a, b, c\}$ and $H = \{f, g\}$.