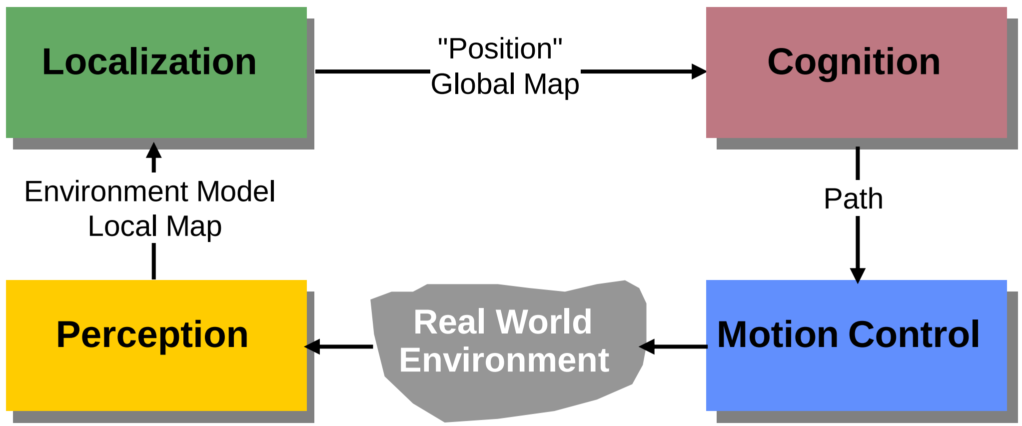


Mobile Robot Kinematics

Mobile Robot Kinematics: Overview

- Mobile robot and manipulator arm characteristics
  - Arm is fixed to the ground and usually comprised of a single chain of actuated links
  - Mobile robot motion is defined through rolling and sliding constra-ints taking effect at the wheel-ground contact points

C Willow Garage

C dexter1232222222222, youtube.com

Mobile Robot Kinematics: Overview

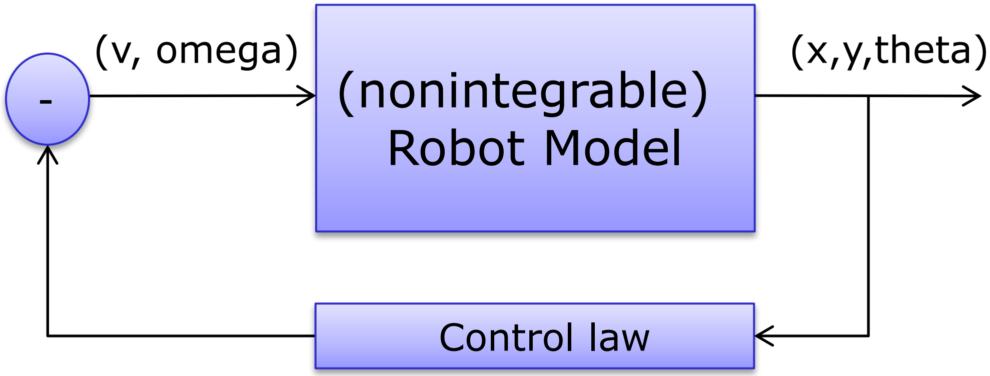
- Definition and Origin
  - From *kinein* (Greek); to move
  - Kinematics is the subfield of Mechanics which deals with motions of bodies

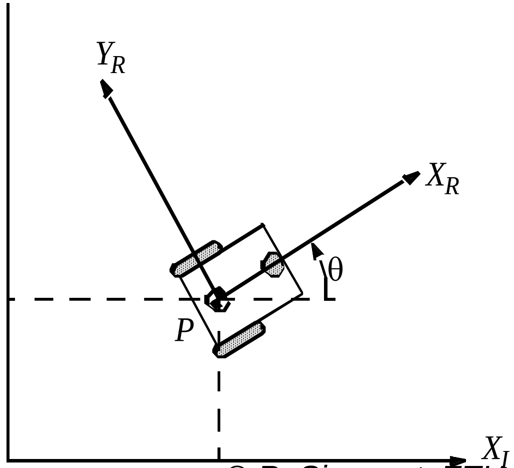
- Manipulator- vs. Mobile Robot Kinematics
  - Both are concerned **with forward and inverse kinematics**
  - However, for mobile robots, encoder values don‘t map to unique robot poses
  - However, **mobile robots** can move unbound with respect to their environment
    - There is **no direct** (=instantaneous) **way to measure the robot’s position**
    - **Position must be integrated over time**, depends on path taken
    - Leads to inaccuracies of the position (motion) estimate
  - Understanding mobile robot motion starts with **understanding wheel con-straints** placed on the robot’s mobility

Forward and Inverse Kinematics

- Forward kinematics:
  - Transformation from joint- to physical space
- Inverse kinematics
  - Transformation from physical- to joint space
  - Required for motion control

- Due to nonholonomic constraints in mobile robotics, we deal with **differential** (inverse) kinematics
  - Transformation between velocities instead of positions
  - Such a differential kinematic model of a robot has the following form:

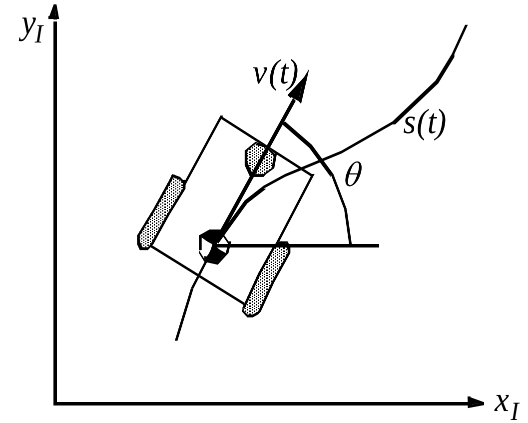




Differential Kinematics Model

- Due to a lack of alternatives:
  - establish the robot speed $\dot{\xi} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$ as a function of the wheel speeds $\dot{\varphi}_i$, steering angles β_i , steering speeds $\dot{\beta}_i$ and the geometric parameters of the robot (*configuration coordinates*).
  - forward kinematics

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m)$$



  - Inverse kinematics

$$\begin{bmatrix} \dot{\varphi}_1 & \cdots & \dot{\varphi}_n & \beta_1 & \ldots & \beta_m & \dot{\beta}_1 & \ldots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$

  - But generally not integrable into

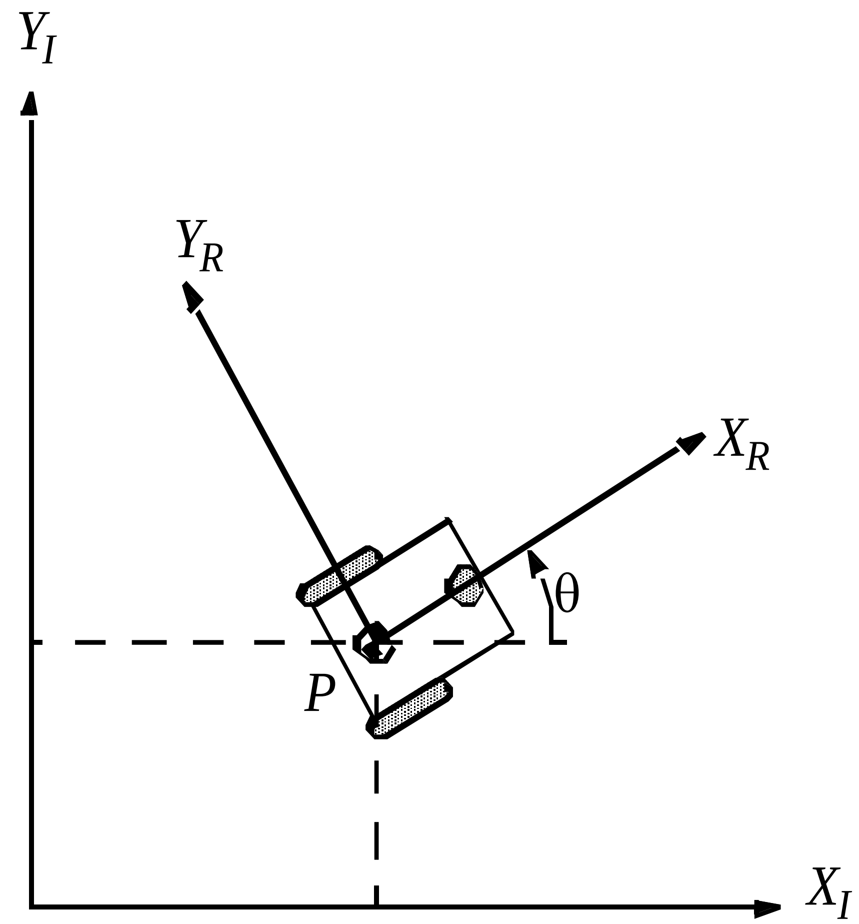
$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = f(\varphi_1, \ldots \varphi_n, \beta_1, \ldots \beta_m)$$

Representing Robot Pose

- Representing the robot within an arbitrary initial frame
  - Inertial frame: $\{X_I, Y_I\}$
  - Robot frame: $\{X_R, Y_R\}$
  - Robot pose: $\xi_I = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$
  - Mapping between the two frames

$$\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta) \cdot \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$$

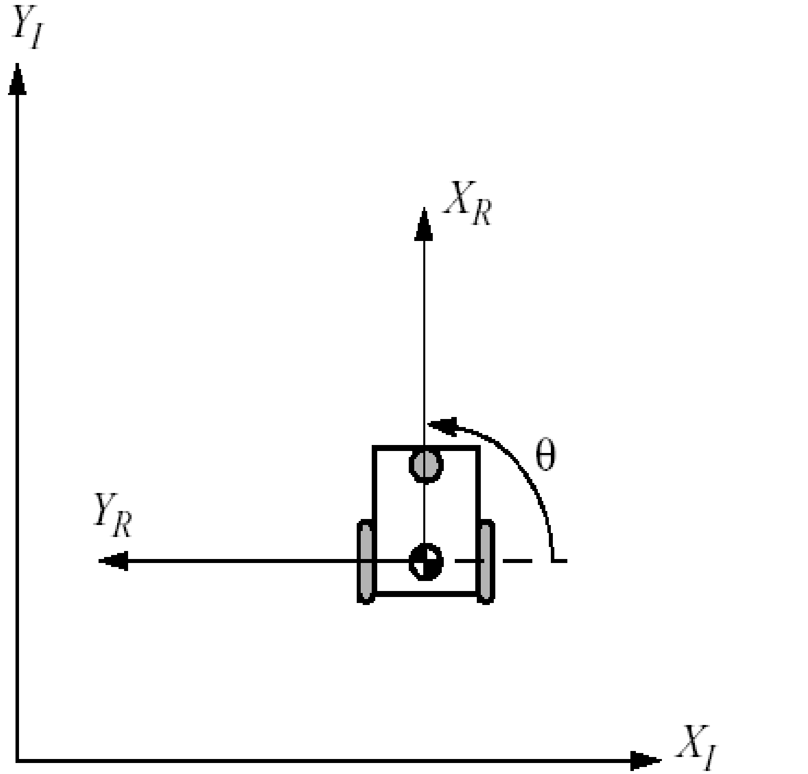
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example: Robot aligned with Y_I

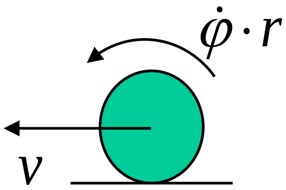
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

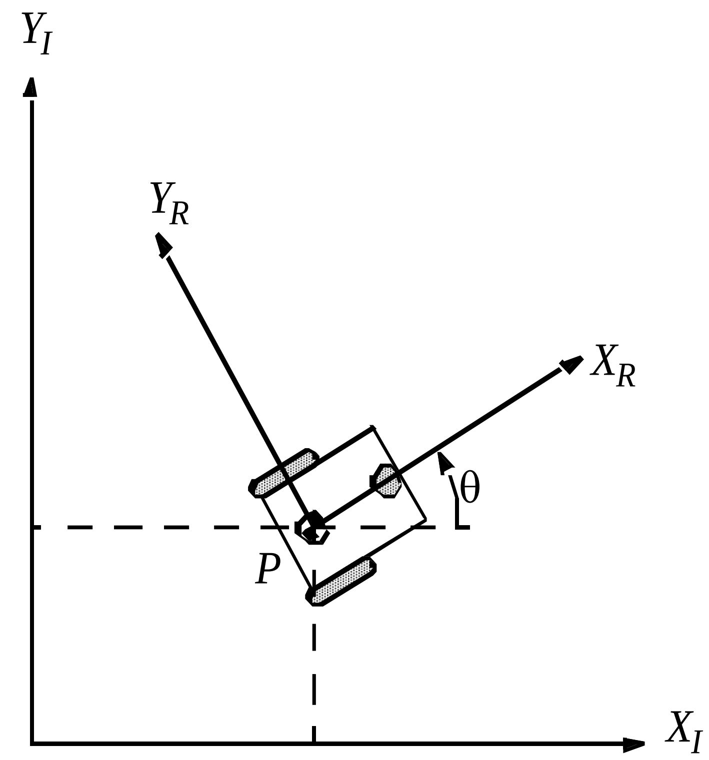
$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$



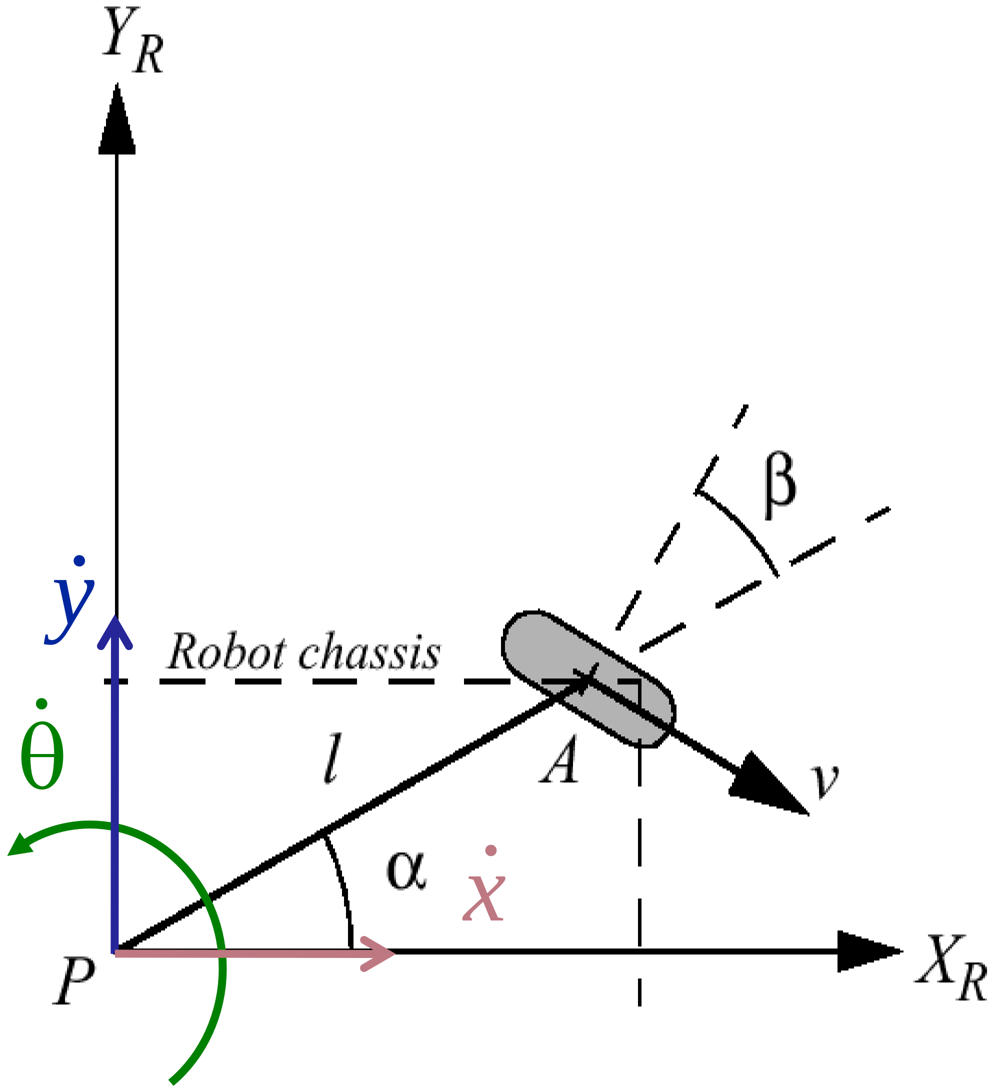
Wheel Kinematic Constraints

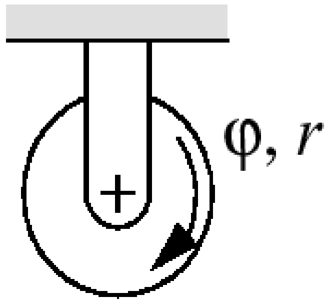
- Assumptions
  - Movement on a horizontal plane
  - Point contact of the wheels
  - Wheels not deformable
  - Pure rolling (v_c = 0 at contact point)
  - No slipping, skidding or sliding
  - No friction for rotation around contact point
  - Steering axes orthogonal to the surface
  - Wheels connected by rigid frame (chassis)

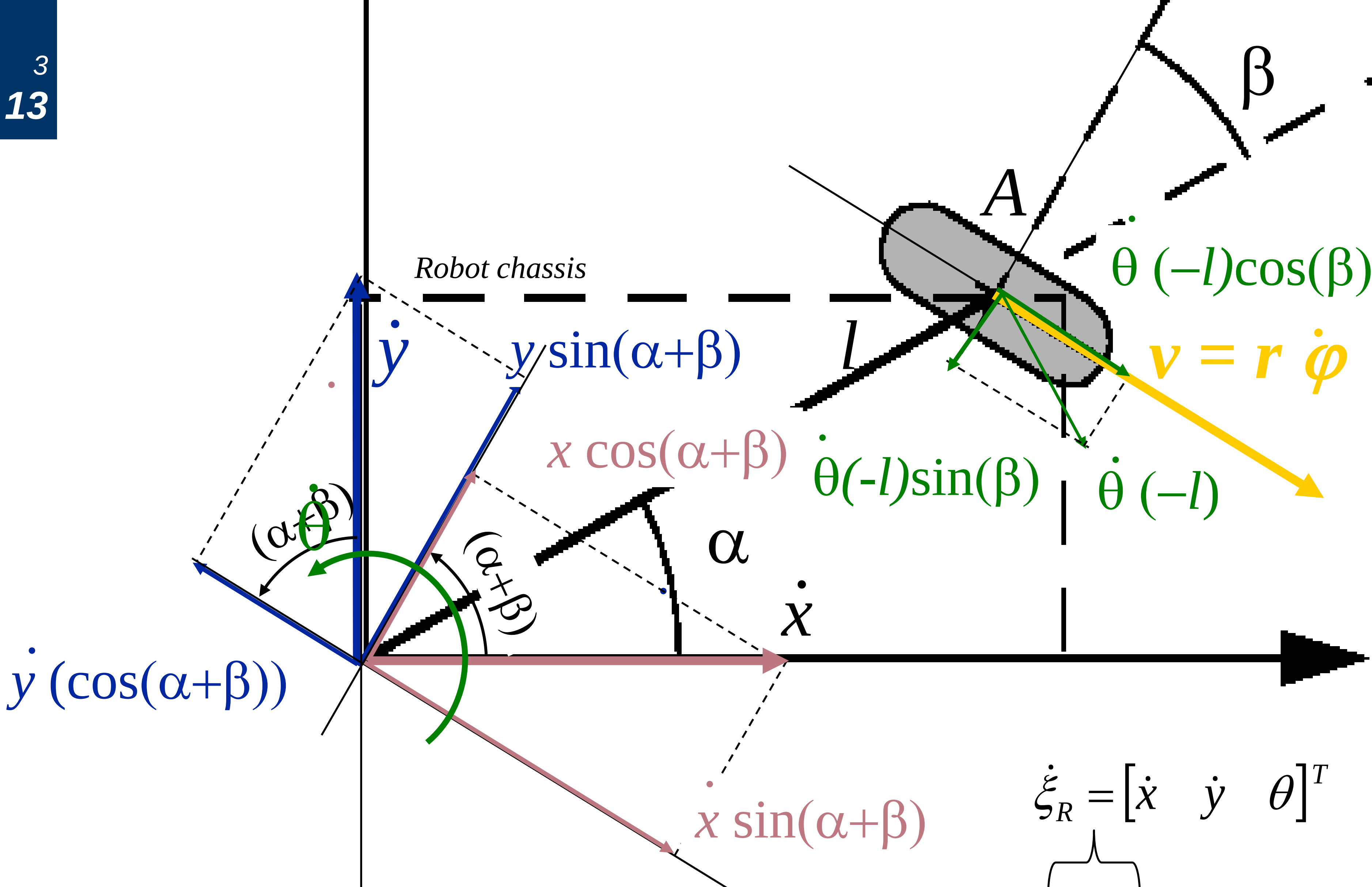




Kinematic Constraints: Fixed Standard Wheel







$$\dot{\xi}_R = \begin{bmatrix} \dot{x} & \dot{y} & \theta \end{bmatrix}^T$$

$$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix} R(\theta)\dot{\xi}_I = 0$$

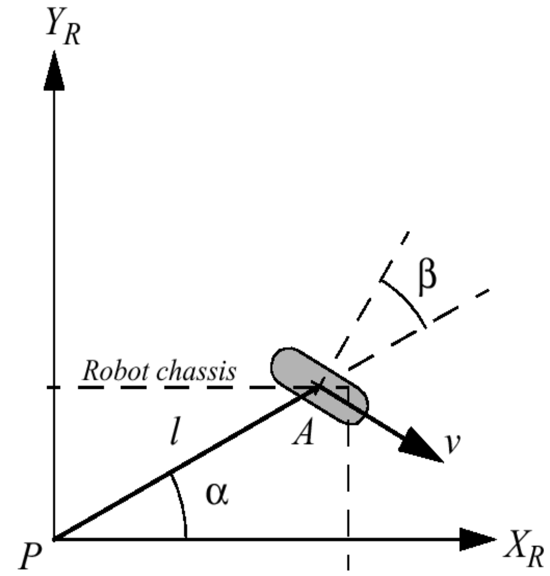
14 Example

$$\left[\sin(\alpha+\beta) \;\; -\cos(\alpha+\beta) \;\; (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta) \;\; \sin(\alpha+\beta) \;\; l\sin\beta\right] R(\theta)\dot{\xi}_I = 0$$

- Suppose that the wheel A is in position such that **α = 0 and β = 0**
- This would place the contact point of the wheel on X_I with the plane of the wheel oriented parallel to Y_I. If $\theta = 0$, then the **sliding constraint** reduces to:

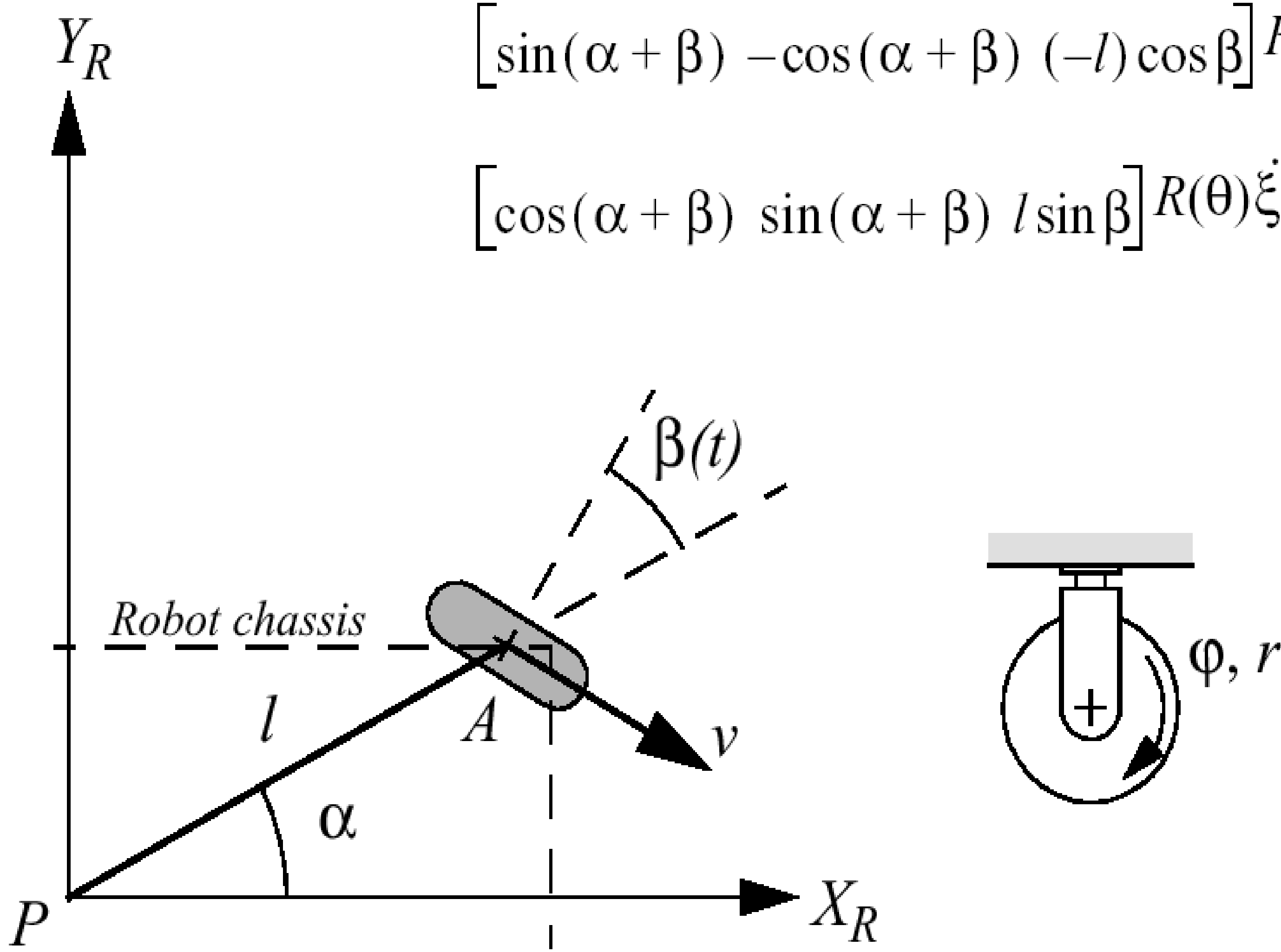
$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$



Kinematic Constraints: Steered Standard Wheel

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

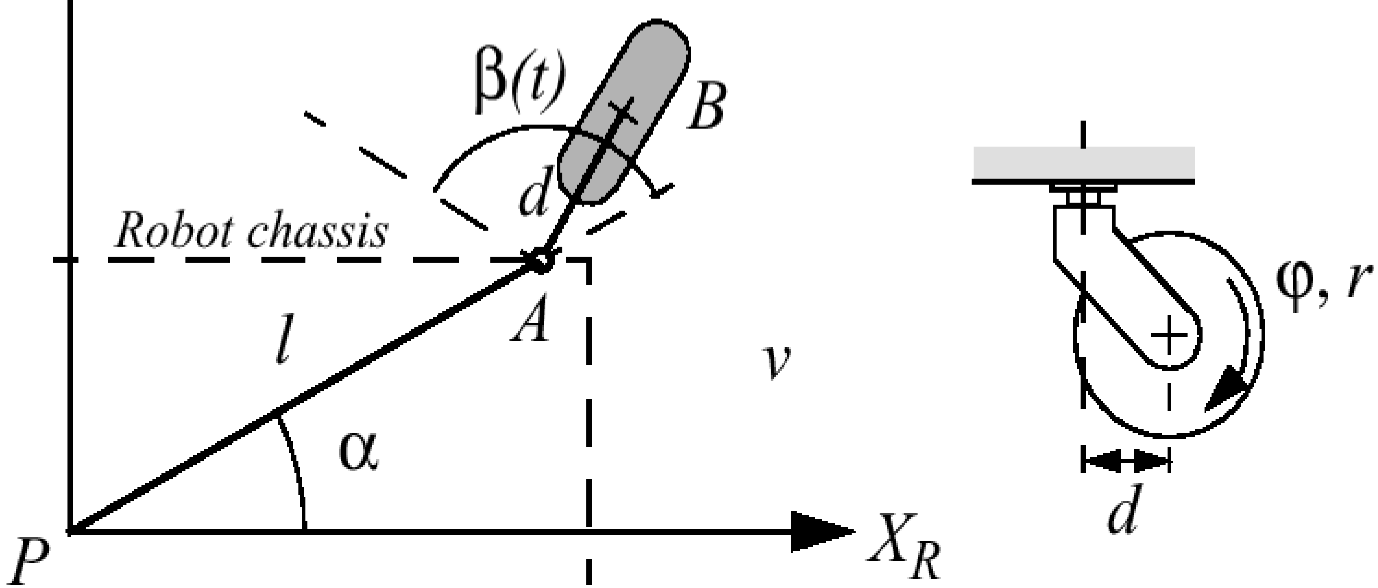
$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0$$



Kinematic Constraints: Castor Wheel

$$\left[\sin(\alpha+\beta) \;\; -\cos(\alpha+\beta) \;\; (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

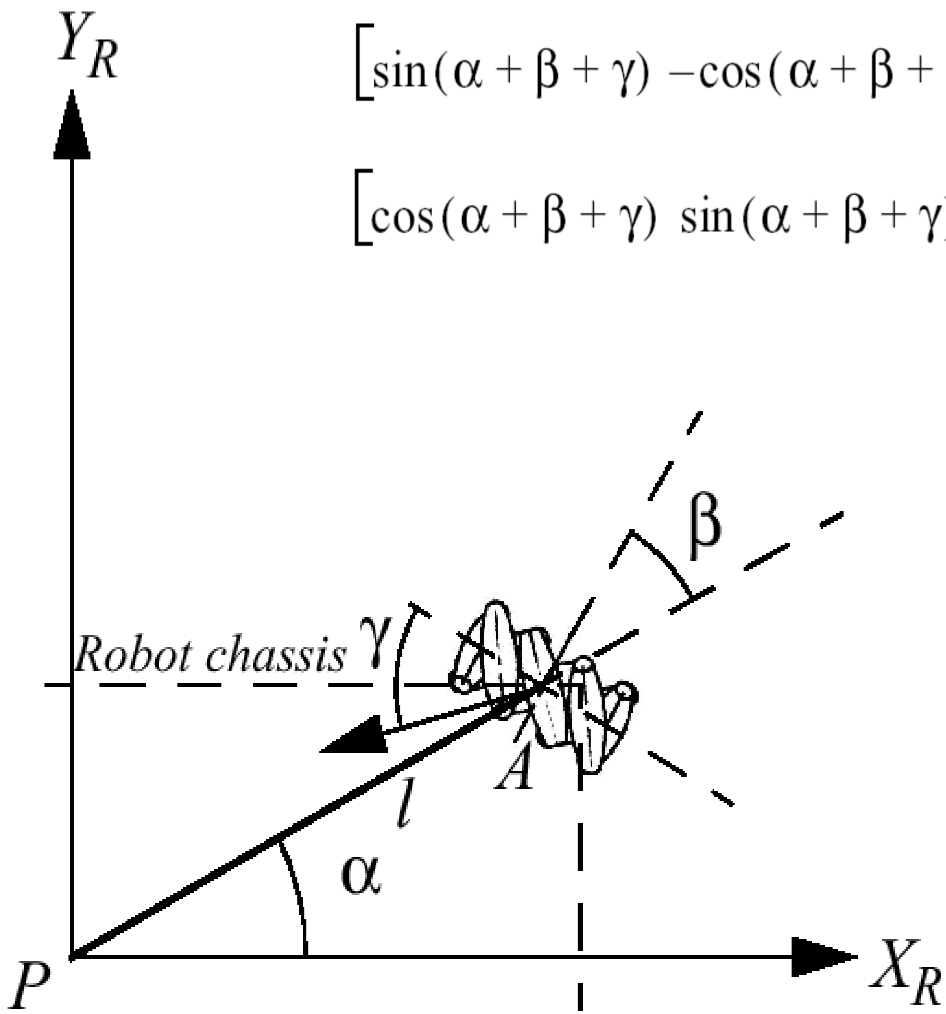
$$\left[\cos(\alpha+\beta) \;\; \sin(\alpha+\beta) \;\; d + l\sin\beta\right] R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$

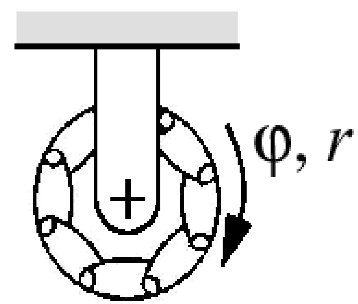


Kinematic Constraints: Swedish Wheel

$$\left[\sin(\alpha+\beta+\gamma)\quad -\cos(\alpha+\beta+\gamma)\quad (-l)\cos(\beta+\gamma)\right] R(\theta)\dot{\xi}_I - r\dot{\varphi}\cos\gamma = 0$$

$$\left[\cos(\alpha+\beta+\gamma)\quad \sin(\alpha+\beta+\gamma)\quad l\sin(\beta+\gamma)\right] R(\theta)\dot{\xi}_I - r\dot{\varphi}\sin\gamma - r_{sw}\dot{\varphi}_{sw} = 0$$

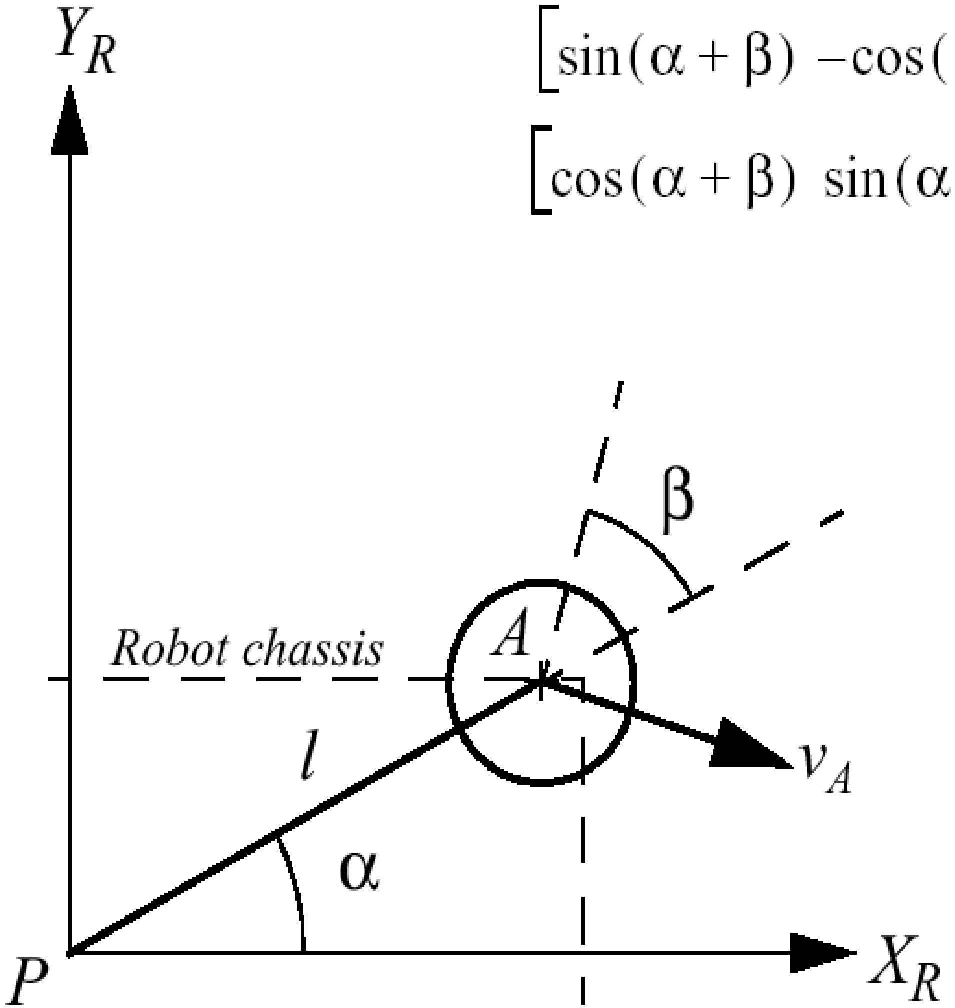


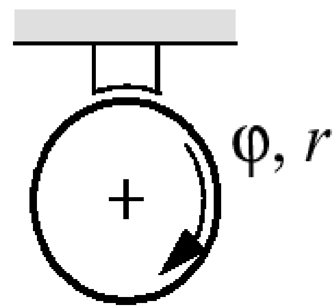


Kinematic Constraints: Spherical Wheel

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0$$





- Rotational axis of the wheel can have an arbitrary direction

Kinematic Constraints: Complete Robot

- Given a robot with *M* wheels
  - each wheel imposes zero or more constraints on the robot motion
  - **only fixed and steerable standard wheels impose constraints**
- What is the maneuverability of a robot considering a combination of different wheels?
- Suppose we have a total of $N=N_f + N_s$ standard wheels
  - We can develop the equations for the constraints in matrix forms:
  - Rolling

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}_{(N_f+N_s)\times 1} \qquad J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3} \qquad J_2 = diag(r_1 \cdots r_N)$$

  - Lateral movement

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3}$$

Mobile Robot Maneuverability

- The maneuverability of a mobile robot is the combination
  - of the mobility available based on the sliding constraints
  - plus additional freedom contributed by the steering

- Three wheels is sufficient for static stability
  - additional wheels need to be synchronized
  - this is also the case for some arrangements with three wheels

- It can be derived using the equation seen before
  - Degree of mobility δ_m
  - Degree of steerability δ_s
  - Robots maneuverability $\delta_M = \delta_m + \delta_s$

Mobile Robot Maneuverability: Degree of Mobility

- To avoid any lateral slip the motion vector $R(\theta)\dot{\xi}_I$ has to satisfy the following constraints:

$$C_{1f} R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

- Mathematically:
  - $R(\theta)\dot{\xi}_I$ must belong to the *null space* of the projection matrix $C_1(\beta_s)$
  - *Null space* of $C_1(\beta_s)$ is the space N such that for any vector n in N

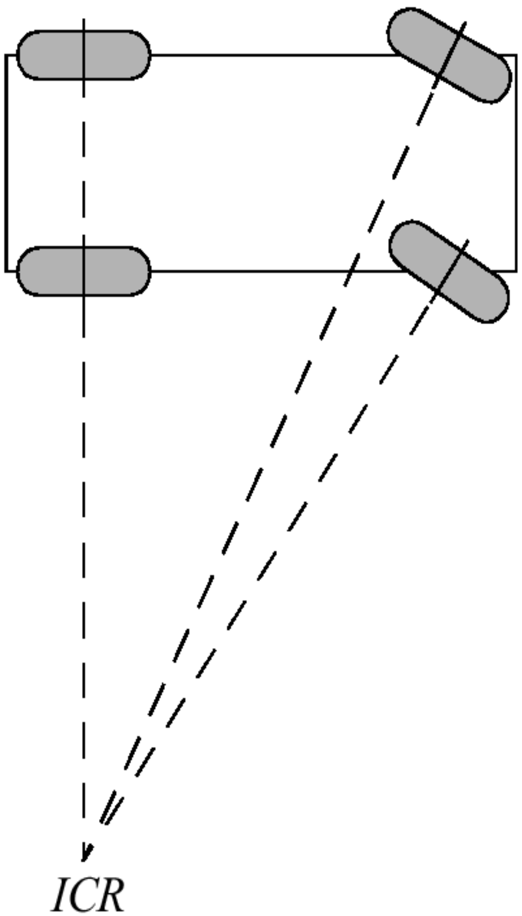
$$C_1(\beta_s) \cdot n = 0$$

  - Geometrically this can be shown by the *Instantaneous Center of Rotation (ICR)*

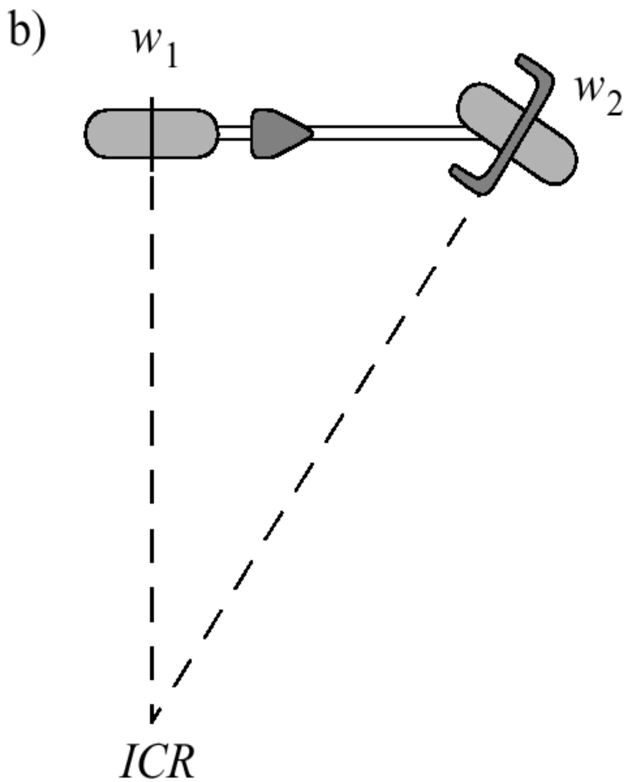
Mobile Robot Maneuverability: ICR

- Instantaneous center of rotation (ICR)

- Ackermann Steering



Bicycle



23 Mobile Robot Maneuverability: More on Degree of Mobility

- Robot chassis kinematics is a function of the set of independent constraints

$$rank\left[C_1(\beta_s)\right] \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix} \qquad \begin{matrix} C_{1f} R(\theta)\dot{\xi}_I = 0 \\ C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0 \end{matrix}$$

  - the greater the rank of $C_1(\beta_s)$ the more constrained is the mobility

- Mathematically

$$\delta_m = \dim N\left[C_1(\beta_s)\right] = 3 - rank\left[C_1(\beta_s)\right] \qquad 0 \le rank\left[C_1(\beta_s)\right] \le 3$$

  - no standard wheels $rank\left[C_1(\beta_s)\right] = 0$
  - all direction constrained $rank\left[C_1(\beta_s)\right] = 3$

- Examples:
  - Unicycle: One single fixed standard wheel
  - Differential drive: Two fixed standard wheels
    - wheels on same axle
    - wheels on different axle

Mobile Robot Maneuverability: Degree of Steerability

- Indirect degree of motion

$$\delta_s = rank\left[C_{1s}(\beta_s)\right]$$

  - The particular orientation at any instant imposes a kinematic constraint
  - However, the ability to change that orientation can lead additional degree of maneuverability
- Range of δ_s: $0 \le \delta_s \le 2$

- Examples:
  - one steered wheel: Tricycle
  - two steered wheels: No fixed standard wheel
  - car (Ackermann steering): $N_f = 2$, $N_s = 2$ -> common axle

Mobile Robot Maneuverability: Robot Maneuverability

- Degree of Maneuverability

$$\delta_M = \delta_m + \delta_s$$

  - Two robots with same δ_M are not necessary equal
  - Example: Differential drive and Tricycle (next slide)

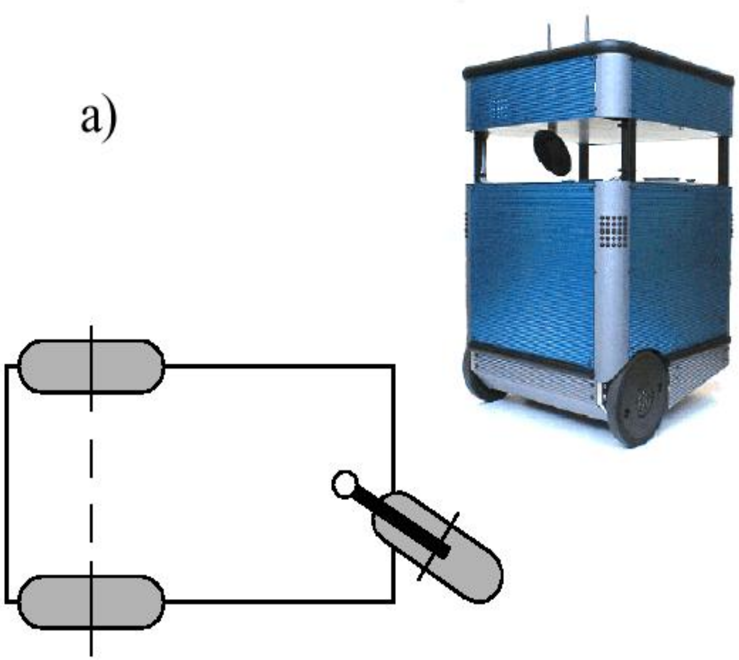
  - For any robot with $\delta_M = 2$ the ICR is always constrained to *lie on a line*
  - For any robot with $\delta_M = 3$ the ICR is not constrained and can *be set to any point on the plane*

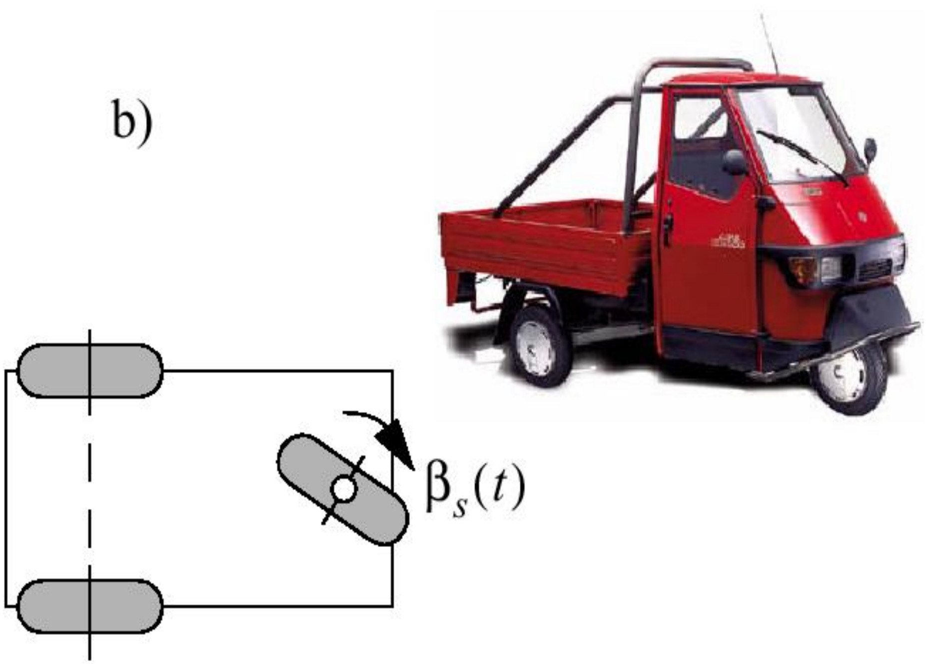
- The Synchro Drive example: $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$

Mobile Robot Maneuverability: Wheel Configurations

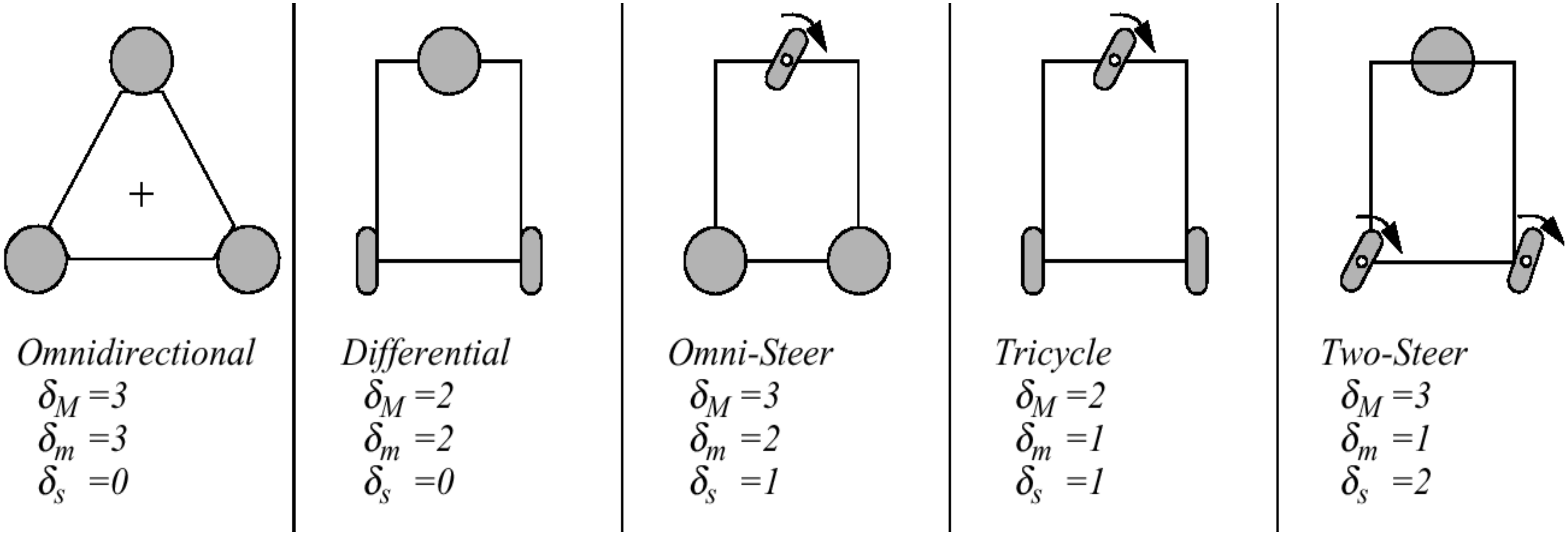
- Differential Drive

Tricycle



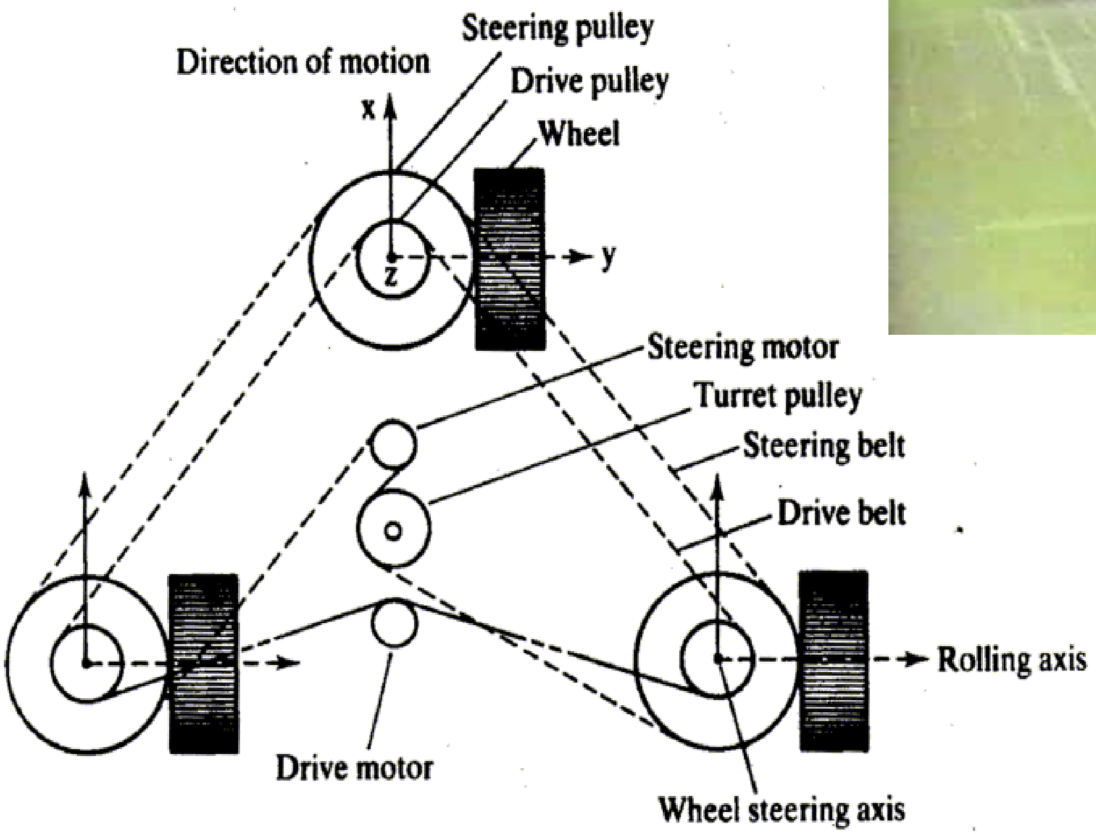


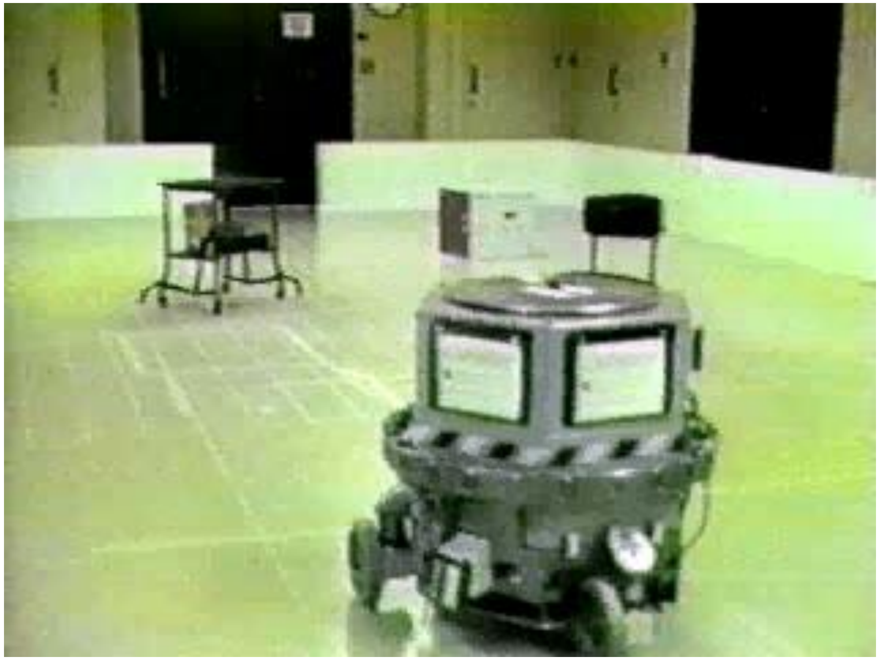
Five Basic Types of Three-Wheel Configurations



Synchro Drive

$$\delta_M = \delta_m + \delta_s = 1 + 1 = 2$$





C J. Borenstein

Mobile Robot Workspace: Degrees of Freedom

- The Degree of Freedom (DOF) is the robot's ability to achieve various poses.

- But what is the degree of vehicle's freedom in its environment?
  - Car example

- Workspace
  - how the vehicle is able to move between different configuration in its workspace?

- The robot's independently achievable velocities
  - = *differentiable degrees of freedom (DDOF) =* δ_m
  - Bicycle: $\delta_M = \delta_m + \delta_s = 1 + 1$ DDOF = 1; DOF=3
  - Omni Drive: $\delta_M = \delta_m + \delta_s = 3 + 0$ DDOF=3; DOF=3

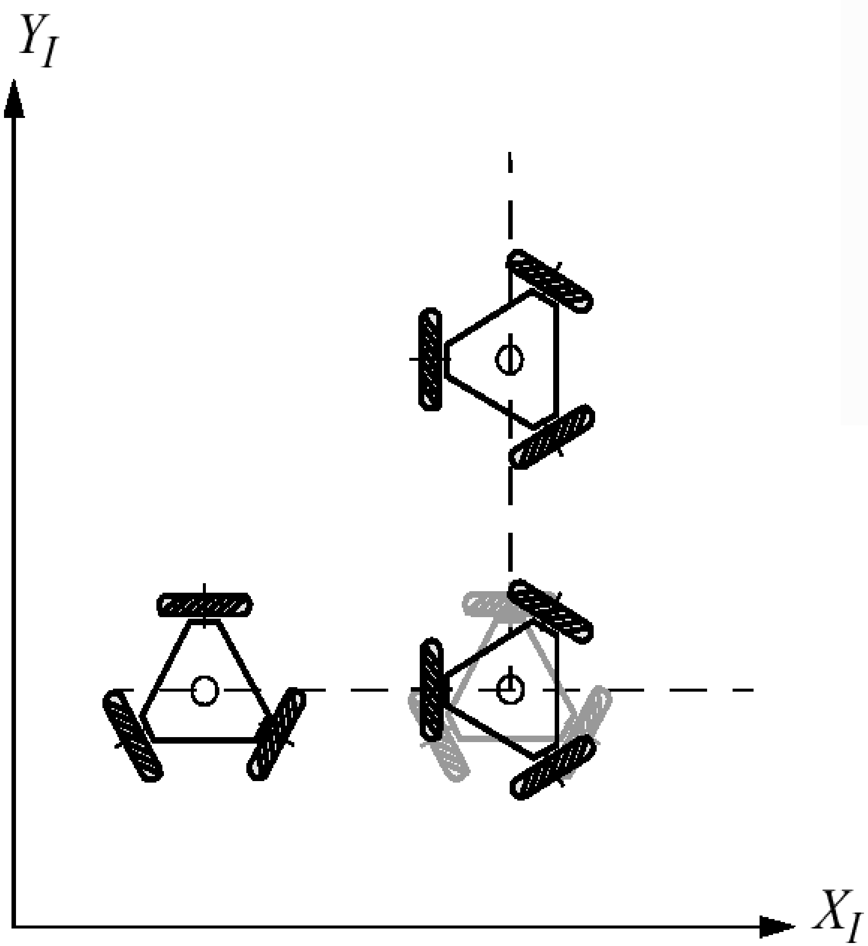
Mobile Robot Workspace: Degrees of Freedom, Holonomy

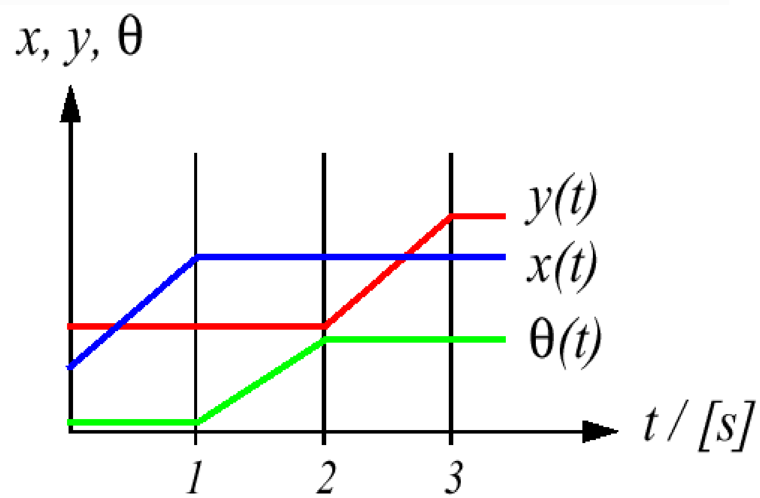
- DOF *degrees of freedom*:
  - Robots ability to achieve various poses
- DDOF *differentiable degrees of freedom*:
  - Robots ability to achieve various trajectories

$$DDOF \leq \delta_M \leq DOF$$

- Holonomic Robots
  - A holonomic kinematic constraint can be expressed as an explicit function of position variables only
  - A non-holonomic constraint requires a different relationship, such as the derivative of a position variable
  - *Fixed and steered standard wheels impose non-holonomic constraints*

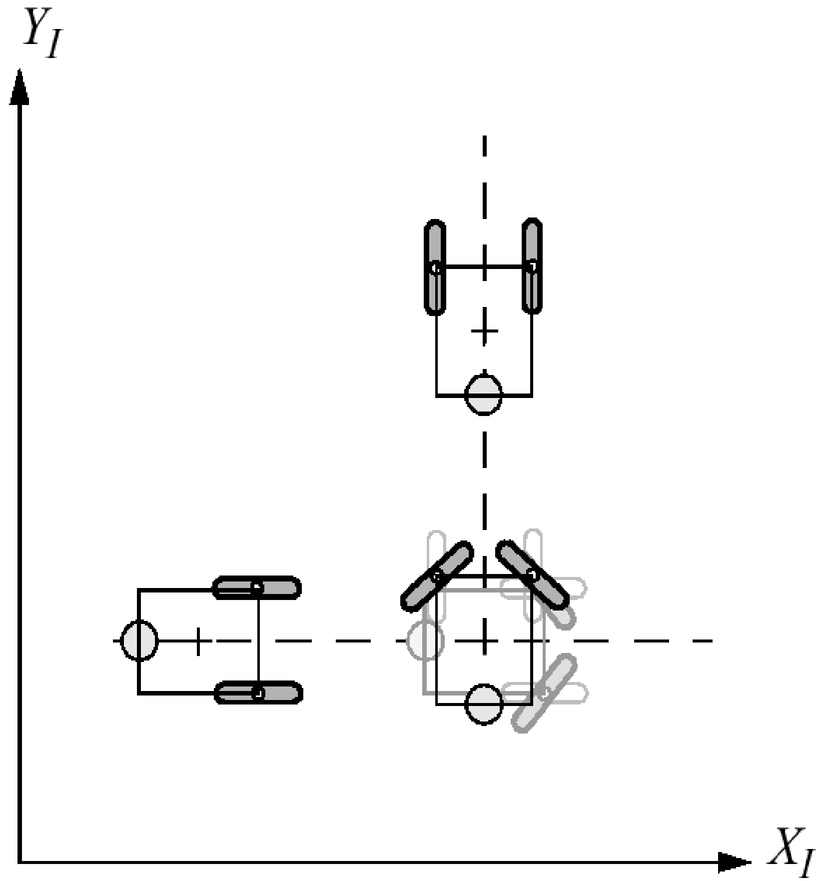
Path / Trajectory Considerations: Omnidirectional Drive

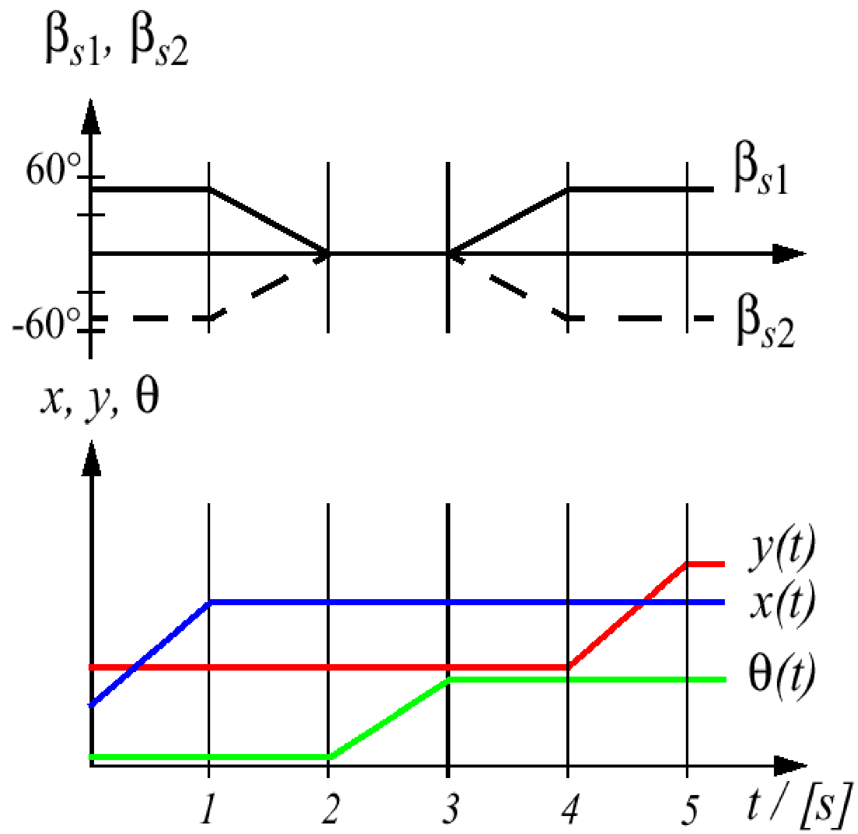




Tribolo ETH/EPFL

Path / Trajectory Considerations: Two-Steer





Beyond Basic Kinematics

- At higher speeds, and in difficult terrain, dynamics become important

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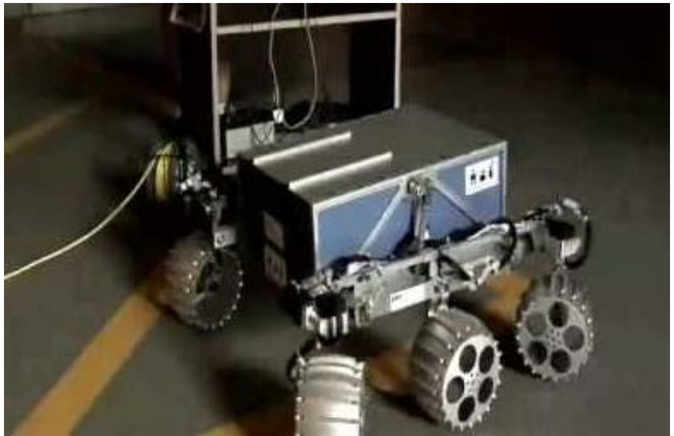
- For other vehicles, the no-sliding constraints, and simple kinematics presented in this lecture do not hold



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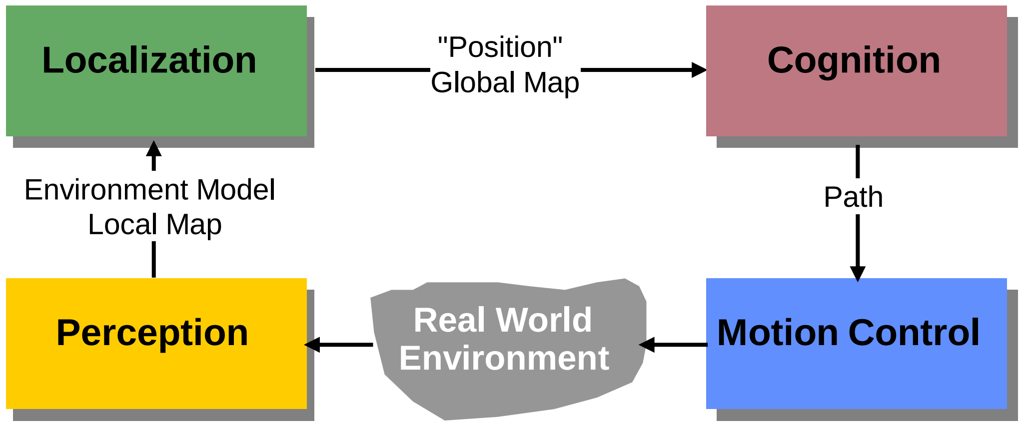


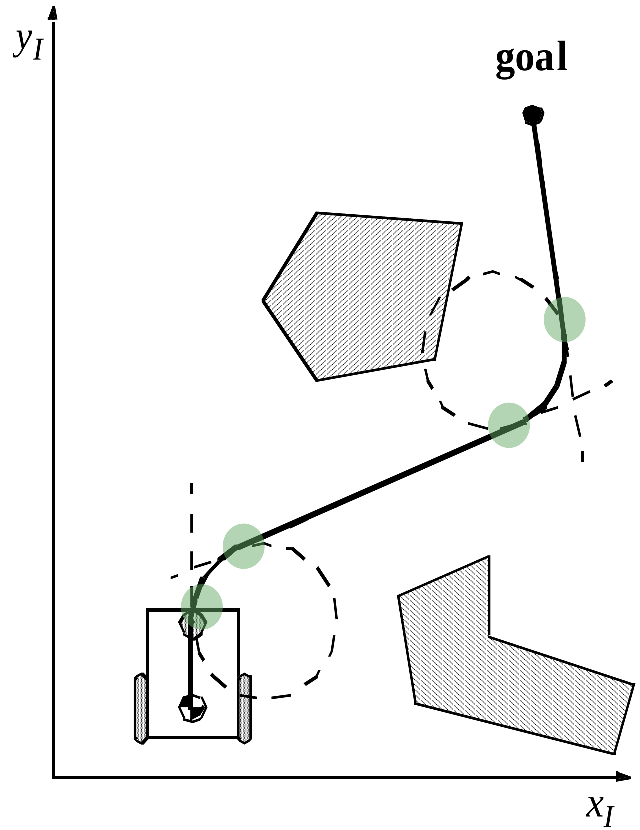
Motion Control
wheeled robots

Wheeled Mobile Robot Motion Control: Overview

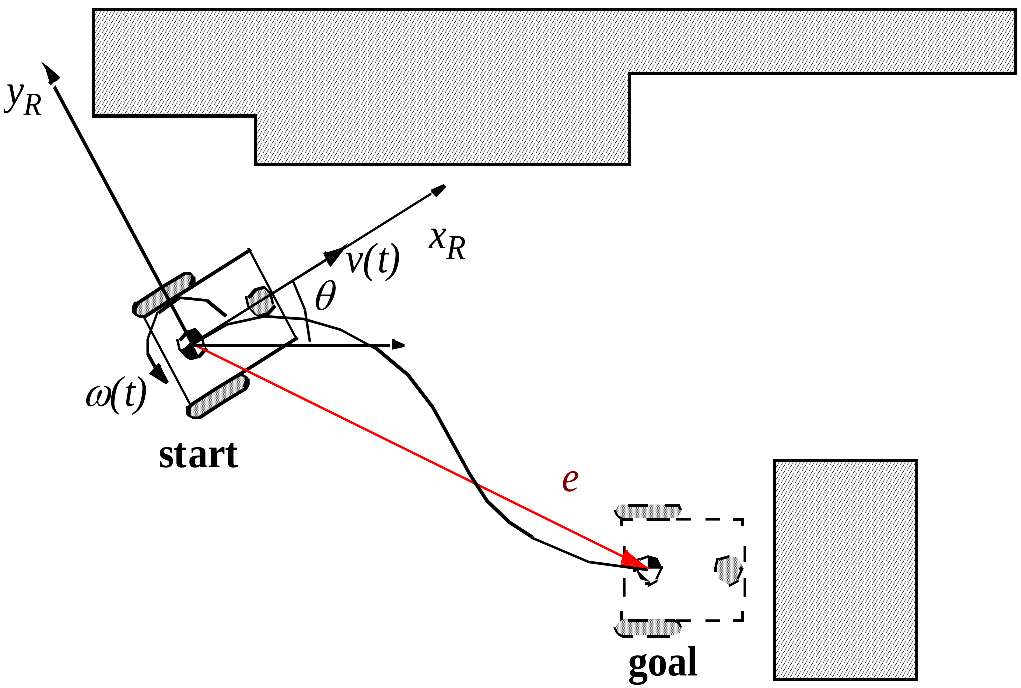
- The objective of a kinematic controller is to follow a trajectory described by its position and/or velocity profiles as function of time.
- Motion control is not straight forward because mobile robots are typically non-holonomic and MIMO systems.
- Most controllers (including the one presented here) are not considering the dynamics of the system

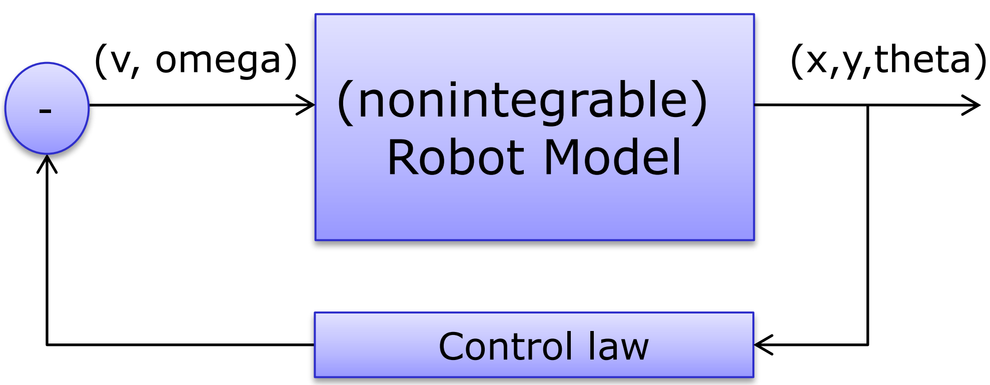
Motion Control: Open Loop Control

- trajectory (path) divided in motion segments of clearly defined shape:
  - straight lines and segments of a circle
  - Dubins car, and Reeds-Shepp car
- control problem:
  - pre-compute a smooth trajectory based on line, circle (and clothoid) segments
- Disadvantages:
  - It is not at all an easy task to pre-compute a feasible trajectory
  - limitations and constraints of the robots velocities and accelerations
  - does not adapt or correct the trajectory if dynamical changes of the environment occur.
  - The resulting trajectories are usually not smooth (in acceleration, jerk, etc.)



Motion Control: Feedback Control





- Find a control matrix *K*, if exists

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

with $k_{ij}=k(t,e)$

- such that the control of *v(t)* and $\omega(t)$

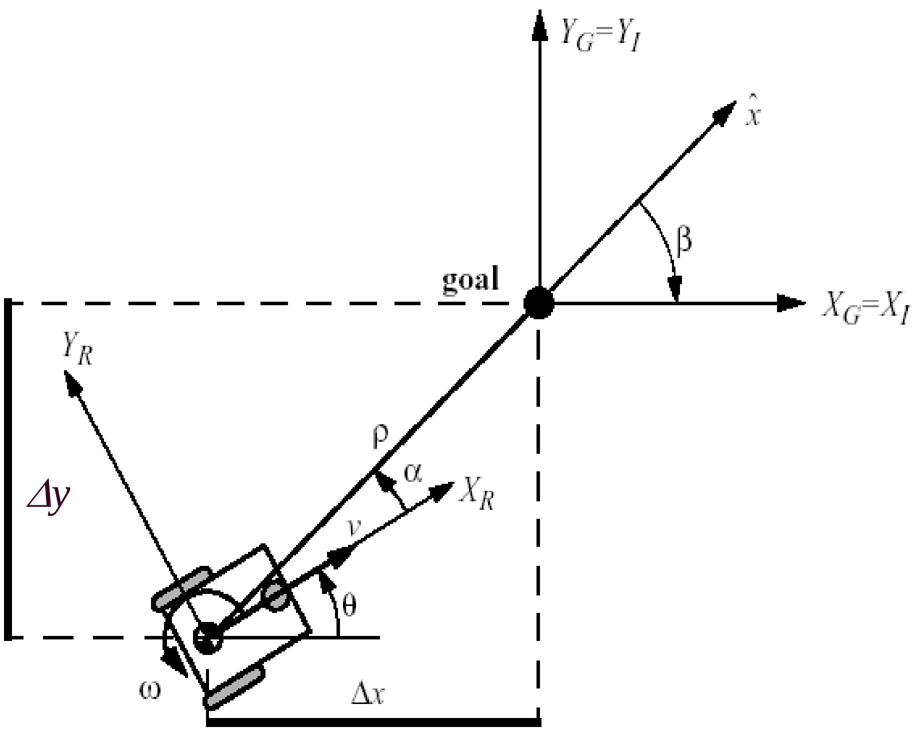
$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \cdot {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

- drives the error e to zero

$$\lim_{t \to \infty} e(t) = 0$$

- MIMO state feedback control

Motion Control: Kinematic Position Control



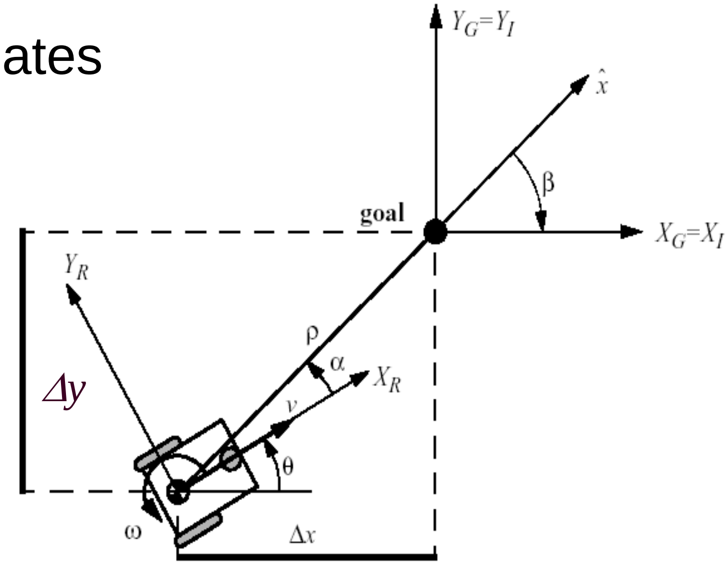
- The kinematics of a differential drive mobile robot described in the inertial frame $\{x_I, y_I, \theta\}$ is given by,

$$
{}^I\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}
$$

- where $\dot{x}$ and $\dot{y}$ are the linear velocities in the direction of the x_I and y_I of the inertial frame.
- Let alpha denote the angle between the x_R axis of the robots reference frame and the vector connecting the center of the axle of the wheels with the final position.

Kinematic Position Control: Coordinates Transformation

- Coordinates transformation into polar coordinates with its origin at goal position:



$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$

- System description, in the new polar coordinates

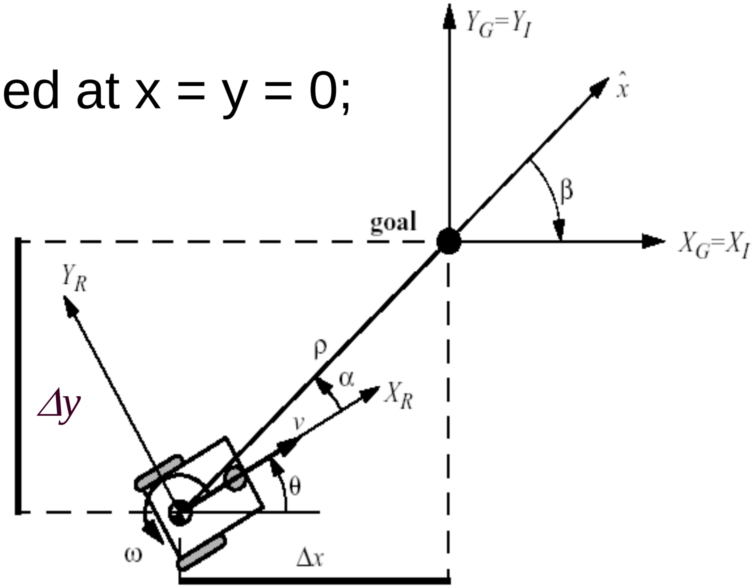
$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ -\frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

for $I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\frac{\sin\alpha}{\rho} & -1 \\ \frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

for $I_2 = (-\pi, -\pi/2] \cup (\pi/2, \pi]$

Kinematic Position Control: Remarks



- The coordinates transformation is not defined at x = y = 0;
- For $\alpha \in I_1$ the forward direction of the robot points toward the goal, for $\alpha \in I_2$ it is the backward direction.

$$\alpha \in I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- By properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at t=0. However this does not mean that a remains in I_1 for all time t.

Kinematic Position Control: The Control Law

- It can be shown, that with

$$v = k_\rho \rho \qquad \omega = k_\alpha \alpha + k_\beta \beta$$

the feedback controlled system

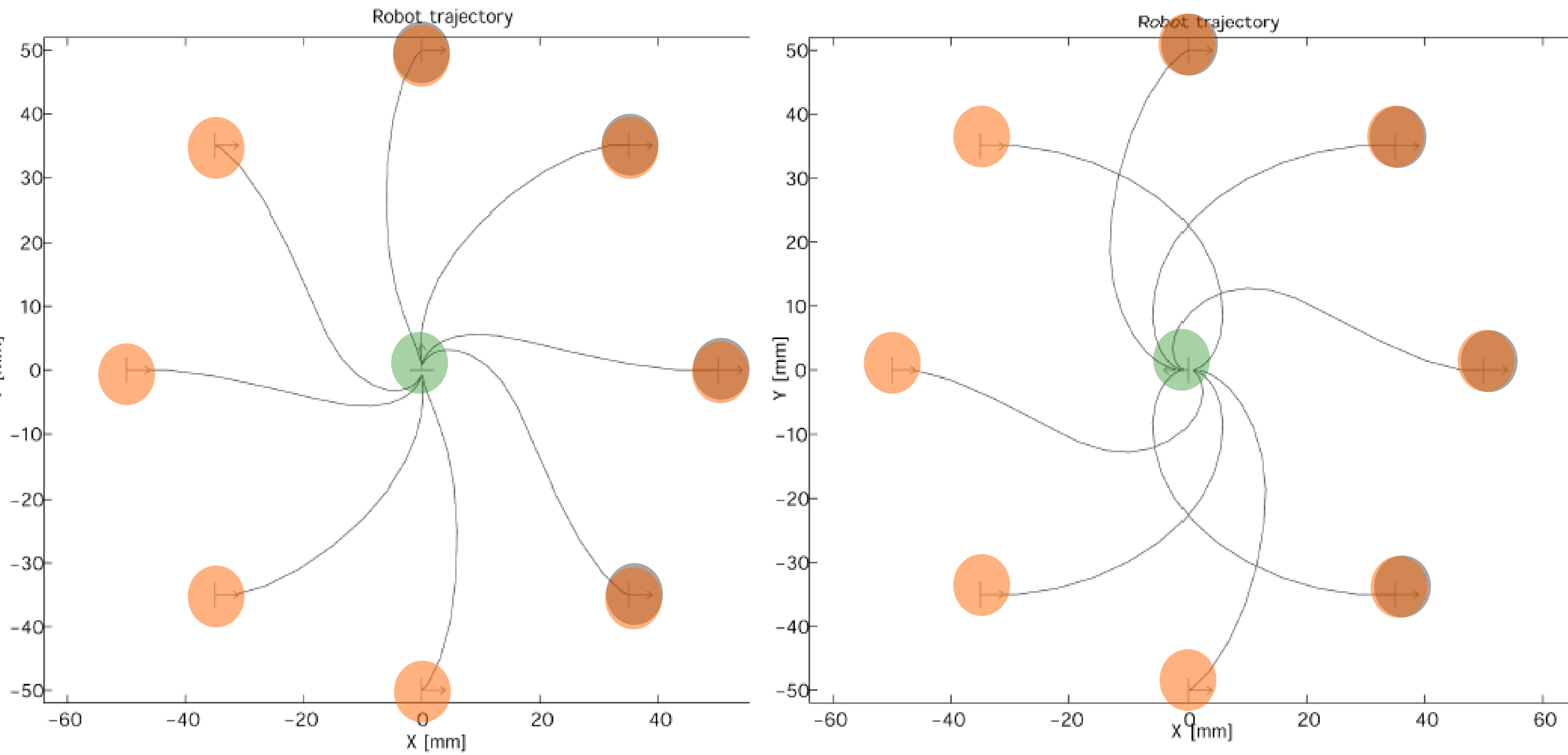
$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

will drive the robot to $(\rho,\alpha,\beta) = (0,0,0)$

- The control signal v has always constant sign,
  - the direction of movement is kept positive or negative during movement
  - parking maneuver is performed always in the most natural way and without ever inverting its motion.

Kinematic Position Control: Resulting Path

- The goal is in the center and the initial position on the circle.



$$k = (k_\rho, k_\alpha, k_\beta) = (3,8,-1.5)$$

Kinematic Position Control: Stability Issue

- It can further be shown, that the closed loop control system is locally exponentially stable if

$$k_\rho > 0 \;\; ; \;\; k_\beta < 0 \;\; ; \;\; k_\alpha - k_\rho > 0$$

$$\mathrm{k} = (\mathrm{k}_\rho, \mathrm{k}_\alpha, \mathrm{k}_\beta) = (3, 8, -1.5)$$

- Proof:
for small x –> cos$x$ = 1, sin$x$ = x

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix} \qquad A = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix}$$

and the characteristic polynomial of the matrix A of all roots

$$(\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha - k_\rho) - k_\rho k_\beta)$$

have negative real parts.