Inference in Bayesian networks

Chapter 14.4–5

Outline

◊ Exact inference by enumeration

◊ Exact inference by variable elimination

◊ Approximate inference by stochastic simulation

◊ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

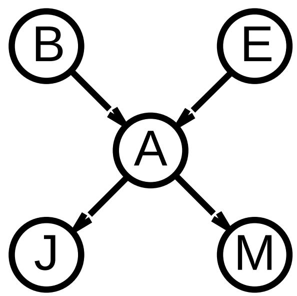
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha \sum_e \sum_a \mathbf{P}(B,e,a,j,m)$

Rewrite full joint entries using product of CPT entries:
$\mathbf{P}(B|j,m)$
$= \alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a)$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

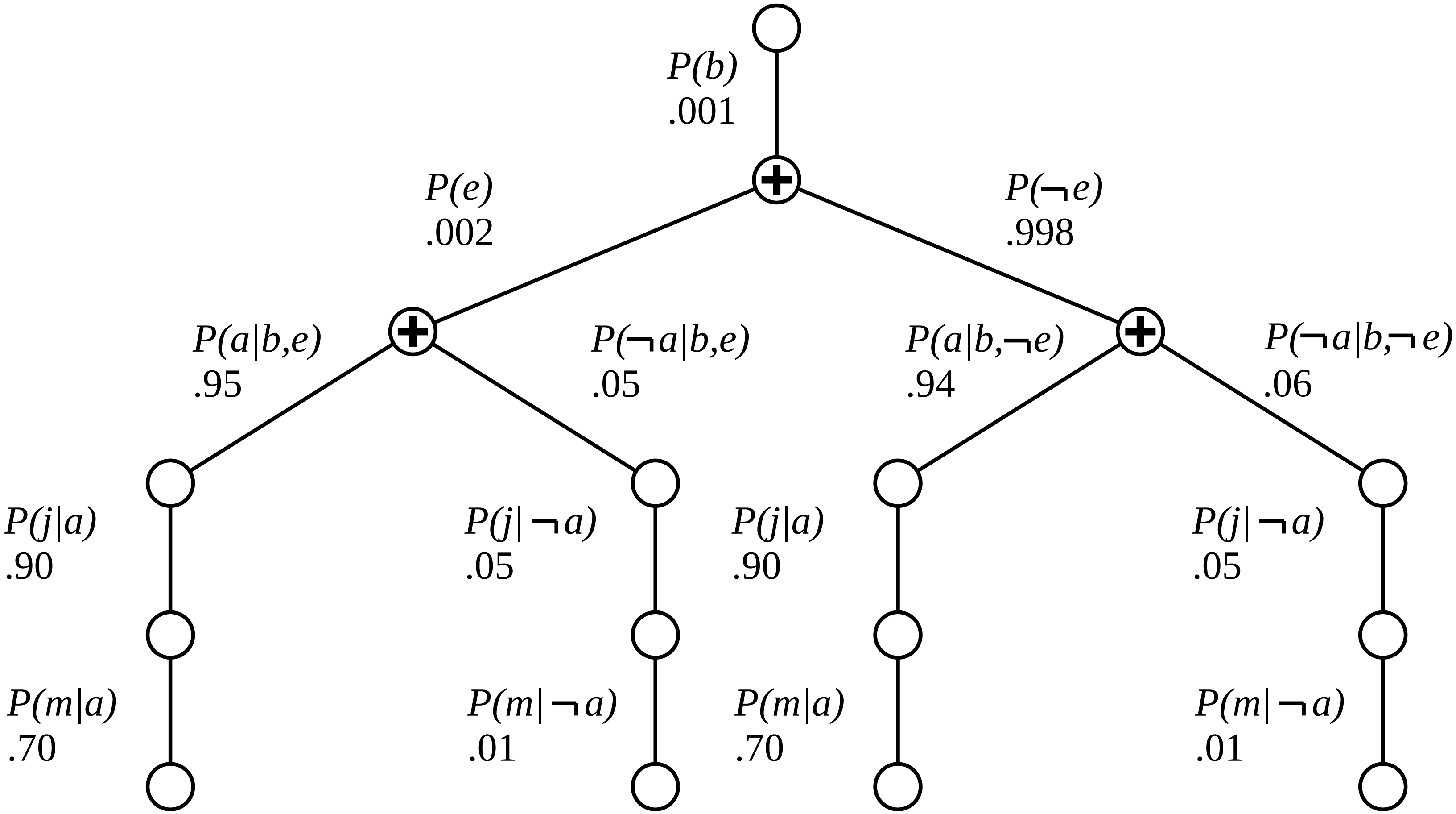
```
function Enumeration-Ask(X, e, bn) returns a distribution over X
   inputs: X, the query variable
           e, observed values for variables E
           bn, a Bayesian network with variables {X} ∪ E ∪ Y

   Q(X) ← a distribution over X, initially empty
   for each value x_i of X do
       extend e with value x_i for X
       Q(x_i) ← Enumerate-All(Vars[bn], e)
   return Normalize(Q(X))
```

```
function Enumerate-All(vars, e) returns a real number
   if Empty?(vars) then return 1.0
   Y ← First(vars)
   if Y has value y in e
       then return P(y | Pa(Y)) × Enumerate-All(Rest(vars), e)
       else return Σ_y P(y | Pa(Y)) × Enumerate-All(Rest(vars), e_y)
           where e_y is e extended with Y = y
```

Evaluation tree

$\mathbf{P}(B|j,m) = \alpha \mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a) = 0.284$



But, enumeration is inefficient; computes $P(j|a)P(m|a)$ for each value of e

Inference by variable elimination

Variable elimination: carry out summations right-to-left,
storing intermediate results (factors) to avoid recomputation.
(This is a form of dynamic programming, working from the bottom up.)

$$
\begin{aligned}
\mathbf{P}(B|j,m) \\
&= \alpha \underbrace{\mathbf{P}(B)}_{B} \textstyle\sum_e \underbrace{P(e)}_{E} \textstyle\sum_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M} \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a f_A(a,b,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)} \\
&= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)} \\
&= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:
 move any constant factors outside the summation
 add up submatrices in pointwise product of remaining factors

$$\Sigma_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \, \Sigma_x \, f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Example:

$$\begin{aligned} f_{\bar{A}JM}(B,E) = \Sigma_a f_A(a,B,E) \times f_J(a) \times f_M(a) \\ = f_A(a,B,E) \times f_J(a) \times f_M(a) \\ + f_A(\neg a,B,E) \times f_J(\neg a) \times f_M(\neg a) \end{aligned}$$

Pointwise product of factors f_1 and f_2:

$$\begin{aligned} f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l) \\ = f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l) \end{aligned}$$

E.g., $f_1(a,b) \times f_2(b,c) = f(a,b,c)$

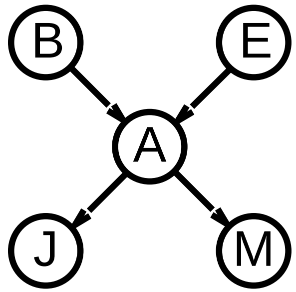
Variable elimination algorithm

```
function ELIMINATION-ASK(X, e, bn) returns a distribution over X
    inputs: X, the query variable
            e, evidence specified as an event
            bn, a belief network specifying joint distribution P(X_1, ..., X_n)

    factors ← [ ]; vars ← REVERSE(VARS[bn])
    for each var in vars do
        factors ← [MAKE-FACTOR(var, e)|factors]
        if var is a hidden variable then factors ← SUM-OUT(var, factors)
    return NORMALIZE(POINTWISE-PRODUCT(factors))
```

Irrelevant variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over m is identically 1; M is **irrelevant** to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so $MaryCalls$ is irrelevant

Complexity of exact inference

Singly connected networks (or polytrees):

– any two nodes are connected by at most one (undirected) path

– time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:

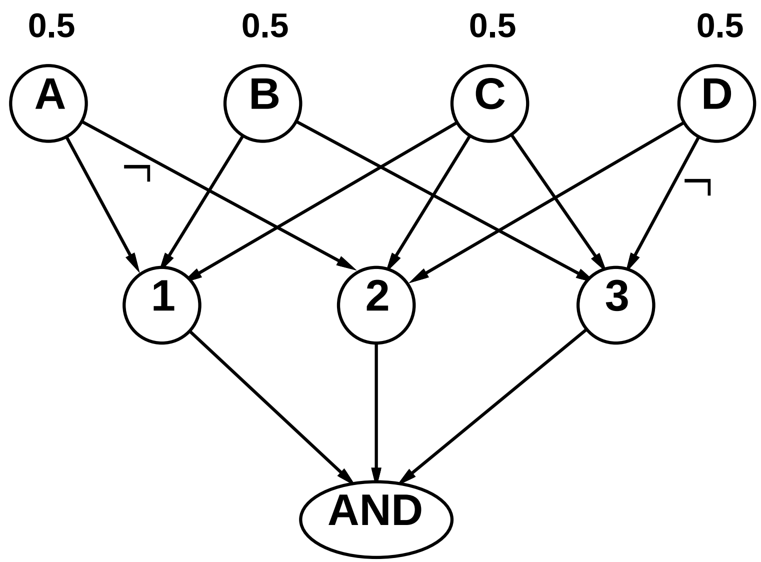
– can reduce 3SAT to exact inference $\Rightarrow$ NP-hard

– equivalent to **counting** 3SAT models $\Rightarrow$ #P-hard

1. A v B v C

2. C v D v ¬A

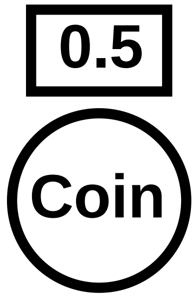
3. B v C v ¬D



Inference by stochastic simulation

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P



Outline:

– Sampling from an empty network
– Rejection sampling: reject samples disagreeing with evidence
– Likelihood weighting: use evidence to weight samples
– Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an empty network

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

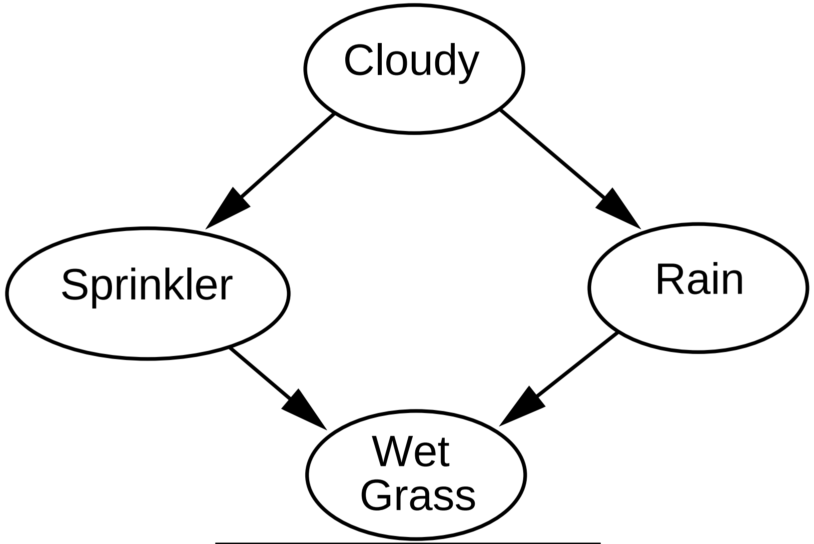
  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in $\mathbf{x}$
  return $\mathbf{x}$

Must sample each variable in turn, in topological order.

The probability distribution from which the value is sampled is conditioned on the values already assigned to the variable's parents.

Example

P(C)
.50



C	P(S|C)
T	.10
F	.50

C	P(R|C)
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S	R	P(W|S,R)
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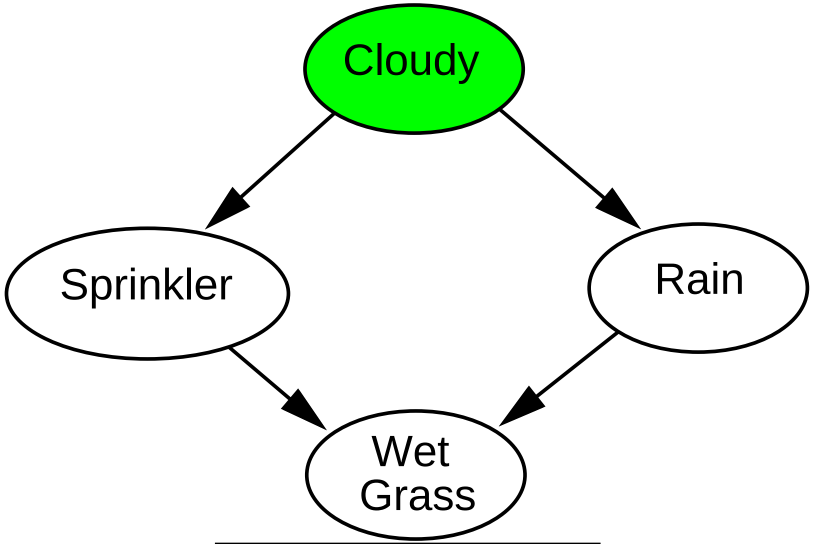

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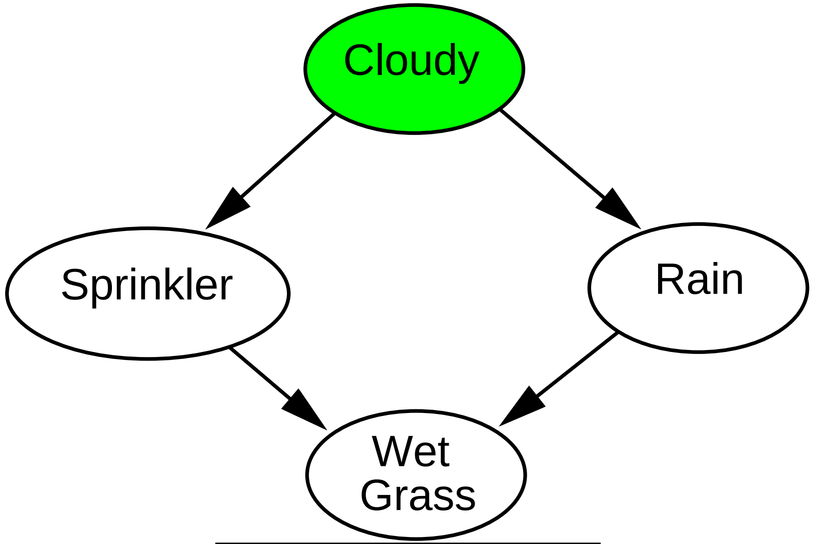

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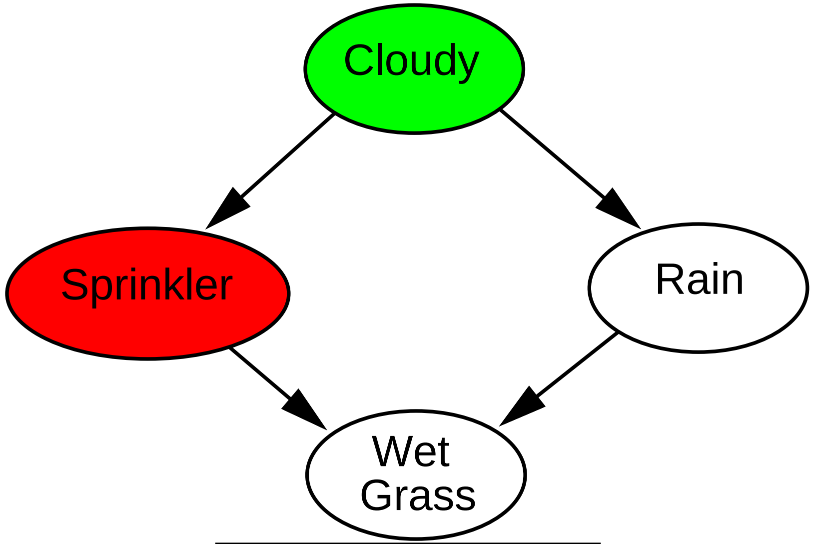

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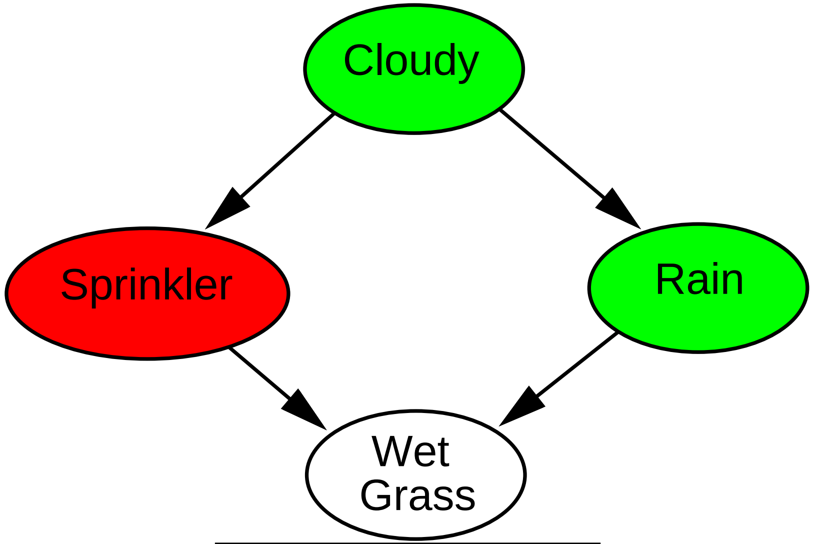

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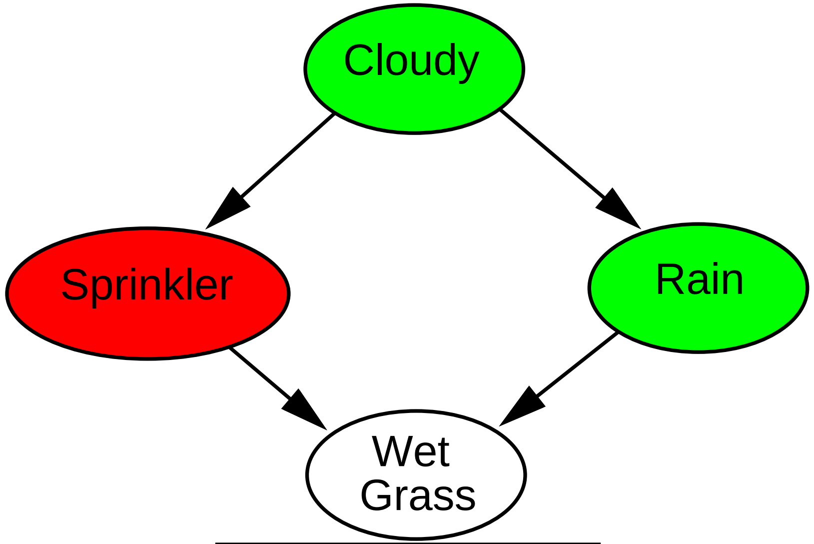

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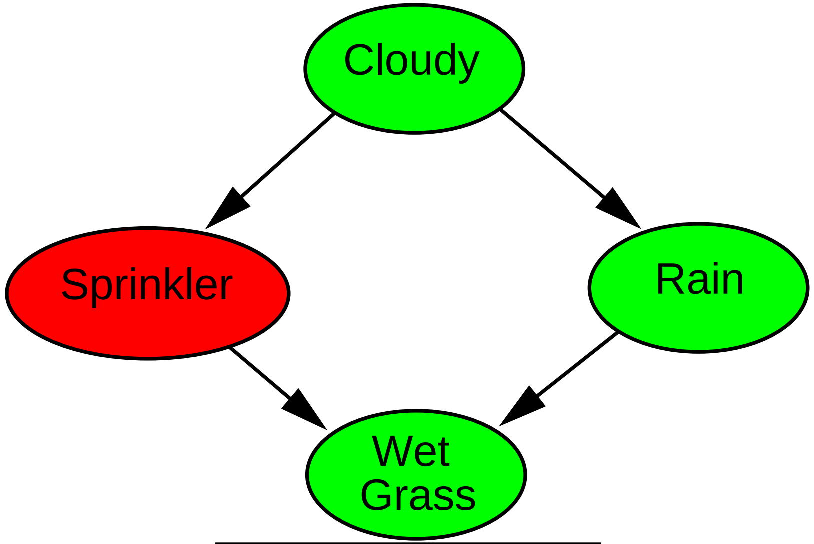

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Sampling from an empty network contd.

Probability that PriorSample generates a particular event

$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned}
\lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\
&= S_{PS}(x_1, \ldots, x_n) \\
&= P(x_1 \ldots x_n)
\end{aligned}$$

That is, estimates derived from PriorSample are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function Rejection-Sampling(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N, a vector of counts over X, initially zero

   for j = 1 to N do
       x ← Prior-Sample(bn)
       if x is consistent with e then
           N[x] ← N[x]+1 where x is the value of X in x
   return Normalize(N[X])
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler=true)$ using 100 samples
  27 samples have $Sprinkler=true$
    Of these, 8 have $Rain=true$ and 19 have $Rain=false$.

$\hat{\mathbf{P}}(Rain|Sprinkler=true) = \text{Normalize}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X, \mathbf{e})$ (algorithm defn.)

$= \mathbf{N}_{PS}(X, \mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)

$\approx \mathbf{P}(X, \mathbf{e})/P(\mathbf{e})$ (property of PriorSample)

$= \mathbf{P}(X|\mathbf{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables,
and weight each sample by the likelihood it accords the evidence

function LIKELIHOOD-WEIGHTING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for j = 1 to N **do**
    $\mathbf{x}$, $w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

---

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for i = 1 **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$, w

Likelihood weighting example

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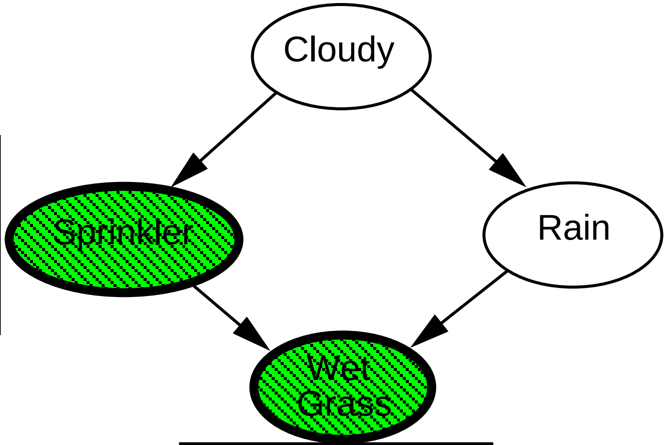

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$w = 1.0$

Likelihood weighting example

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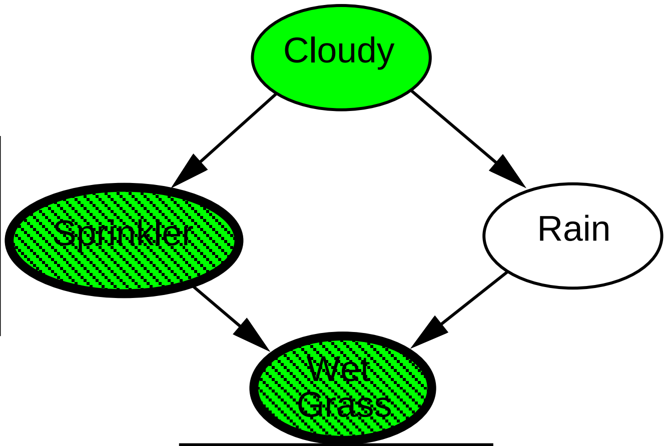

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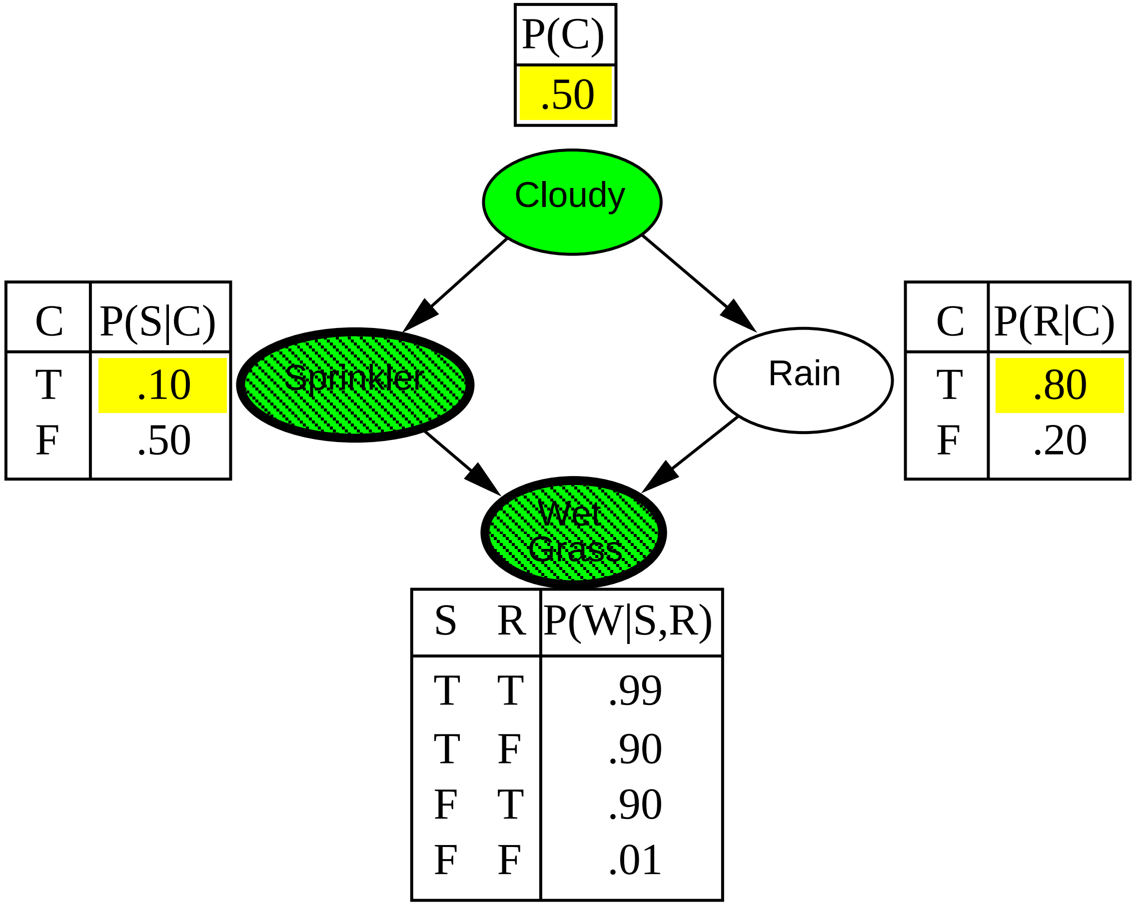

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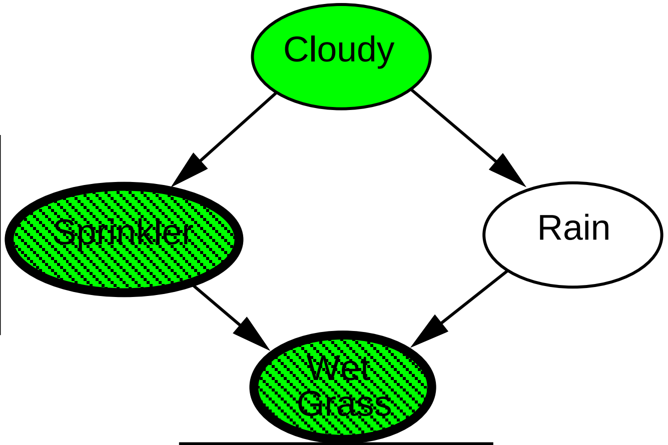

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$w = 1.0 \times 0.1$

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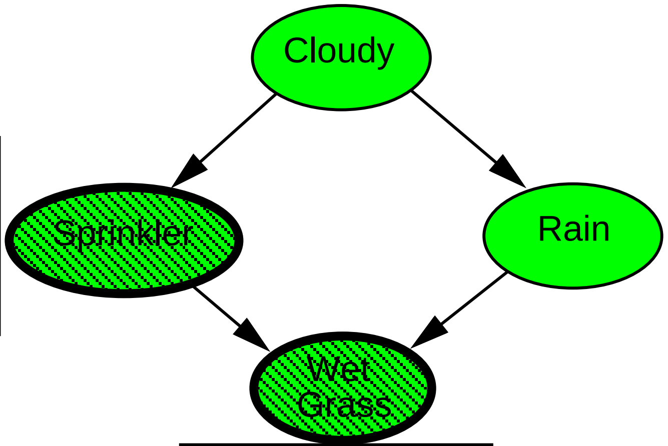

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Likelihood weighting example

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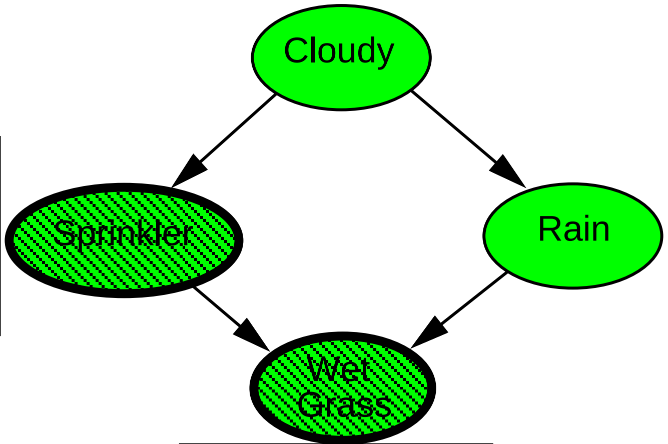

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Likelihood weighting example

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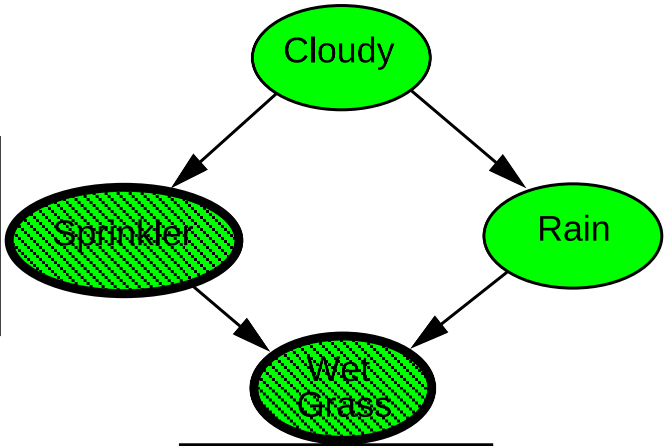

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$w = 1.0 \times 0.1 \times 0.99 = 0.099$. Weight is low because the event describes a cloudy day, which makes the sprinkler unlikely to be on.

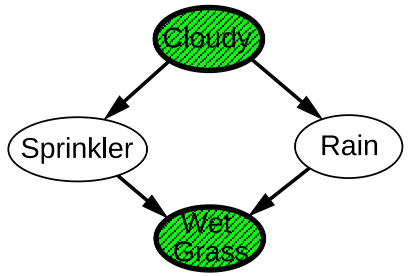
Likelihood weighting analysis

Sampling probability for WeightedSample is

$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | parents(Z_i))$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | parents(E_i))$

Weighted sampling probability is

$$S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e})$$
$$= \prod_{i=1}^{l} P(z_i | parents(Z_i)) \ \prod_{i=1}^{m} P(e_i | parents(E_i))$$
$$= P(\mathbf{z}, \mathbf{e})$$ (by standard global semantics of network)

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

Approximate inference using MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one non-evidence variable given its Markov blanket; Sample each variable in turn, keeping evidence fixed

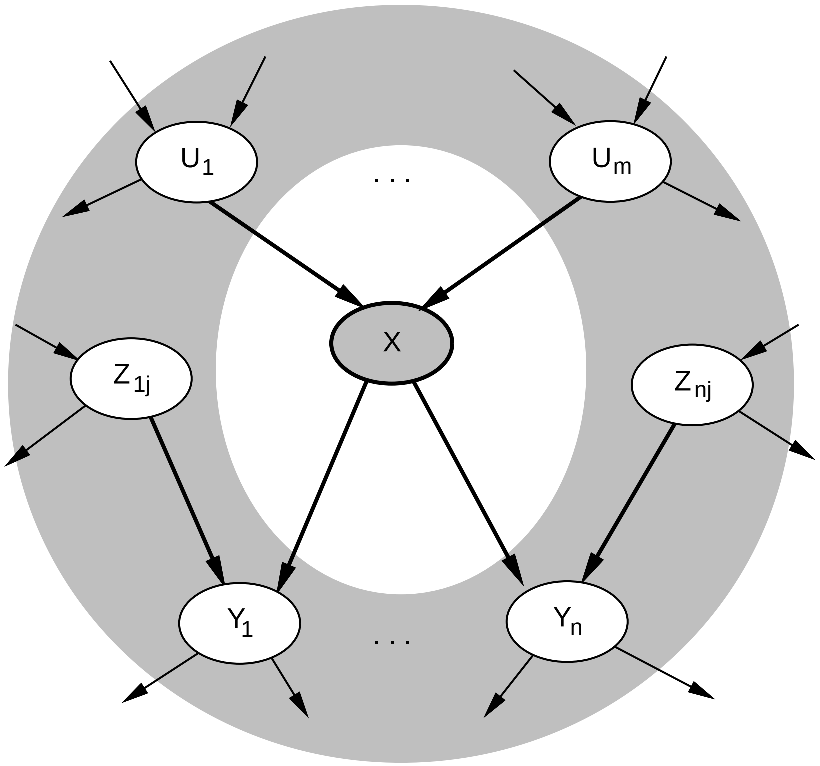
```
function MCMC-Ask(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N[X], a vector of counts over X, initially zero
                    Z, the nonevidence variables in bn
                    x, the current state of the network, initially copied from e

   initialize x with random values for the variables in Z
   for j = 1 to N do
       for each Z_i in Z do
           sample the value of Z_i in x from P(Z_i|mb(Z_i))
               given the values of MB(Z_i) in x
           N[x] ← N[x] + 1 where x is the value of X in x
   return Normalize(N[X])
```

Can also choose a variable to sample at random each time

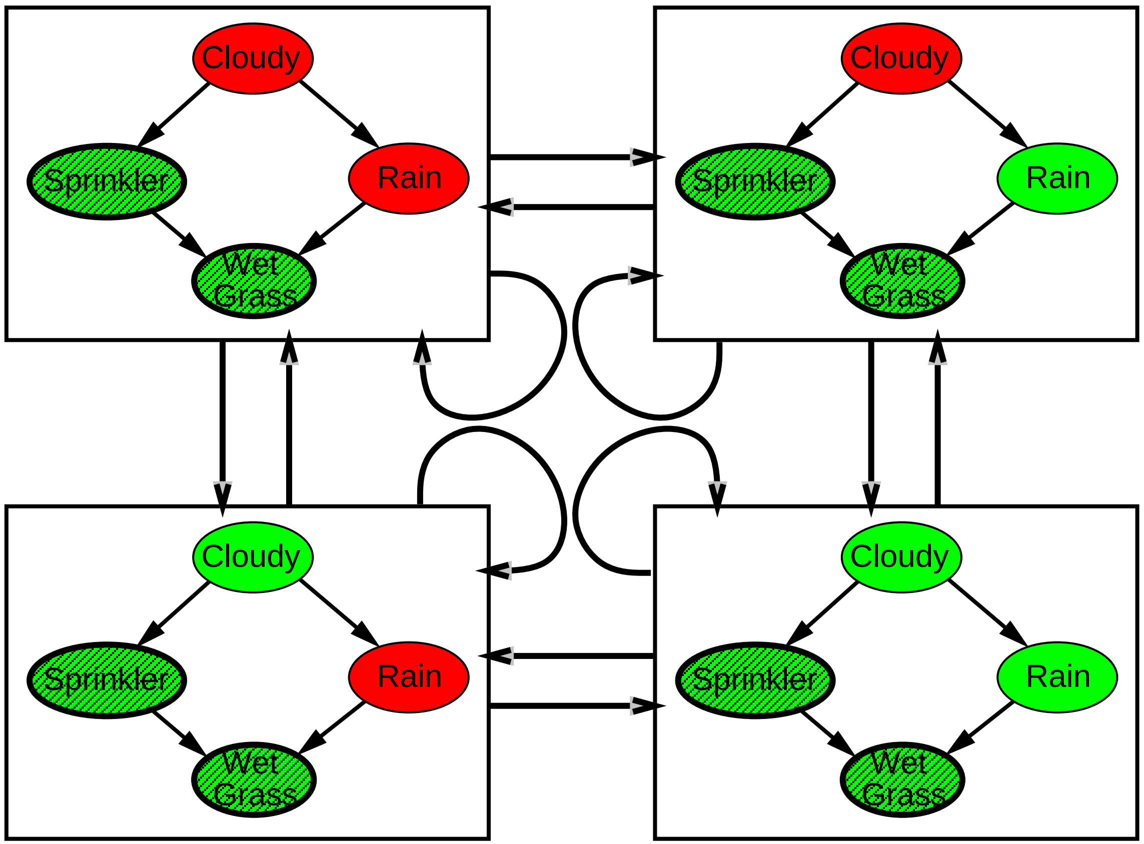
Recall: Markov blanket

Each node is conditionally independent of all others given its
Markov blanket: parents + children + children's parents



The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

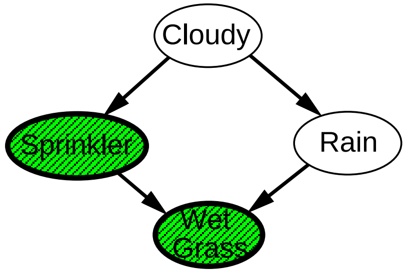
E.g., visit 100 states
  31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
  $= \text{NORMALIZE}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$

Theorem: chain approaches stationary distribution:
  long-run fraction of time spent in each state is exactly
  proportional to its posterior probability

Markov blanket sampling

Markov blanket of $Cloudy$ is
$Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
$Cloudy$, $Sprinkler$, and $WetGrass$



Probability given the Markov blanket is calculated as follows:

$$P(x_i'|mb(X_i)) = P(x_i'|parents(X_i))\prod_{Z_j \in Children(X_i)} P(z_j|parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
   $P(X_i|mb(X_i))$ won't change much (law of large numbers)

Summary

Exact inference by variable elimination:
- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

Approximate inference by LW, MCMC:
- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables