2. Lattices

Definition 2.1 (meet and join). For \(x, y \in P\), their meet is defined to be their greatest lower bound (if it exists): \(x \wedge y = \text{glb}\{x, y\}\). Likewise, the join is defined \(x \vee y = \text{lub}\{x, y\}\).

Problem 2.1. For the poset \((\mathbb{R}, \leq)\) describe the results of the meet and join operations.

Problem 2.2. Let \(P\) be a set of sets. What are the meet and join operations in the poset \((P, \subseteq)\)?

Problem 2.3. Let \(P\) be a set of sets. What are the meet and join operations in the poset \((P, \supseteq)\)? (Be careful! This means that \(x \subseteq y\) if and only if \(x \supseteq y\).)

Problem 2.4. What are the meet and join operations in the poset of truth values \((2, \Rightarrow)\)?

Definition 2.2 (lattice). A poset \(P\) is called a lattice if for each \(x, y \in P\), both \(x \wedge y\) and \(x \vee y\) exist.

Definition 2.3 (complete lattice). A lattice is complete if each of its subsets has both an lub and a glb.

Problem 2.5. Give an example of a lattice that is not complete.

Problem 2.6. Which of the example posets in Worksheet 1 are lattices? Which are complete?

Theorem 2.1. Any nonempty complete lattice has a greatest element \(\top\) and a least element \(\bot\).

Theorem 2.2. The dual of a lattice is a lattice; the dual of a complete lattice is a complete lattice.

Theorem 2.3. Let \(\mathcal{P}(S)\) be the set of all subsets of \(S\). Then \((\mathcal{P}(S), \subseteq)\) is a complete lattice. (What are its top and bottom elements?)

Whenever we define new operators, we should investigate immediately their properties. The meet and join operations satisfy a number of algebraic properties.

Theorem 2.4. In any poset, the meet and join operations, whenever they exist, satisfy the following algebraic laws:
L1 (Idempotent): $x \land x = x$, $x \lor x = x$.

L2 (Commutative): $x \land y = y \land x$, $x \lor y = y \lor x$.

L3 (Associative): $x \land (y \land z) = (x \land y) \land z$, $x \lor (y \lor z) = (x \lor y) \lor z$.

L4 (Absorption): $x \land (x \lor y) = x = x \lor (x \land y)$.

**Theorem 2.5** (consistency). $x \sqsubseteq y$ if and only if $x \land y = x$ if and only if $x \lor y = y$.

**Theorem 2.6.** If a poset $P$ has a top element $\top$, then for all $x \in P$, $x \land \top = x$ and $x \lor \top = \top$. Similarly for $\bot$.

**Theorem 2.7** (isotone property). The meet and join operations in a lattice are isotone; that is, if $y \sqsubseteq z$, then $x \land y \sqsubseteq x \land z$ and $x \lor y \sqsubseteq x \lor z$.

**Theorem 2.8** (distributive inequalities). In any lattice,

- $x \land (y \lor z) \sqsubseteq (x \land y) \lor (x \land z)$,
- $x \lor (y \land z) \sqsubseteq (x \lor y) \land (x \lor z)$.

**Problem 2.7.** You might be surprised that these are inequalities and not equalities. Find a lattice for which equality does not apply.

**Theorem 2.9** (modular inequality). In a lattice, $x \sqsubseteq z$ implies $x \lor (y \land z) \sqsubseteq (x \lor y) \land z$.

**Theorem 2.10.** In a lattice, $(a \lor b) \land (c \lor d) \sqsubseteq (a \land c) \lor (b \land d)$.

**Definition 2.4** (semilattice). A semilattice $(X, \diamond)$ is a set $X$ and a binary operation $\diamond$ on $X$ that is idempotent, commutative, and associative.

**Theorem 2.11.** If $P$ is a poset in which every pair of elements has a meet, then $(P, \land)$ is a semilattice. Likewise for $\lor$.

**Theorem 2.12.** In a semilattice $(X, \diamond)$ define $x \sqsubseteq y$ to mean $x \diamond y = x$. Then $(X, \sqsubseteq)$ is a poset with $x \diamond y = \text{glb}\{x, y\}$.

**Theorem 2.13.** A set with two binary operations obeying laws L1–L4 (Thm. 2.4) is a lattice, and conversely.

**Definition 2.5** (sublattice). If $L$ is a lattice, then $S \subseteq L$ is a sublattice if every pair of elements of $S$ has both a meet and a join in $S$ (i.e., using the same meet and join as $L$).

**Theorem 2.14.** Both the empty set and the singleton sets are sublattices of a lattice. (Always check “degenerate” cases such as these.)

**Problem 2.8.** Give examples of (non-degenerate) sublattices of the example lattices from Worksheet 1.
Theorem 2.15. If $L$ is a complete lattice and $S \subseteq L$, and if (1) $\top \in S$ and (2) $\text{glb} R \in S$ for every $R \subseteq S$, then $S$ is a complete lattice.

Problem 2.9. Give counter-examples showing that each of the two conditions in the preceding theorem are required.

Definition 2.6 (direct product of posets). If $P, Q$ are posets, their direct product $P \times Q$ is defined $(x, y) \sqsubseteq (x', y')$ if and only if $x \sqsubseteq x'$ in $P$ and $y \sqsubseteq y'$ in $Q$.

Theorem 2.16. The direct product of two lattices is a lattice.