The following simple arithmetic problems are for practice writing inductive proofs. All can be proved inductively, although some can also be proved noninductively. Most of the problems come from the *Introduction to Arithmetic* of Nicomachus of Gerasa (fl. c. 100 CE).

1. Show that the sum of the first \( n \) powers of 2 is \( 2^n - 1 \); that is \( 2^n - 1 = \sum_{k=0}^{n-1} 2^k \).

2. Show that \( r - 1 \) times the sum of the first \( n \) powers of \( r \) is \( r^n - 1 \), i.e., \( r^n - 1 = (r - 1) \sum_{k=0}^{n-1} r^k \). For example, \( 2 \times (1 + 3 + 9 + 27) = 80 = 81 - 1 = 3^4 - 1 \).

3. Show
\[
\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^n} = \frac{r^n - 1}{r - 1} + \frac{r^n}{r^n}.
\]

The following problems concern *figurate numbers*, that is, numbers in the shapes of triangles, squares, etc.

**Definition:** The triangular numbers are defined
\[
\begin{align*}
\triangle_1 &= 1, \\
\triangle_n &= \triangle_{n-1} + n, \quad n > 1
\end{align*}
\]

Thus the triangles are 1, 3, 6, 10, . . .

4. Show \( \triangle_n = n(n + 1)/2 \).

5. Show
\[
\frac{n + 1}{n} = \frac{\triangle_{2n+1}}{\triangle_{2n}}, \quad n \geq 1.
\]

6. Show
\[
\sum_{k=2}^{n} \frac{1}{\triangle_k} = \frac{\triangle_{n-1}}{\triangle_n}, \quad n \geq 2.
\]

**Definition:** The square numbers are defined \( \square_n = n^2 \). Thus the squares are 1, 4, 9, 16, . . .

7. Show that \( \square_n \) is the sum of the first \( n \) odd integers; for example, \( 3^2 = 1 + 3 + 5 \).

8. Show that the \( n \)th square is the sum of the \( n \)th and \( n - 1 \)st triangles, \( \square_n = \triangle_n + \triangle_{n-1} \). (For what values of \( n \) does this statement make sense?)
9. Show that one more than 8 times a triangle is a square, that is, \(8\triangle_n + 1\) is square.

10. Show

\[
\square_n = 1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1.
\]

For example, \(3^2 = 9 = 1 + 2 + 3 + 2 + 1\).

**Definition:** The \(n\)th oblong number is \(0_n = n(n + 1)\). Thus the oblong numbers are \(2, 6, 12, 20, \ldots\)

11. Show that the \(n\)th oblong number is the sum of the first \(n\) even numbers. For example, \(0_3 = 12 = 2 + 4 + 6\).

12. Show that the \(2n\)th triangle is the sum of the \(n\)th square and the \(n\) oblong, \(\triangle_{2n} = \square_n + 0_n\).

13. List the square and oblong numbers in an alternating sequence,

\[
\square_1, 0_1, \square_2, 0_2, \square_3, 0_3, \ldots
\]

That is,

\[
1, 2, 4, 6, 9, 12, 16, 20, \ldots
\]

Show that the sum of any two consecutive numbers in this sequence is a triangular number.

**Definition:** The *pentagonal numbers* are defined

\[
\Pi_1 = 1,
\]
\[
\Pi_n = \Pi_{n-1} + 3n - 2, \quad n > 1
\]

Thus the pentagons are \(1, 5, 12, 22, \ldots\)

14. Show that the \(n\)th pentagonal number is the sum of the first \(n\) numbers in the series \(1, 4, 7, 10, \ldots\) (increasing by 3s). For example, \(\Pi_3 = 12 = 1 + 4 + 7\).

15. Show that the \(n\)th pentagonal number is the sum of the \(n\)th square and the \(n - 1\)st triangle, i.e., \(\Pi_n = \square_n + \triangle_{n-1}\), for \(n > 1\).

16. Show that 1 more than 24 times a pentagon is a square, that is, \(24\Pi_n + 1\) is square. *Hint:* Write out the cases \(n = 1, 2, 3, 4\) to get a formula for the square.

**Definition:** The \(n\)th *hexagonal number* \(H_n\) is equal to the sum of the first \(n\) numbers in the sequence \(1, 5, 9, 13, \ldots\) (increasing by 4s). Thus the hexagons are \(1, 6, 15, 28, \ldots\)

\section*{2}
17. Show that the $n$th hexagon is the sum of the $n$th pentagon and the $n - 1$st triangle, $H_n = P_n + \triangle_{n-1}$.

**Definition:** Let $\Phi^s_n$ represent the $n$th figurate number with $s$ sides, so, for example, $\triangle_n = \Phi^3_n$, $\square_n = \Phi^4_n$, etc. In general,

\[
\Phi^s_1 = 1, \\
\Phi^s_{n+1} = \Phi^s_n + (s-2)n + 1, \quad n \geq 1.
\]

(Actually, the definition of $\Phi^s_n$ can be extended to $n = 0$. What should be the value of $\Phi^0_n$?)

18. Show $\Phi^s_{n+1} = \Phi^s_n + \triangle_{n-1}$, for $s \geq 2$.

19. Show $\Phi^s_n = n + (s-2)\triangle_{n-1}$ for $s \geq 2$.

20. Show that $\Phi^s_n$ is the sum of the first $n$ numbers in the sequence

\[
1, \quad (s-2) + 1, \quad 2(s-2) + 1, \quad 3(s-2) + 1, \quad \ldots
\]

(increasing by $s-2$).

21. List the odd numbers:

\[
1, 3, 5, 7, 9, 11, 13, \ldots
\]

The first is a cube, 1. The sum of the next two is a cube, $3 + 5 = 8 = 2^3$. The sum of the next three is a cube, $7 + 9 + 11 = 27 = 3^3$. And so forth. Express the general proposition mathematically and prove or disprove it.