B. Stochastic Neural Networks

(in particular, the stochastic Hopfield network)

Motivation

- **Idea:** with low probability, go against the local field
  - move up the energy surface
  - make the “wrong” microdecision
- **Potential value for optimization:** escape from local optima
- **Potential value for associative memory:** escape from spurious states
  - because they have higher energy than imprinted states

The Stochastic Neuron

Deterministic neuron: \( x'_i = \text{sgn}(h_i) \)
\[
\Pr\{x'_i = +1\} = \Theta(h_i) \\
\Pr\{x'_i = -1\} = 1 - \Theta(h_i)
\]

Stochastic neuron:
\[
\Pr\{x'_i = +1\} = \sigma(h_i) \\
\Pr\{x'_i = -1\} = 1 - \sigma(h_i)
\]

Logistic sigmoid: \( \sigma(h) = \frac{1}{1 + \exp(-2h/T)} \)
Properties of Logistic Sigmoid

\[ \sigma(h) = \frac{1}{1 + e^{-2h/T}} \]

- As \( h \to +\infty \), \( \sigma(h) \to 1 \)
- As \( h \to -\infty \), \( \sigma(h) \to 0 \)
- \( \sigma(0) = 1/2 \)

Logistic Sigmoid With Varying \( T \)

\( T \) varying from 0.05 to \( \infty \) (\( 1/T = \beta = 0, 1, 2, \ldots, 20 \))

Logistic Sigmoid

\( T = 0.5 \)

Slope at origin = \( 1/2T \)

Logistic Sigmoid

\( T = 0.01 \)

Logistic Sigmoid

\( T = 0.1 \)

Logistic Sigmoid

\( T = 1 \)
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Logistic Sigmoid

\[ T = 10 \]

\[ T = 100 \]

Pseudo-Temperature

- Temperature = measure of thermal energy (heat)
- Thermal energy = vibrational energy of molecules
- A source of random motion
- Pseudo-temperature = a measure of nondirected (random) change
- Logistic sigmoid gives same equilibrium probabilities as Boltzmann-Gibbs distribution

Transition Probability

Recall, change in energy \( \Delta E = -\Delta s_i h_i \)

\[ = 2s_i h_i \]

\[ Pr\{s_i = \pm 1\} = \sigma(\pm h_i) = \sigma(-s_i h_i) \]

\[ Pr\{s_i \rightarrow -s_i\} = \frac{1}{1 + \exp(2s_i h_i/T)} \]

\[ = \frac{1}{1 + \exp(\Delta E/T)} \]

Stability

- Are stochastic Hopfield nets stable?
- Thermal noise prevents absolute stability
- But with symmetric weights:
  - average values \( \langle s_i \rangle \) become time-invariant

Does “Thermal Noise” Improve Memory Performance?

- Experiments by Bar-Yam (pp. 316-20):
  - \( n = 100 \)
  - \( p = 8 \)
  - Random initial state
  - To allow convergence, after 20 cycles set \( T = 0 \)
  - How often does it converge to an imprinted pattern?
Analysis of Stochastic Hopfield Network

- Complete analysis by Daniel J. Amit & colleagues in mid-80s
- The analysis is beyond the scope of this course
Simulated Annealing

(Kirkpatrick, Gelatt & Vecchi, 1983)

Dilemma

• In the early stages of search, we want a high temperature, so that we will explore the space and find the basins of the global minimum

• In the later stages we want a low temperature, so that we will relax into the global minimum and not wander away from it

• Solution: decrease the temperature gradually during search

Quenching vs. Annealing

• Quenching:
  – rapid cooling of a hot material
  – may result in defects & brittleness
  – local order but global disorder
  – locally low-energy, globally frustrated

• Annealing:
  – slow cooling (or alternate heating & cooling)
  – reaches equilibrium at each temperature
  – allows global order to emerge
  – achieves global low-energy state

Multiple Domains

Moving Domain Boundaries

Effect of Moderate Temperature
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Effect of High Temperature

Effect of Low Temperature

Annealing Schedule

- Controlled decrease of temperature
- Should be sufficiently slow to allow equilibrium to be reached at each temperature
- With sufficiently slow annealing, the global minimum will be found with probability 1
- Design of schedules is a topic of research

Typical Practical Annealing Schedule

- Initial temperature $T_0$ sufficiently high so all transitions allowed
- Exponential cooling: $T_{k+1} = \alpha T_k$
  - typical $0.8 < \alpha < 0.99$
  - fixed number of trials at each temp.
  - expect at least 10 accepted transitions
- Final temperature: three successive temperatures without required number of accepted transitions

Summary

- Non-directed change (random motion) permits escape from local optima and spurious states
- Pseudo-temperature can be controlled to adjust relative degree of exploration and exploitation

Quantum Annealing

- See for example D-wave Systems
  - <www.dwavesys.com>
Part 3B: Stochastic Neural Networks

Hopfield Network for Task Assignment Problem

- Six tasks to be done (I, II, …, VI)
- Six agents to do tasks (A, B, …, F)
- They can do tasks at various rates
  - A (10, 5, 4, 6, 5, 1)
  - B (6, 4, 9, 7, 3, 2)
  - etc.
- What is the optimal assignment of tasks to agents?

Continuous Hopfield Net

\[ \dot{U}_i = \sum_{j=1}^{n} T_{ij} V_j + I_i - \frac{U_i}{\tau} \]
\[ V_i = \sigma(U_i) \in (0,1) \]

k-out-of-n Rule

Network for Task Assignment

NetLogo Implementation of Task Assignment Problem

Run TaskAssignment.nlogo