Homework 1 — CS 594 only! — Due: Sept. 29, 2004

General Instructions

- Please work on this assignment independently.
- You can hand in your assignment electronically (email to ytang@cs.utk.edu) or as hardcopy (put it in Yifan Tang’s mailbox).

Introduction

In class (Lecture 7) we studied an activation-inhibition pattern generation system defined by a CA with the update rule:

\[
{s_i(t + 1) = \text{sign} \left[ h + J_1 \sum_{r_i < R_1} s_j(t) + J_2 \sum_{R_1 < r_i < R} s_j(t) \right].}
\]

Recall that \( r_{ij} \) represents the distance between cells \( i \) and \( j \), so the first summation is over all cells within a distance of \( R_1 \) to cell \( i \), and the second summation is over all cells with a distance between \( R_1 \) and \( R_2 \). (It doesn’t matter what you do with the \( r_{ij} = R_1 \) case; include it in the first or second summation as you like.) For simplicity, assume that the \( R_1 \) neighborhood does not include the center cell \( i \).

The state of a CA can be updated either \textit{synchronously} or \textit{asynchronously}. With synchronous updating, which is what we usually do, all the states are updated simultaneously. With asynchronous updating the cells are updated one at a time (usually in some random order).

This homework assignment explores the stability of this activation-inhibition system; that is, does it inevitably reach a stable state?
Problems

Problem 1
Prove that if the states are updated asynchronously, then the CA must reach a stable state.

Hint: Define the following function (called an energy or Lyapunov function) of the total state of a CA:¹

\[
E(s(t)) = -\frac{1}{2} \sum_i s_i(t) \left[ 2h + J_1 \sum_{r_i<R_i} s_j(t) + J_2 \sum_{r_i<R_i} s_j(t) \right].
\]

Show that updating any single cell, according to the state update rule, cannot increase this function (that is, \(\Delta E \leq 0\)). What else do you need to show in order to guarantee convergence to a stable state?

Extra Credit: Assume that the \(R_i\) neighborhood does include the center cell, and explore any additional assumptions that might be needed to guarantee convergence.

Problem 2
Prove, by exhibiting a counter-example, that if synchronous updating is used, then the CA may not reach a stable state.

Hint: Construct a very simple CA, obeying the above state update equation, that cycles between two different states.

¹ For this energy function, look in Bar-Yam on p. 630 (section 7.2.2) and p. 170 (sec. 1.6.6).