**Conditions for Stability**

Stability of entire pattern:

\[ x^n = \text{sgn} \left( x^n + \sum_{k=1}^{n} x^k \cos \theta_{km} \right) \]

Stability of a single bit:

\[ x_i^m = \text{sgn} \left( x_i^m + \sum_{k=1}^{n} x^k \cos \theta_{km} \right) \]

**Sufficient Conditions for Instability (Case 1)**

Suppose \( x_i^m = -1 \). Then unstable if:

\[ (-1) + \frac{1}{n} \sum_{k=1}^{n} x^k \cos \theta_{km} > 0 \]

\[ \frac{1}{n} \sum_{k=1}^{n} x^k \cos \theta_{km} > 1 \]

**Sufficient Conditions for Instability (Case 2)**

Suppose \( x_i^m = +1 \). Then unstable if:

\[ (+1) + \frac{1}{n} \sum_{k=1}^{n} x^k \cos \theta_{km} < 0 \]

\[ \frac{1}{n} \sum_{k=1}^{n} x^k \cos \theta_{km} < -1 \]

**Sufficient Conditions for Stability**

\[ \left| \frac{1}{n} \sum_{k=1}^{n} x^k \cos \theta_{km} \right| \leq 1 \]

The crosstalk with the sought pattern must be sufficiently small
Capacity of Hopfield Memory

- Depends on the patterns imprinted
- If orthogonal, $p_{\text{max}} = n$
  - but every state is stable ⇒ trivial basins
- So $p_{\text{max}} < n$
- Let load parameter $\alpha = p / n$

Single Bit Stability Analysis

- For simplicity, suppose $x^i_k$ are random
- Then $x^i_k \cdot x^n$ are sums of $n$ random $\pm 1$
  - binomial distribution ≈ Gaussian
  - in range $-n, \ldots, +n$
  - with mean $\mu = 0$
  - and variance $\sigma^2 = n$
- Probability sum $> t$:
  $$\frac{1}{\sqrt{2\pi}} \left[ 1 - \text{erf} \left( \frac{t}{\sqrt{2n}} \right) \right]$$
[See “Review of Gaussian (Normal) Distributions” on course website]

Approximation of Probability

Let crosstalk $C_i^n = \frac{1}{n} \sum_{i=1}^{n} (x^i_k \cdot x^n)$
We want $\Pr\{C_i^n > 1\} = \Pr\{nC_i^n > n\}$

Note: $nC_i^n = \sum_{j=1}^{n} x^i_j x^n_j$
A sum of $n(p - 1) \approx np$ random $\pm 1$
Variance $\sigma^2 = np$

Probability of Bit Instability

$$\Pr\{nC_i^n > n\} = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{n}{\sqrt{2np}} \right) \right] = \frac{1}{2} \left[ 1 - \text{erf} \left( \sqrt{n/2p} \right) \right]$$

Tabulated Probability of Single-Bit Instability

<table>
<thead>
<tr>
<th>( P_{\text{error}} )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1%</td>
<td>0.105</td>
</tr>
<tr>
<td>0.36%</td>
<td>0.138</td>
</tr>
<tr>
<td>1%</td>
<td>0.185</td>
</tr>
<tr>
<td>5%</td>
<td>0.37</td>
</tr>
<tr>
<td>10%</td>
<td>0.61</td>
</tr>
</tbody>
</table>

Spurious Attractors

- **Mixture states:**
  - sums or differences of odd numbers of retrieval states
  - number increases combinatorially with \( \rho \)
  - shallower, smaller basins
  - basins of mixtures swamp basins of retrieval states \( \Rightarrow \) overload
  - useful as combinatorial generalizations?
  - self-coupling generates spurious attractors

- **Spin-glass states:**
  - not correlated with any finite number of imprinted patterns
  - occur beyond overload because weights effectively random

Basins of Mixture States

\[ x_i^{\text{mix}} = \text{sgn}(x_i^1 + x_i^2 + x_i^3) \]

Fraction of Unstable Imprints

(\( n = 100 \))
Summary of Capacity Results

- Absolute limit: $p_{\text{max}} < \alpha_n = 0.138 n$
- If a small number of errors in each pattern permitted: $p_{\text{max}} \propto n$
- If all or most patterns must be recalled perfectly: $p_{\text{max}} \propto n / \log n$
- Recall: all this analysis is based on random patterns
- Unrealistic, but sometimes can be arranged

Stochastic Neural Networks

(in particular, the stochastic Hopfield network)
Part 3: Autonomous Agents

Trapping in Local Minimum

Escape from Local Minimum

Escape from Local Minimum