Conditions for Stability

Stability of entire pattern:

$$\mathbf{x}^m = \operatorname{sgn}\left(\mathbf{x}^m + \tfrac{1}{n}\sum_{k \neq m} \mathbf{x}^k \cos\theta_{km}\right)$$

Stability of a single bit:

$$x_i^m = \operatorname{sgn}\left(x_i^m + \tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km}\right)$$

11/10/04 1

Sufficient Conditions for Instability (Case 1)

Suppose $x_i^m = -1$. Then unstable if:

$$(-1) + \tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} > 0$$

$$\tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} > 1$$

11/10/04 2

Sufficient Conditions for Instability (Case 2)

Suppose $x_i^m = +1$. Then unstable if:

$$(+1) + \tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} < 0$$

$$\tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} < -1$$

11/10/04 3

Sufficient Conditions for Stability

$$\left|\tfrac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km}\right| \leq 1$$

The crosstalk with the sought pattern must be sufficiently small

11/10/04 4

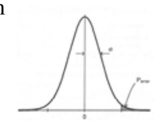
Capacity of Hopfield Memory

- Depends on the patterns imprinted
- If orthogonal, $p_{\max} = n$
  - but every state is stable $\Rightarrow$ trivial basins
- So $p_{\max} < n$
- Let load parameter $\alpha = p / n$

11/10/04 equations 5

Single Bit Stability Analysis

- For simplicity, suppose $\mathbf{x}^k$ are random
- Then $\mathbf{x}^k \cdot \mathbf{x}^m$ are sums of n random ± 1
  - binomial distribution $\approx$ Gaussian
  - in range $-n, \ldots, +n$
  - with mean $\mu = 0$
  - and variance $\sigma^2 = n$
- Probability sum $> t$: $\frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{t}{\sqrt{2n}}\right)\right]$



[See "Review of Gaussian (Normal) Distributions" on course website]

11/10/04 6

Approximation of Probability

Let crosstalk $C_i^m = \frac{1}{n} \sum_{k \neq m} x_i^k \left(\mathbf{x}^k \cdot \mathbf{x}^m\right)$

We want $\Pr\left\{C_i^m > 1\right\} = \Pr\left\{nC_i^m > n\right\}$

Note: $nC_i^m = \sum_{\substack{k=1 \\ k \neq m}}^{p} \sum_{j=1}^{n} x_i^k x_j^k x_j^m$

A sum of $n(p-1) \approx np$ random ± 1

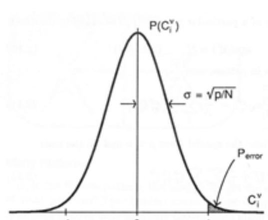
Variance $\sigma^2 = np$

11/10/04 7

Probability of Bit Instability

$$\Pr\left\{nC_i^m > n\right\} = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{n}{\sqrt{2np}}\right)\right]$$

$$= \frac{1}{2}\left[1 - \operatorname{erf}\left(\sqrt{n/2p}\right)\right]$$



11/10/04 (fig. from Hertz & al. *Intr. Theory Neur. Comp.*) 8

Tabulated Probability of Single-Bit Instability

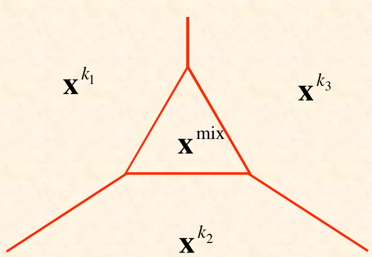
P_{error}	α
0.1%	0.105
0.36%	0.138
1%	0.185
5%	0.37
10%	0.61

(table from Hertz & al. *Intr. Theory Neur. Comp.*)

Spurious Attractors

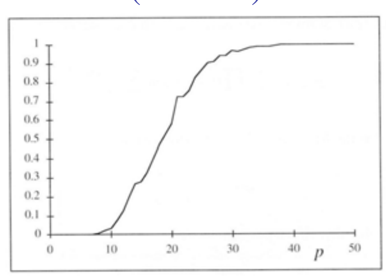
- Mixture states:
  - sums or differences of odd numbers of retrieval states
  - number increases combinatorially with *p*
  - shallower, smaller basins
  - basins of mixtures swamp basins of retrieval states ⇒ overload
  - useful as combinatorial generalizations?
  - self-coupling generates spurious attractors
- Spin-glass states:
  - not correlated with any finite number of imprinted patterns
  - occur beyond overload because weights effectively random

Basins of Mixture States



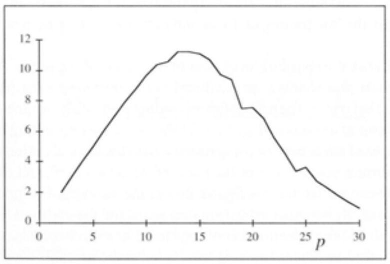
$$x_i^{mix} = \text{sgn}\left(x_i^{k_1} + x_i^{k_2} + x_i^{k_3}\right)$$

Fraction of Unstable Imprints (n = 100)



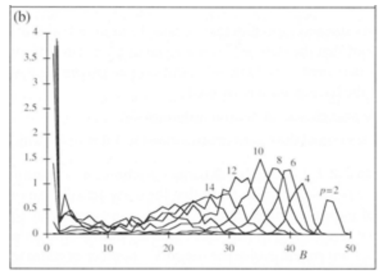
(fig from Bar-Yam)

Number of Stable Imprints (n = 100)



11/10/04 (fig from Bar-Yam) 13

Number of Imprints with Basins of Indicated Size (n = 100)



11/10/04 (fig from Bar-Yam) 14

Summary of Capacity Results

- Absolute limit: $p_{max} < \alpha_c n = 0.138\ n$
- If a small number of errors in each pattern permitted: $p_{max} \propto n$
- If all or most patterns must be recalled perfectly: $p_{max} \propto n / \log n$
- Recall: all this analysis is based on *random* patterns
- Unrealistic, but sometimes can be arranged

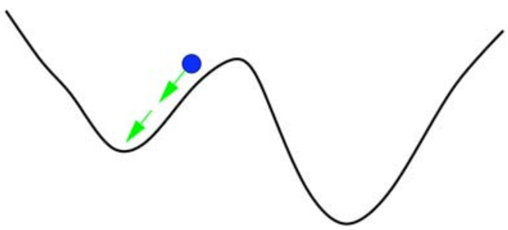
11/10/04 15

Stochastic Neural Networks

(in particular, the stochastic Hopfield network)

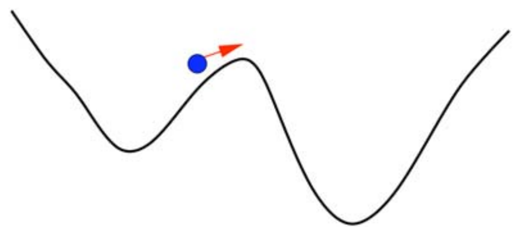
11/10/04 16

Trapping in Local Minimum



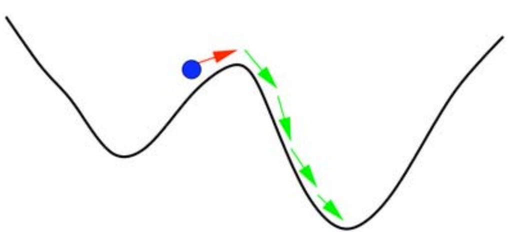
11/10/04 17

Escape from Local Minimum



11/10/04 18

Escape from Local Minimum



11/10/04 19