Lecture 4

Shannon Information
(very briefly!)

- Information varies directly with surprise
- Information varies inversely with probability
- Information is additive
- $\therefore$ The information content of a message is proportional to the negative log of its probability

$$I\{s\} = -\lg \Pr\{s\}$$

Entropy

- Suppose have source S of symbols from ensemble $\{s_1, s_2, \ldots, s_N\}$
- Average information per symbol:

$$\sum_{k=1}^{N} \Pr\{s_k\} I\{s_k\} = \sum_{k=1}^{N} \Pr\{s_k\}\left(-\lg \Pr\{s_k\}\right)$$

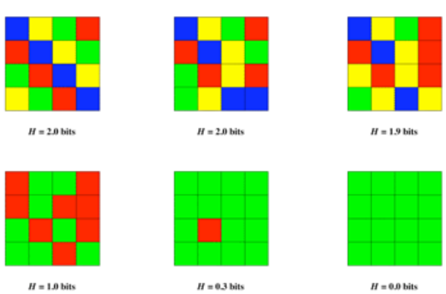
- This is the *entropy* of the source:

$$H\{S\} = -\sum_{k=1}^{N} \Pr\{s_k\} \lg \Pr\{s_k\}$$

Maximum and Minimum Entropy

- Maximum entropy is achieved when all signals are equally likely
  No ability to guess; maximum surprise
  $H_{\max} = \lg N$
- Minimum entropy occurs when one symbol is certain and the others are impossible
  No uncertainty; no surprise
  $H_{\min} = 0$

Entropy Examples



Entropy of Transition Rules

- Among other things, a way to measure the uniformity of a distribution

$$H = -\sum_i p_i \lg p_i$$

- Distinction of quiescent state is arbitrary
- Let n_k = number mapping into state k
- Then $p_k = n_k / T$

$$H = \lg T - \frac{1}{T}\sum_{k=1}^{K} n_k \lg n_k$$

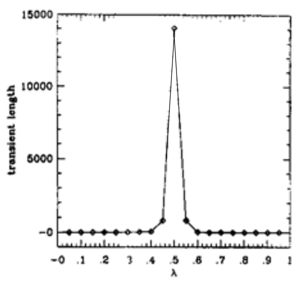
Entropy Range

- Maximum entropy ($\lambda = 1 - 1/K$):
  uniform as possible
  all $n_k = T/K$
  $H_{max} = \lg K$
- Minimum entropy ($\lambda = 0$ or $\lambda = 1$):
  non-uniform as possible
  one $n_s = T$
  all other $n_r = 0$ ($r \neq s$)
  $H_{min} = 0$

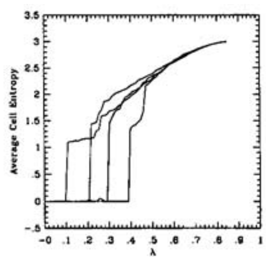
Further Investigations by Langton

- 2-D CAs
- $K = 8$
- $N = 5$
- 64×64 lattice
- periodic boundary conditions

Avg. Transient Length vs. λ (K=4, N=5)

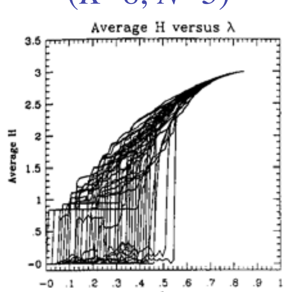


Avg. Cell Entropy vs. λ (K=8, N=5)

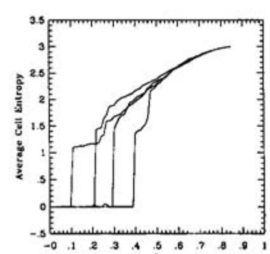


$$H(A) = \sum_{k=1}^{K} p_k \lg p_k$$

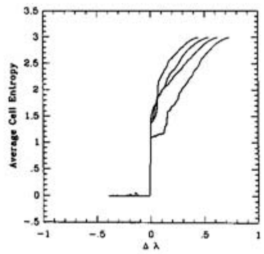
Avg. Cell Entropy vs. λ (K=8, N=5)



Avg. Cell Entropy vs. λ (K=8, N=5)



Avg. Cell Entropy vs. Δλ (*K*=8, *N*=5)



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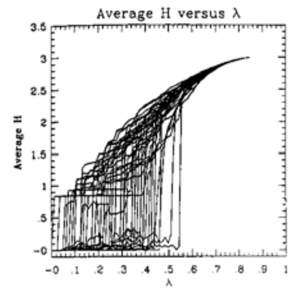
Avg. Cell Entropy vs. λ (*K*=8, *N*=5)



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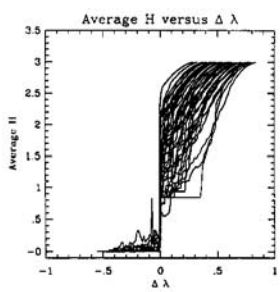
Avg. Cell Entropy vs. Δλ (*K*=8, *N*=5)



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Entropy of Independent Systems

- Suppose sources *A* and *B* are independent
- Let $p_j = \Pr\{a_j\}$ and $q_k = \Pr\{b_k\}$
- Then $\Pr\{a_j, b_k\} = \Pr\{a_j\} \Pr\{b_k\} = p_j q_k$

$$H(A,B) = \sum_{j,k} \Pr(a_j, b_k) \lg \Pr(a_j, b_k)$$
$$= \sum_{j,k} p_j q_k \lg(p_j q_k) = \sum_{j,k} p_j q_k (\lg p_j + \lg q_k)$$
$$= \sum_j p_j \lg p_j + \sum_k q_k \lg q_k = H(A) + H(B)$$

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Mutual Information

- *Mutual information* measures the degree to which two sources are not independent
- A measure of their correlation

$$I(A,B) = H(A) + H(B) - H(A,B)$$

- $I(A,B) = 0$ for completely independent sources
- $I(A,B) = H(A) = H(B)$ for completely correlated sources

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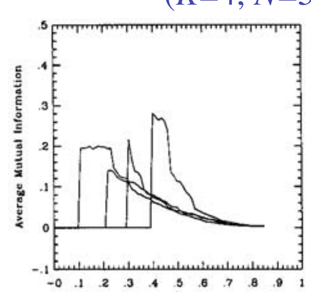
Avg. Mutual Info vs. λ (*K*=4, *N*=5)



$I(A,B) =$
$H(A) + H(B)$
$- H(A,B)$

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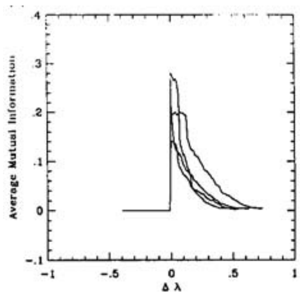
Avg. Mutual Info vs. Δλ (K=4, N=5)



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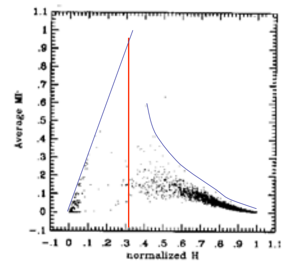
Mutual Information vs. Normalized Cell Entropy



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Critical Entropy Range

- Information *storage* involves *lowering* entropy
- Information *transmission* involves *raising* entropy
- *Computation* requires a tradeoff between low and high entropy

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Suitable Media for Computation

- How can we identify/synthesize novel computational media?
  - especially nanostructured materials for massively parallel computation
- Seek materials/systems exhibiting Class IV behavior
  - may be identifiable via entropy, mut. info., etc.
- Find physical properties (such as λ) that can be controlled to put into Class IV

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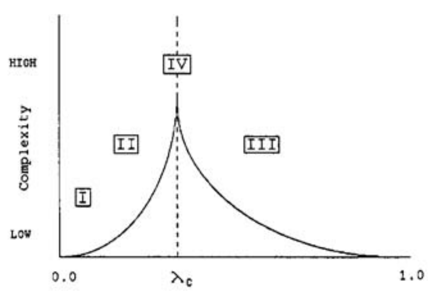
Complexity vs. λ



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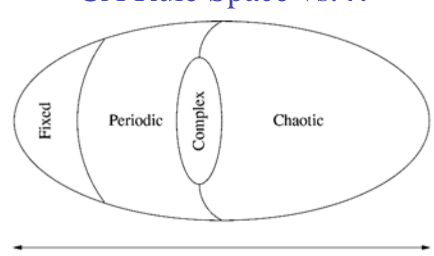
Schematic of CA Rule Space vs. λ



9/4/07 Fig. from Langton, "Life at Edge of Chaos" 24

Some of the Work in this Area

- Wolfram: *A New Kind of Science*
  - www.wolframscience.com/nksonline/toc.html
- Langton: Computation/life at the edge of chaos
- Crutchfield: Computational mechanics
- Mitchell: Evolving CAs
- and many others…

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Some Other Simple Computational Systems Exhibiting the Same Behavioral Classes

- CAs (1D, 2D, 3D, totalistic, etc.)
- Mobile Automata
- Turing Machines
- Substitution Systems
- Tag Systems
- Cyclic Tag Systems
- Symbolic Systems (combinatory logic, lambda calculus)
- Continuous CAs (coupled map lattices)
- PDEs
- Probabilistic CAs
- Multiway Systems

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Universality

- A system is *computationally universal* if it can compute anything a Turing machine (or digital computer) can compute
- The Game of Life is universal
- Several 1D CAs have been proved to be universal
- Are all complex (Class IV) systems universal?
- Is universality rare or common?

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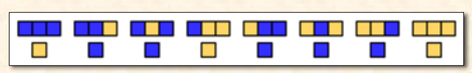
Rule 110: A Universal 1D CA

- $K = 2$, $N = 3$
- New state $= \neg(p \wedge q \wedge r) \wedge (q \vee r)$

  where p, q, r are neighborhood states
- Proved by Wolfram



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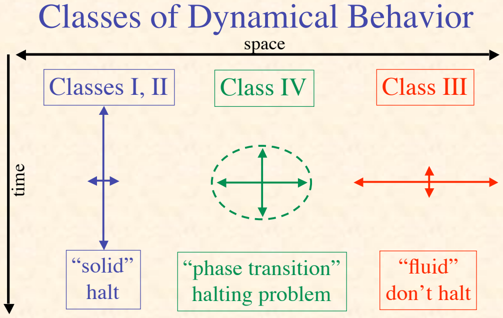
Fundamental Universality Classes of Dynamical Behavior



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Wolfram's Principle of Computational Equivalence

- "a fundamental unity exists across a vast range of processes in nature and elsewhere: despite all their detailed differences every process can be viewed as corresponding to a computation that is ultimately equivalent in its sophistication" (*NKS* 719)
- Conjecture: "among all possible systems with behavior that is not obviously simple an overwhelming fraction are universal" (*NKS* 721)

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Computational Irreducibility

- "systems one uses to make predictions cannot be expected to do computations that are any more sophisticated than the computations that occur in all sorts of systems whose behavior we might try to predict" (*NKS* 741)
- "even if in principle one has all the information one needs to work out how some particular system will behave, it can still take an irreducible amount of computational work to do this" (*NKS* 739)
- That is: for Class IV systems, you can't (in general) do better than simulation.

Additional Bibliography

1. Langton, Christopher G. "Computation at the Edge of Chaos: Phase Transitions and Emergent Computation," in *Emergent Computation*, ed. Stephanie Forrest. North-Holland, 1990.
2. Langton, Christopher G. "Life at the Edge of Chaos," in *Artificial Life II*, ed. Langton et al. Addison-Wesley, 1992.
3. Emmeche, Claus. *The Garden in the Machine: The Emerging Science of Artificial Life*. Princeton, 1994.
4. Wolfram, Stephen. *A New Kind of Science*. Wolfram Media, 2002.

Part 2B