Hopfield Network

- Symmetric weights: \( w_{ij} = w_{ji} \)
- No self-action: \( w_{ii} = 0 \)
- Zero threshold: \( \theta = 0 \)
- Bipolar states: \( s_i \in \{-1, +1\} \)
- Discontinuous bipolar activation function:
  \[
  o(h) = \text{sgn}(h) = \begin{cases} 
  -1, & h < 0 \\
  +1, & h > 0 
  \end{cases}
  \]

What to do about \( h = 0 \)?

- There are several options:
  - \( o(0) = +1 \)
  - \( o(0) = -1 \)
  - \( o(0) = -1 \) or \(+1\) with equal probability
  - \( h_i = 0 \Rightarrow \) no state change \( (s'_i = s_i) \)
  - Not much difference, but be consistent
  - Last option is slightly preferable, since symmetric

Positive Coupling

- Positive sense (sign)
- Large strength

Negative Coupling

- Negative sense (sign)
- Large strength

Weak Coupling

- Either sense (sign)
- Little strength
State = −1 & Local Field < 0

State = −1 & Local Field > 0

State Reverses

State = +1 & Local Field > 0

State = +1 & Local Field < 0

State Reverses
Part 3: Autonomous Agents

NetLogo Demonstration of Hopfield State Updating

Run Hopfield-update.nlogo

Hopfield Net as Soft Constraint Satisfaction System
• States of neurons as yes/no decisions
• Weights represent soft constraints between decisions
  – hard constraints must be respected
  – soft constraints have degrees of importance
• Decisions change to better respect constraints
• Is there an optimal set of decisions that best respects all constraints?

Demonstration of Hopfield Net Dynamics I

Run Hopfield-dynamics.nlogo

Demonstration of Hopfield Net Dynamics II

Run initialized Hopfield.nlogo

Convergence
• Does such a system converge to a stable state?
• Under what conditions does it converge?
• There is a sense in which each step relaxes the “tension” in the system
• But could a relaxation of one neuron lead to greater tension in other places?

Quantifying “Tension”
• If \( w_{ij} > 0 \), then \( s_i \) and \( s_j \) want to have the same sign \( (s_i s_j = +1) \)
• If \( w_{ij} < 0 \), then \( s_i \) and \( s_j \) want to have opposite signs \( (s_i s_j = -1) \)
• If \( w_{ij} = 0 \), their signs are independent
• Strength of interaction varies with \( |w_{ij}| \)
• Define disharmony (“tension”) \( D_{ij} \) between neurons \( i \) and \( j \):
  \[ D_{ij} = -s_i w_{ij} s_j \]
• \( D_{ij} < 0 \) ⇒ they are happy
• \( D_{ij} > 0 \) ⇒ they are unhappy
Total Energy of System

The “energy” of the system is the total “tension” (disharmony) in it:

\[ E[s] = \sum_{i,j} D_{ij} = -\sum s_i w_{ij} s_j \]

\[ = -\frac{1}{2} \sum s_i w_{ij} s_j \]

\[ = -\frac{1}{2} \sum s_i w_{ij} s_j \]

\[ = -\frac{1}{2} s^T W s \]

Another View of Energy

The energy measures the number of neurons whose states are in disharmony with their local fields (i.e. of opposite sign):

\[ E[s] = -\frac{1}{2} \sum s_i w_{ij} s_j \]

\[ = -\frac{1}{2} \sum s_i w_{ij} s_j \]

\[ = -\frac{1}{2} \sum s_i h_i \]

\[ = -\frac{1}{2} s^T h \]

Energy Does Not Increase

- In each step in which a neuron is considered for update:
  \[ E(s(t + 1)) - E(s(t)) \leq 0 \]
- Energy cannot increase
- Energy decreases if any neuron changes
- Must it stop?

Review of Some Vector Notation

\[ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \ldots, x_n)^T \] (column vectors)

\[ x^T y = \sum_{i=1}^n x_i y_i = x \cdot y \] (inner product)

\[ x y^T = \begin{pmatrix} x_1 y_1 & \cdots & x_n y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{pmatrix} \] (outer product)

\[ x^T My = \sum_{j=1}^n \sum_{k=1}^m x_j M_{jk} y_k \] (quadratic form)

Do State Changes Decrease Energy?

- Suppose that neuron \( k \) changes state
- Change of energy:
  \[ \Delta E = E[s'] - E[s] \]
  \[ = \sum_{i,j} D_{ij} = -\sum s_i' w_{ij} s_j' \]
  \[ = -\frac{1}{2} \sum s_i' w_{ij} s_j' \]
  \[ = -\frac{1}{2} \sum s_i w_{ij} s_j \]
  \[ = -\Delta s_i h_i \]
  \[ < 0 \]

Proof of Convergence in Finite Time

- There is a minimum possible energy:
  - The number of possible states \( s \in \{-1, +1\}^n \) is finite
  - Hence \( E_{min} = \min \{ E(s) \mid s \in \{-1, +1\}^n \} \) exists
- Must show it is reached in a finite number of steps
Steps are of a Certain Minimum Size

If $h_k > 0$, then (let $h_{min} = \min$ of possible positive $h$)

$$h_i \geq \min \left( \left\{ h = \sum_{j \neq i} W_{ij}s_j \wedge s \in \{ \pm 1 \}^n \wedge h > 0 \right\} \right) = h_{min}$$

$$\Delta E = -\Delta s_i h_i = -2h_i \leq -2h_{min}$$

If $h_k < 0$, then (let $h_{max} = \max$ of possible negative $h$)

$$h_i \geq \max \left( \left\{ h = \sum_{j \neq i} W_{ij}s_j \wedge s \in \{ \pm 1 \}^n \wedge h < 0 \right\} \right) = h_{max}$$

$$\Delta E = -\Delta s_i h_i = 2h_i \leq 2h_{max}$$

Conclusion

- If we do asynchronous updating, the Hopfield net must reach a stable, minimum energy state in a finite number of updates
- This does not imply that it is a global minimum

Lyapunov Functions

- A way of showing the convergence of discrete- or continuous-time dynamical systems
- For discrete-time system:
  - Need a Lyapunov function $E$ ("energy" of the state)
  - $E$ is bounded below ($E(s) > E_{min}$)
  - $\Delta E < (\Delta E)_{max} \leq 0$ (energy decreases a certain minimum amount each step)
  - Then the system will converge in finite time
- Problem: finding a suitable Lyapunov function

Example Limit Cycle with Synchronous Updating

The Hopfield Energy Function is Even

- A function $f$ is odd if $f(-x) = -f(x)$, for all $x$
- A function $f$ is even if $f(-x) = f(x)$, for all $x$
- Observe:

$$E(-s) = -\frac{1}{2}(-s)^T W(-s) = -\frac{1}{2}s^T Ws = E(s)$$

Conceptual Picture of Descent on Energy Surface
Energy Surface (fig. from Haykin Neur. Netw.)

Energy Surface + Flow Lines (fig. from Haykin Neur. Netw.)

Flow Lines
Basins of Attraction (fig. from Haykin Neur. Netw.)