C. QUANTUM INFORMATION

C.7 Universal quantum gates

This lecture follows NC §4.5.

¶1. **Classical logic circuits:** Both the Fredkin (controlled swap) and Toffoli (controlled-controlled-NOT) gates are sufficient for classical logic circuits.

¶2. But note that they can operate on qubits in superposition.

¶3. **Single-qubit unitary operators:** Single-qubit unitary operators can be approximated arbitrarily closely by the Hadamard and $T$ ($\pi/8$) gates.

¶4. **$\pi/8$ or $T$ gate:** The $T$ or $\pi/8$ gate is defined:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \approx \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$$

(Illustration 19)

(ignoring global phase).

¶5. For an $m$-gate circuit and an accuracy of $\epsilon$, $\mathcal{O}(m \log^c(m/\epsilon))$, where $c \approx 2$, gates are needed (Solovay-Kitaev theorem).

¶6. **Two-level unitary operations:** A two-level operation is one on a $d$-dimensional Hilbert space that non-trivially affects only two qubits out of $n$ (where $d = 2^n$).

¶7. Any two-level unitary operation can be computed by a combination of CNOTs and single-qubit operations.

¶8. This requires $\mathcal{O}(n^2)$ single-qubit and CNOT gates.

¶9. **Arbitrary unitary matrix:** An arbitrary $d$-dimensional unitary matrix can be decomposed into a product of two-level unitary matrices.

¶10. At most $d(d - 1)/2$ are required.

Therefore an operator on an $n$-qubit system requires at most $2^{n-1}(2^n - 1)$ two-level matrices.

¶11. **Conclusions:** The $H$ (Hadamard), CNOT, and $\pi/8$ gates are sufficient.
¶12. **Fault-tolerance:** For fault-tolerance, either the *standard set* $H$ (Hadamard), CNOT, $\pi/8$, and $S$ (phase) can be used, or $H$, CNOT, Toffoli, and $S$.

¶13. **$S$ or phase gate:** The *phase gate* is defined:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$  \hspace{1cm} (III.20)

Note $S = T^2$. 