C.7 Universal quantum gates

This lecture follows Nielsen & Chuang (2010, §4.5).

¶1. Classical logic circuits: Both the Fredkin (controlled swap) and Toffoli (controlled-controlled-NOT) gates are sufficient for classical logic circuits.

¶2. But note that they can operate on qubits in superposition.

¶3. Single-qubit unitary operators: Single-qubit unitary operators can be approximated arbitrarily closely by the Hadamard and $T$ ($\pi/8$) gates.

¶4. $\pi/8$ or $T$ gate: The $T$ or $\pi/8$ gate is defined:

$$
T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \cong \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \quad \text{(III.19)}
$$

(ignoring global phase).

¶5. For an $m$-gate circuit and an accuracy of $\epsilon$, $O(m \log c(m/\epsilon))$, where $c \approx 2$, gates are needed (Solovay-Kitaev theorem).

¶6. Two-level unitary operations: A two-level operation is one on a $d$-dimensional Hilbert space that non-trivially affects only two qubits out of $n$ (where $d = 2^n$).

¶7. Any two-level unitary operation can be computed by a combination of CNOTs and single-qubit operations.

¶8. This requires $O(n^2)$ single-qubit and CNOT gates.

¶9. Arbitrary unitary matrix: An arbitrary $d$-dimensional unitary matrix can be decomposed into a product of two-level unitary matrices.

¶10. At most $d(d - 1)/2$ are required.

Therefore an operator on an $n$-qubit system requires at most $2^{n-1}(2^n - 1)$ two-level matrices.

¶11. Conclusions: The $H$ (Hadamard), CNOT, and $\pi/8$ gates are sufficient.
¶12. **Fault-tolerance:** For fault-tolerance, either the *standard set* $H$ (Hadamard), CNOT, $\pi/8$, and $S$ (phase) can be used, or $H$, CNOT, Toffoli, and $S$.

¶13. **$S$ or phase gate:** The *phase gate* is defined:

$$ S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. $$  \hspace{1cm} (III.20)

Note $S = T^2$. 