B Filtering models

1. Filtering models: Operate by filtering out of the solution molecules that are not part of the solution.

2. A solution can be treated mathematically as a finite bag or multi-set of molecules, and filtering operations can be treated as operations to produce multi-sets from multi-sets.

3. Initial multi-set: Typically, for a problem of size $n$, strings of size $O(n)$ are required. Should contain enough strings to include many copies of all possible solutions. Therefore, for an exponential problem, we will have $O(k^n)$ strings.

4. This is essentially a brute-force method.

B.1 Adleman: HPP

B.1.a Review of HPP

1. Hamiltonian Path Problem (HPP): The Hamiltonian Path Problem is to determine, for a given directed graph $G = (V, E)$ and two of its vertices $v_{in}, v_{out} \in V$, whether there is a HP from $v_{in}$ to $v_{out}$, that is, a path that goes through each vertex exactly once.

2. NP-complete: HPP is an NP-complete problem.

3. We will see that for Adleman’s algorithm the number of algorithm steps is linear in problem size.

4. Laboratory demonstration: Leonard Adleman gave a laboratory demonstration of the procedure in 1994 (for $n = 7$). (By the way, he is the “A” of “RSA.”)

5. “In 2002, he and his research group managed to solve a ‘nontrivial’ problem using DNA computation. Specifically, they solved a 20-variable SAT problem having more than 1 million potential solutions.
5 Physical Implementations
5.4 Adleman’s Implementation

Adleman utilized the incredible storage capacity of DNA to implement a brute-force algorithm for the directed Hamiltonian Path Problem (HPP). Recall that the HPP involves finding a path through a graph that visits each vertex exactly once. The instance of the HPP that Adleman solved is depicted in Fig. 5.2, with the unique Hamiltonian Path (HP) highlighted by a dashed line.

Adleman’s approach was simple:

1. Generate strands encoding random paths such that the Hamiltonian Path (HP) is represented with high probability. The quantities of DNA used far exceeded those necessary for the small graph under consideration, so it is likely that many strands encoding the HP were present.

2. Remove all strands that do not encode the HP.

3. Check that the remaining strands encode a solution to the HPP.

The individual steps were implemented as follows:

Stage 1: Each vertex and edge was assigned a distinct 20-mer sequence of DNA (Fig. 5.3a). This implies that strands encoding a HP were of length 140 b.p. Sequences representing edges act as ‘splints’ between strands representing their endpoints (Fig. 5.3b).

In formal terms, the sequence associated with an edge \( i \rightarrow j \) is the 3’ 10-mer of the sequence representing \( v_i \) followed by the 5’ 10-mer of the sequence representing \( v_j \). The oligonucleotides were then combined to form strands encoding random paths through the graph. An (illegal) example path (\( v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \)) is depicted in Fig. 5.4.

Fixed amounts (50 pmol) of each oligonucleotide were mixed together in a single ligase reaction. At the end of this reaction, it is assumed that a.

Figure IV.6: HPP solved by Adleman. The HP is indicated by the dotted edges. [source: Amos, Fig. 5.2]

They did it in a manner similar to the one Adleman used in his seminal 1994 paper.”

B.1.b Problem Representation

1. The heart of Adleman’s algorithm is a clever way to encode candidate paths in DNA.

2. Vertices: Vertices were represented by single-stranded 20mers, that is, sequences of 20nt (nucleotides).

3. They were generated at random and assigned to the vertices, but with the restriction that none of them were too similar or complementary.

4. Each vertex code can be considered a catenation of two 10mers: \( v_i = a_i b_i \).
   (I.e., \( a_i \) is the 5’ 10mer and \( b_i \) is the 3’ 10mer.)

5. Edges: Also represented by 20mers.
   The edge from vertex \( i \) to vertex \( j \) is represented by
   \[ e_{i \rightarrow j} = b_i a_j, \] where \( v_i = a_i b_i \), and \( v_j = a_j b_j \).

6. Paths: Paths are represented by using complements of the vertex 20mers to stitch together the edge 20mers.
(Of course, using the complements of the edges to stitch together the vertices works as well.)
For example, a path $2 \rightarrow 3 \rightarrow 4$ is represented:

\[
\begin{array}{c}
\text{\textcolor{red}{b}_2} \quad \text{\textcolor{blue}{a}_3} \\
\text{\textcolor{blue}{b}_3} \quad \text{\textcolor{red}{a}_4}
\end{array}
\]

7. Initial and final edges: Edges from $v_{in}$ and to $v_{out}$ have special representations as 30mers:

\[
e_{in\rightarrow j} = v_{in}a_j, \text{ where } v_j = a_jb_j;
\]
\[
e_{i\rightarrow out} = b_iv_{out}, \text{ where } v_i = a_ib_i.
\]

8. Note that the special representation of the initial and terminal edges results in blunt ends for complete paths.

9. Therefore, for the $n = 7$ problems, candidate solutions were 140bp in length.
   There are $n - 1$ edges, but the first and last edges are 30mers.
   Hence $2 \times 30 + (n - 3) \times 20 = 140$.

10. Ligation is used to remove the nicks in the backbone.

B.1.c Algorithm

1. Step 1: Generate multiple representations of all possible paths through the graph.

2. This is done by combining the oligos for the edges with the oligos for the complements of the vertices in a single ligation reaction.

3. Step 2: Amplify the concentration of paths beginning with $v_{in}$ and ending with $v_{out}$.

4. This is done by PCR using $v_{in}$ and $v_{out}$ as primers.
   Remember that denaturation separates the sense and antisense strands.
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PCR moves in the 3' direction from $v_{in}$ on the sense strand, and in the 3' direction from $v_{out}$ on the antisense strand.

¶5. We now have paths of all sorts from $v_{in}$ to $v_{out}$.

¶6. **Step 3:** Only paths with the correct length are retained. For $n = 7$ this is 140bp.

¶7. This is accomplished by gel electrophoresis.
The band corresponding to 140bp is determined by comparison with a *marker lane*.
The DNA is extracted from this band and amplified by PCR.

¶8. Gel electrophoresis and PCR are repeated to get a sufficient quantity.

¶9. We now have paths from $v_{in}$ to $v_{out}$, but they might not be Hamiltonian.

¶10. **Step 4 (affinity purification):** Select for paths that contain all the vertices (and are thus necessarily Hamiltonian).
This is done by selecting all those that contain $v_1$, and of those, all that contain $v_2$, and so forth.

¶11. To do select for $v_i$, first heat the solution to separate the strands.
Then add the $v_i$ bound to a magnetic bead.
Rehybridize, and use a magnet to extract all the paths containing $v_i$.
Repeat for each vertex.

¶12. **Step 5:** If there any paths left, they are Hamiltonian.
Therefore amplify them by PCR.
Inspect the result by gel electrophoresis to see if there are any strands of the correct length.
If there are, then there is a HP; if not, then not.

¶13. **Determining the path:** The precise HP can be determined by a graduated PCR procedure.
Run $n - 1$ PCR reactions.
In the $i$th lane, $v_{in}$ is the left primer and $v_i$ is the right primer.
“[T]he unique HP . . . should produce bands of length 40, 60, 80, 100, 120, and 140 b.p. in lanes 1 through 6.” [Amos, p. 114]
That is, the primer that produced 40 is the second vertex in the HP; the primer that produced 60 is the third, etc.
B. FILTERING MODELS

14. The final process depends on there being only one HP and is error-prone due to its dependence on PCR.

B.1.d Discussion

1. Linear: Adleman’s algorithm is linear in the number of nodes, since the only iteration is Step 4, which is repeated for each vertex.

2. Adleman’s experiment took about a week, but with a more automated approach it could be done in a few hours.

3. On the other hand, the PCR process cannot be significantly shortened.

4. Molecular resources: The number of different oligos required is proportional to $n$.

5. Strands: The number of strands is much larger, since there must be a number of representatives of each possible path.
   If $d$ is the average degree of the graph, then there are about $d^n$ possible paths (exponential in $n$).
   For example, if $d = 10$ and $n = 80$, then the $10^{80}$ DNA molecules is more than the estimated number of atoms in the universe.

6. Hartmanis calculated that for $n = 200$ the weight of the DNA would exceed the weight of the earth.

7. So this brute-force approach is still defeated by exponential explosion.

8. Lipton (1995) estimates that it is feasible for $n \leq 70$, but this is also within the range of conventional computer.

   Step 1 (generation of all possible paths) took about an hour for $n = 7$.
   Adleman estimates that about $10^{14}$ ligation operations were performed, and that it could be scaled up to $10^{20}$ operations.
   Therefore, speeds of about $10^{15}$ to $10^{16}$ ops/sec (1–10 peta-operations/s) should be achievable.
   That is, digital supercomputer range.
10. **Energy:** Adlemen estimates that $2 \times 10^{19}$ ligation operations were performed per joule of energy. Contemporary supercomputer perform only $10^9$ operations per joule, so MC is $10^{10}$ more energy efficient.

11. It is near the thermodynamic limit of $34 \times 10^{19}$ operations per joule. Recall (Ch. II, Sec. B.1) $kT \ln 2 \approx 3 \times 10^{-9} \text{pJ} = 3 \times 10^{-21} \text{J}$, so there can be about $3.3 \times 10^{20}$ bit changes/J.

12. **Space:** One bit of information occupies about 1 cubic nm. Contemporary disks store a bit in about $10^{10}$ cubic nm. That is, a $10^{10}$ improvement in density.

13. **Inherent error:** A more pervasive problem is the inherent error in the filtering processes (due to incorrect hybridization). Some strands we don’t want, get through; and some that we do want, don’t. With many filtering stages the errors accumulate to the extent that the algorithms fail.

14. There are some approaches to error-resistant DNA computing.