Frequency Sampling Structures
Specifying Frequency Response

Set the frequency response of an FIR filter at M equally-spaced points \( \Omega_k = 2\pi k / M \), \( k = 0,1,2,\ldots,M - 1 \) and find the impulse response \( h[n] \).

Frequency response and impulse response are related by

\[
H(e^{j\Omega}) = \sum_{n=0}^{M-1} h[n] e^{-j\Omega n}
\]

and it follows that

\[
H(e^{j\Omega_k}) = H(e^{j2\pi k / M}) = \sum_{n=0}^{M-1} h[n] e^{-j2\pi kn / M}, \quad k = 0,1,2,\ldots,M - 1
\]

This is the DFT of \( h[n] \).
Specifying Frequency Response

Since $H(e^{j2\pi k/M})$ is the DFT of $h[n]$, $h[n]$ must be the inverse DFT of $H(e^{j2\pi k/M})$.

$$h[n] = \frac{1}{M} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) e^{j2\pi kn/M}, \quad n = 0, 1, 2, \ldots, M - 1$$

The transfer function and impulse response are related by

$$H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) e^{j2\pi kn/M} z^{-n}.$$ 

Exchanging the order of summation,

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) \sum_{n=0}^{M-1} e^{j2\pi kn/M} z^{-n}$$
Specifying Frequency Response

Using the formula for summing a geometric series,

\[
H(z) = \frac{1}{M} \sum_{k=0}^{M-1} H\left(e^{j2\pi k/M}\right) \frac{1 - z^{-M} e^{j2\pi}}{1 - z^{-1} e^{j2\pi k/M}} = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} H\left(e^{j2\pi k/M}\right)
\]

We can think of this as the cascade of two filters, one with transfer function

\[
H_1(z) = \frac{1 - z^{-M}}{M}
\]

and one with transfer function

\[
H_2(z) = \sum_{k=0}^{M-1} \frac{H\left(e^{j2\pi k/M}\right)}{1 - z^{-1} e^{j2\pi k/M}}.
\]
Specifying Frequency Response

\[ H_1(z) \] has zeros at

\[ 1 - z^{-M} = 0 \Rightarrow z^{-M} = 1 \]

\[ z = e^{j2\pi q/M}, \quad q = 0, 1, 2, \ldots, M - 1. \]
Specifying Frequency Response

$H_2(z)$ is the sum of the transfer functions of $M$ subsystems, each of which has a pole where

$$1 - z^{-1} e^{j2\pi k/M} = 0 \Rightarrow z^{-1} e^{j2\pi k/M} = 1 \Rightarrow z = e^{j2\pi k/M}, \ k = 0,1,2,\ldots,M-1$$

So the transfer function can be written as

$$H(z) = \frac{1}{M} \left[ \frac{M-1}{\prod_{q=0}^{M-1} 1 - z^{-1} e^{j2\pi q/M}} \right] \sum_{k=0}^{M-1} \frac{H(e^{j2\pi k/M})}{1 - z^{-1} e^{j2\pi k/M}}$$

or

$$H(e^{j2\pi p/M}) = \frac{1}{M} \sum_{k=0}^{M-1} \frac{M-1}{\prod_{q=0}^{M-1} 1 - e^{-j2\pi(p-q)/M}} H(e^{j2\pi k/M}) \ , \ 0 \leq p < M$$
Specifying Frequency Response

The poles of $H_2(z)$ are at the same locations as the zeros of $H_1(z)$. So for any particular zero of $H_1(z)$, all the terms in the summation are zero except the one for which the pole and zero cancel. Therefore

$$H(e^{j2\pi p/M}) = \frac{H(e^{j2\pi p/M})}{M} \prod_{q=0 \atop q \neq p}^{M-1} 1 - e^{-j2\pi(p-q)/M}, \ 0 \leq p < M$$

This result implies that

$$\prod_{q=0 \atop q \neq p}^{M-1} 1 - e^{-j2\pi(p-q)/M} = M, \ 0 \leq p < M$$
Specifying Frequency Response

The poles and zeros could theoretically be anywhere on the unit circle, but for practical designs they must occur in complex conjugate pairs and

\[ H(e^{j2\pi k/M}) = H^*(e^{-j2\pi k/M}) = H^*(e^{j2\pi(M-k)/M}) \]
Specifying Frequency Response

Taking advantage of the complex-conjugate symmetry, for $M$ even,

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(e^{j2\pi k/M})}{1-z^{-1} e^{j2\pi k/M}} = \frac{H(e^{j0})}{1-z^{-1}} + \sum_{k=1}^{M/2-1} \frac{H(e^{j2\pi k/M})}{1-z^{-1} e^{j2\pi k/M}} + \frac{H(e^{j2\pi(M-k)/M})}{1-z^{-1} e^{j2\pi(M-k)/M}}$$

$$H_2(z) = \frac{H(e^{j0})}{1-z^{-1}} + \sum_{k=1}^{M/2-1} \frac{A(k) - B(k) z^{-1}}{1 - 2 \cos(2\pi k / M) z^{-1} + z^{-2}}$$

where $A(k) = H(e^{j2\pi k/M}) + H(e^{j2\pi(M-k)/M})$

and $B(k) = e^{-j2\pi k/M} H(e^{j2\pi k/M}) + e^{j2\pi k/M} H(e^{j2\pi(M-k)/M})$

Similarly, for $M$ odd,

$$H_2(z) = \frac{H(e^{j0})}{1-z^{-1}} + \frac{H(e^{j\pi})}{1+z^{-1}} + \sum_{k=1}^{(M-1)/2} \frac{A(k) - B(k) z^{-1}}{1 - 2 \cos(2\pi k / M) z^{-1} + z^{-2}}$$
Example

Design a filter whose frequency response goes through these points.

\[
\begin{align*}
  k & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
  H(e^{j2\pi k/M}) & \quad 0 \quad 1 \quad j \quad 0 \quad 0 \quad 0 \quad -j \quad 1
\end{align*}
\]

\[
H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(e^{j2\pi k/M})}{1 - z^{-1}e^{j2\pi k/M}}
\]

\[
H(z) = \frac{1 - z^{-8}}{8} \left[ \frac{1}{1 - z^{-1}e^{j2\pi/8}} + \frac{j}{1 - z^{-1}e^{j4\pi/8}} + \frac{-j}{1 - z^{-1}e^{j12\pi/8}} + \frac{1}{1 - z^{-1}e^{j14\pi/8}} \right]
\]
Example

Taking advantage of the periodicity of $e^{j2\pi k/8}$

$$H(z) = \frac{1-z^{-8}}{8} \left[ \frac{1}{1-z^{-1}e^{j2\pi/8}} + \frac{1}{1-z^{-1}e^{-j2\pi/8}} + \frac{j}{1-z^{-1}e^{j4\pi/8}} + \frac{-j}{1-z^{-1}e^{-j4\pi/8}} \right]$$

or

$$H(z) = \frac{1-z^{-8}}{8} \left[ \frac{2-2z^{-1}\cos(\pi/4)}{1-2z^{-1}\cos(\pi/4)+z^{-2}} - \frac{2z^{-1}\sin(\pi/2)}{1-2z^{-1}\cos(\pi/2)+z^{-2}} \right]$$

and the frequency response is

$$H(e^{j\Omega}) = \frac{1-e^{-j8\Omega}}{8} \left[ \frac{2-2e^{-j\Omega}\cos(\pi/4)}{1-2e^{-j\Omega}\cos(\pi/4)+e^{-j2\Omega}} - \frac{2e^{-j\Omega}\sin(\pi/2)}{1-2e^{-j\Omega}\cos(\pi/2)+e^{-j2\Omega}} \right]$$

The impulse response is

$$h[n] = \begin{cases} 0.25, -0.0732, 0, 0.0732, \\ -0.25, -0.4268, 0, 0.4268 \end{cases} \xrightarrow{\mathcal{DFT}} H(e^{j2\pi k/M}) = \{0,1,j,0,0,0,-j,1\}$$