Graph Coloring and the Immersion Order

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Overview

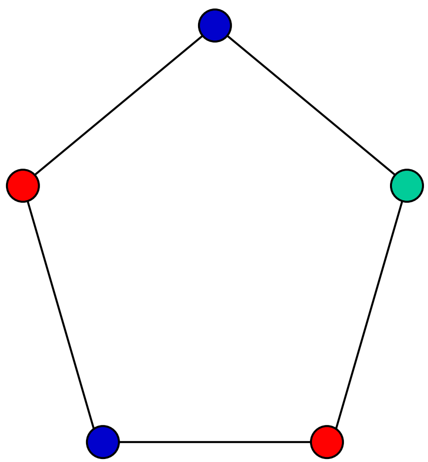
Background: Historical Perspective, Containment, Well-Quasi Orders

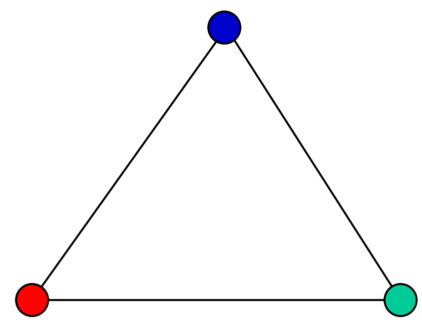
Main Results: Kempe Chains, Finite Basis Characterizations, Cutsets, Connectivity

Open Questions: General Case

Colorability versus Containment

Subgraph not a reliable indicator: C_5 vs K_3

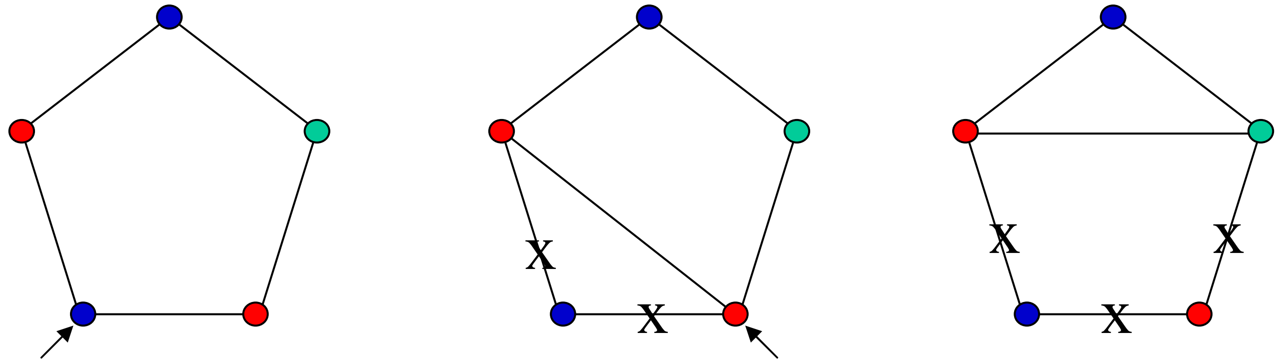




Hajós Conjecture

1940s: If G requires t colors, then G contains a topological K_t (subgraph + subdivision removal)

C_5, for example, contains a topological K_3



Hajós Conjecture

1940s: If G requires t colors, then G contains a topological K_t

Trivially true for t in [1,3]

1952: Dirac proved it true for $t = 4$

Intensely studied, and seemed plausible for another quarter century

Hajós Conjecture

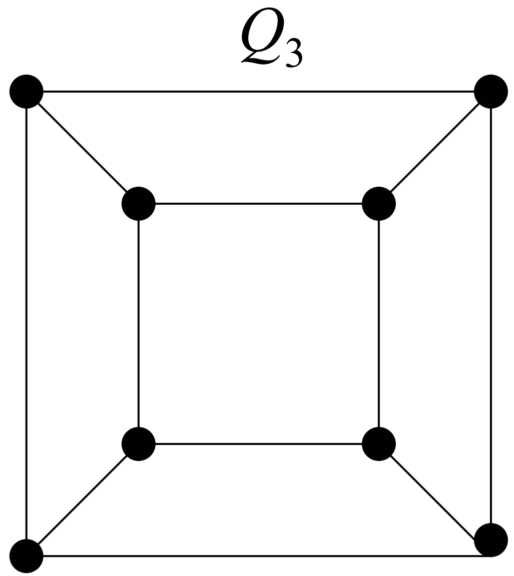
1940s: If G requires t colors, then G contains a topological K_t

1979: Catlin showed it false for all $t >= 7$

1981: Erdos and Fajlowitz showed it false for "almost all" graphs!

Hadwiger's Conjecture

1943: If G requires t colors, then G contains a K_t minor (subgraph + edge contraction)



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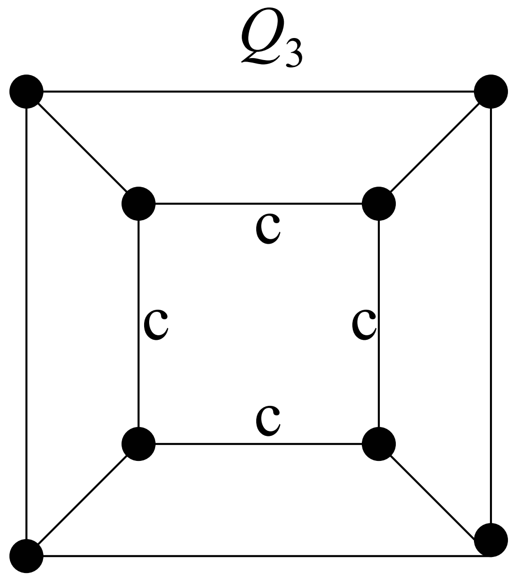

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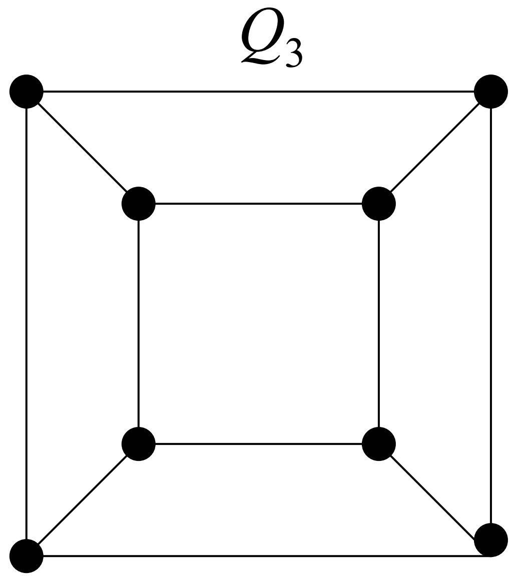
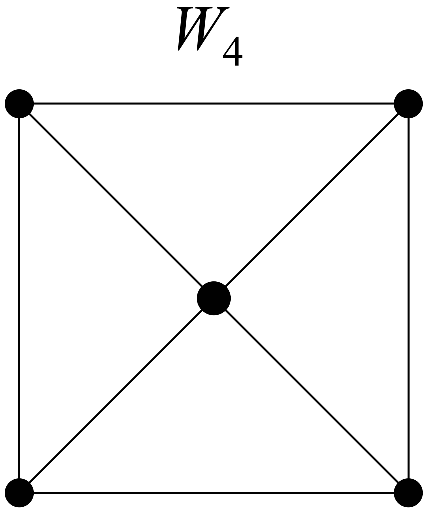




Hadwiger's Conjecture

1943: If G requires t colors, then G contains a K_t minor

Easy for t in [1,4]

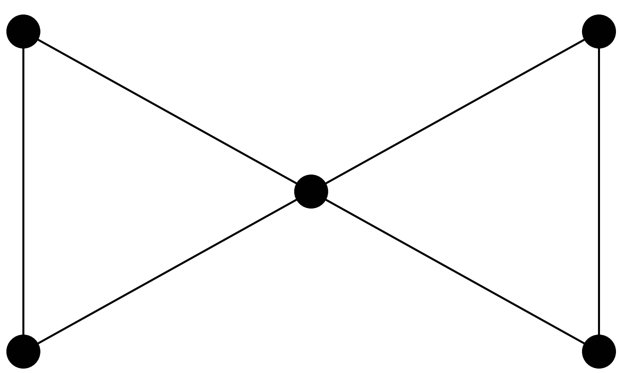
1964: Wagner, equivalent to the 4CT for $t = 5$

1993: Robertson, Seymour and Thomas proved it true for $t = 6$

The Immersion Order and A New Conjecture

If G requires t colors, then G contains an immersed K_t (subgraph + lift)

K_1+2K_2



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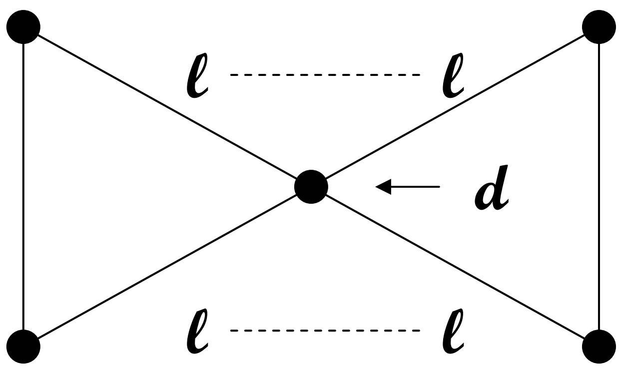

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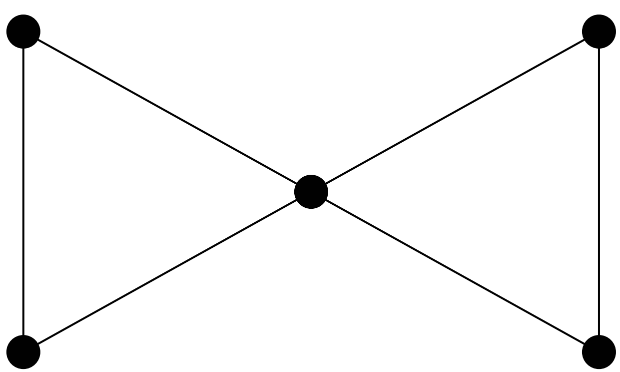
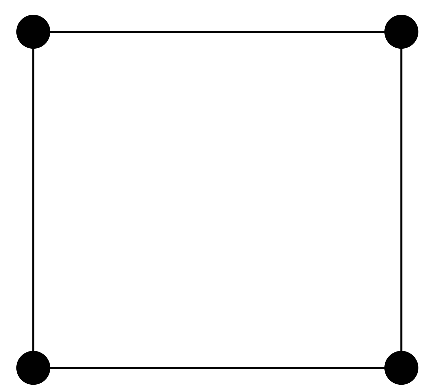
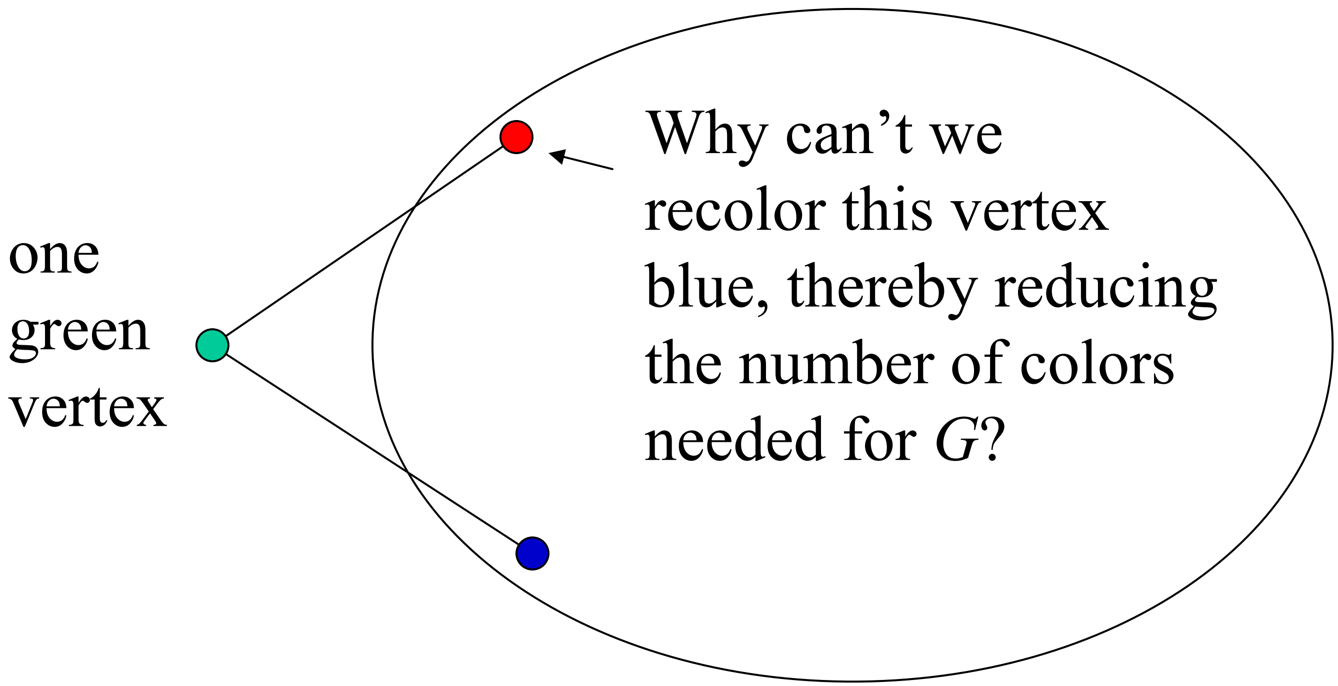


C_4



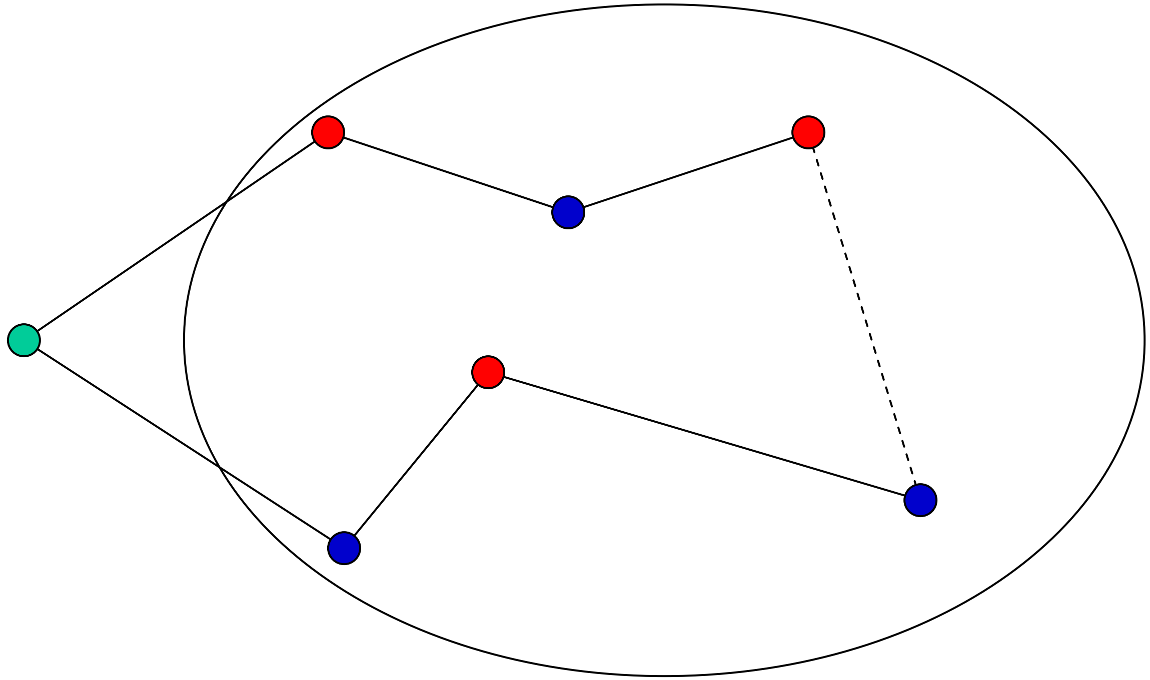
Coloring Arguments

Suppose G requires t colors:



The Kempe Chain

one
green
vertex



Useful Notions

Suppose G requires t colors. Then G is:

t-color-critical if every proper subgraph of G needs fewer than t colors.

t-immersion-critical if every graph properly immersed in G needs fewer than t colors.

Theorem: For each fixed t, there are finitely many t-immersion-critical graphs.

Proof sketch.

Observe that K_t is t-immersion critical. Use the fact that each such graph is an obstruction to a lower ideal in the immersion order. Thus they form an antichain, and all antichains are of finite cardinality because the immersion order is a well-quasi-order.

Vertex Connectivity

For Hadwiger's Conjecture, any t-minor-critical graph other than K_t must be at least :

1968: Mader, 7-vertex-connected

2002: Kawarabayashi, $t/3$-vertex-connected

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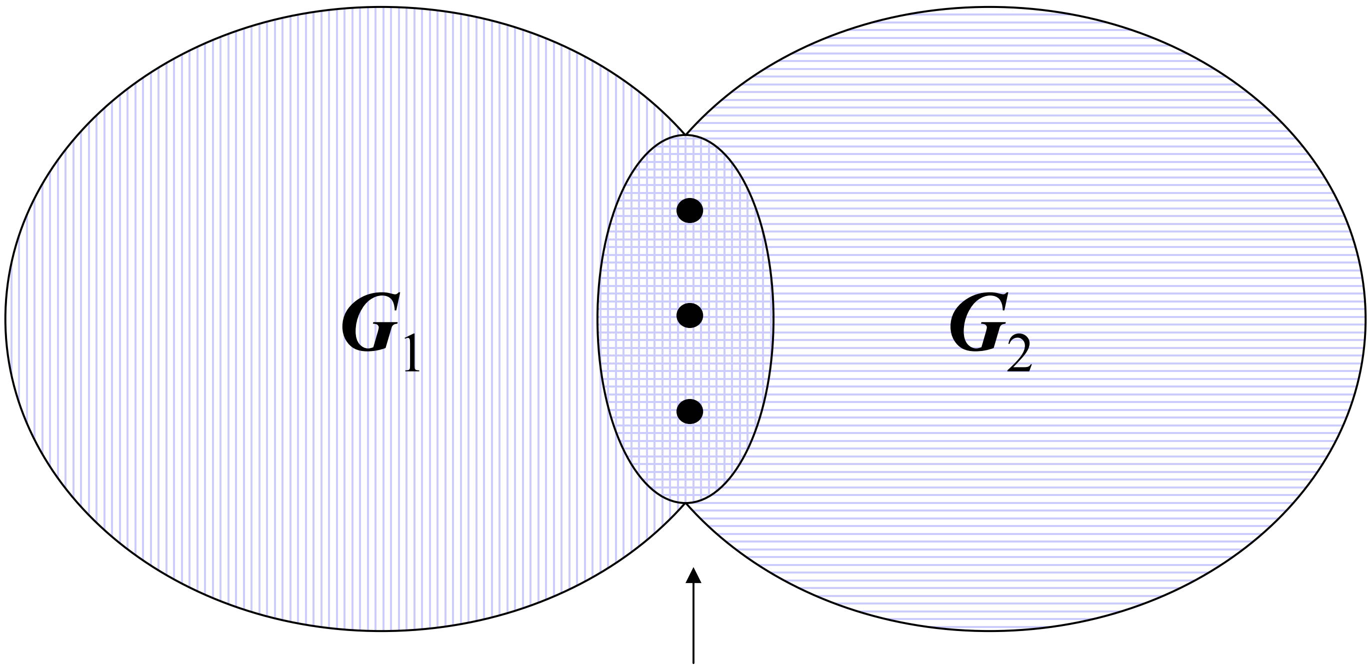
2002: Kawarabayashi, *t*/3-vertex-connected

For our conjecture, we have proved that any *t*-immersion-critical graph other than K_t must be at least 4-vertex-connected.

Proof Structure

G, a t-immersion critical graph, with t at least 5



S, a minimum cardinality vertex cutset

Lemma: Every t-immersion-critical graph is 3-vertex-connected.

Proof. Suppose $S=\{a,b\}$. Cutsets can't be cliques. It must be that t-1-colorings of G_1 and G_2 are not equivalent on S. But by lifting in G_2, we obtain H_1, with edge ab, and properly immersed in G. So G_1 can be t-1 colored with a and b having different colors. And vice versa for G_2. Thus, G is the union of two graphs with t-1 colorings that are equivalent on S, impossible.

Lemma: If $|S|=3$, then G_1 and G_2 admit t-1 colorings that assign more than one color to the elements of S.

Proof. Straightforward, using lifting.

Lemma: If $|S|=3$, then G_1 and G_2 admit t-1 colorings that assign more than one color to the elements of S.

Lemma: If $|S|=3$, then neither G_1 nor G_2 admits a t-1 coloring that assign three different colors to the elements of S.

Proof. Uses three Kempe Chains. Somewhat long, but very pretty application of these chains.

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Lemma: If $|S|=3$, then neither G_1 nor G_2 admits a t-1 coloring that assign three different colors to the elements of S.

Theorem: Every t-immersion critical graph is 4-vertex-connected.

Proof. Repeated use of these Lemmas.

Edge Connectivity

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1972: Toft, *t*-edge-connected

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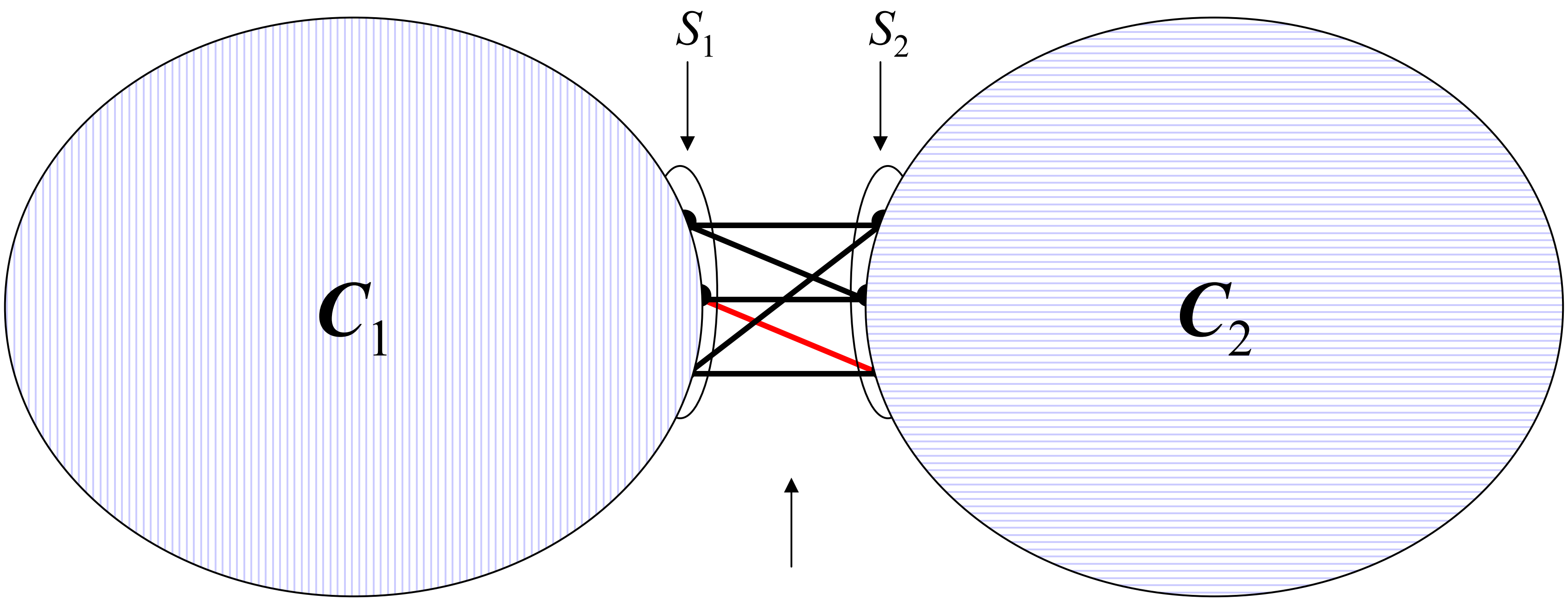
For our conjecture, we have also proved that any *t*-immersion-critical graph other than K_t must be *t*-edge-connected. The techniques are dissimilar.

Proof Structure

G, a *t*-immersion critical graph, with *t* at least 5

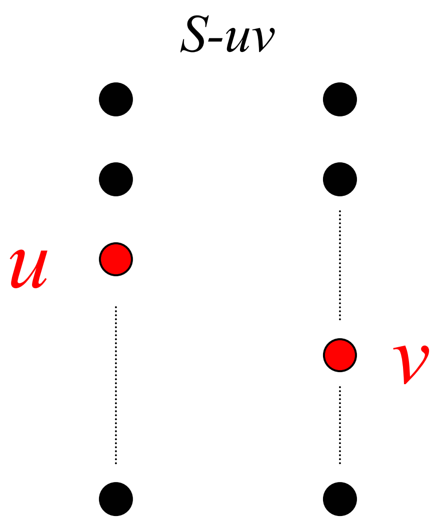


S, a minimum cardinality edge cutset

H = G - uv

Lemma: If G is not t-edge-connected, then every (t-1)-coloring of H assigns either one color to S_1 and all t-1 colors to S_2 or vice versa.

Proof.



S contains exactly t-1 edges (easy).

Vertices u and v must be the same color, else G needs only t-1 colors.

Thus they must be the ends of t-2 Kempe chains.

Kempe chains are edge disjoint, which gives a 1-1 correspondence between chains and S-uv. Case analysis follows.

Lemma: If G is not t-edge-connected, then every (t-1)-coloring of H assigns either one color to S_1 and all t-1 colors to S_2 or vice versa.

Theorem: Any t-immersion-critical graph other than K_t must be t-edge-connected.

Proof. Use of this Lemma with yet more Kempe chains.

New Result

We are within one edge of settling the $t = 5$ case.

Theorem: If G requires 5 colors, then G contains an immersed K_5^-.

Open Issues

- The general case. How do we tackle it? What other tools are needed?
- Applications and use of immersion pre-tests for colorability (no immersed K_t => t-1 colors suffice).
- The immersion order can be defined in terms of, edge-disjoint paths, and Kempe chains are edge disjoint. So will this be easier than Hadwiger's?