Graph Coloring and the Immersion Order*†

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Abstract

The relationship between graph coloring and the immersion order is considered. Vertex connectivity, edge connectivity and related issues are explored. These lead to the conjecture that, if G requires at least t colors, then G must have immersed within it K_t, the complete graph on t vertices. Evidence in support of such a proposition is presented. For each fixed value of t, there can be only a finite number of minimal counterexamples. These counterexamples are characterized based on Kempe chains, connectivity, cutsets and degree bounds. It is proved that minimal counterexamples must, if any exist, be both 4-vertex-connected and t-edge-connected. The $t = 5$ case is examined in additional detail. The historical context and probable difficulty of settling this conjecture, as well as specific hurdles to its final resolution, are also discussed.

Key Words

Graph Theory and Algorithms, Chromatic Number, Immersion Containment

1 Overview

The applications of graph coloring are legion. The usual goal, and the one we consider here, is to assign colors to vertices so that no two adjacent vertices are given the same color. Graph coloring has a long and storied history. The study of four-coloring planar graphs alone has generated interest for over 150 years [26]. Despite all this effort, graph coloring in general remains a notoriously difficult combinatorial problem.

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Comparatively less is known about the immersion order and its applications. A pair of adjacent edges uv and vw, with $u \neq v \neq w$, is *lifted* by deleting the edges uv and vw, and adding the edge uw. A graph H is said to be *immersed* in a graph G if and only if a graph isomorphic to H can be obtained from G by lifting pairs of edges and taking a subgraph. This notion is illustrated in Figure 1, which shows that K_4 is immersed in $K_1 + P_5$.

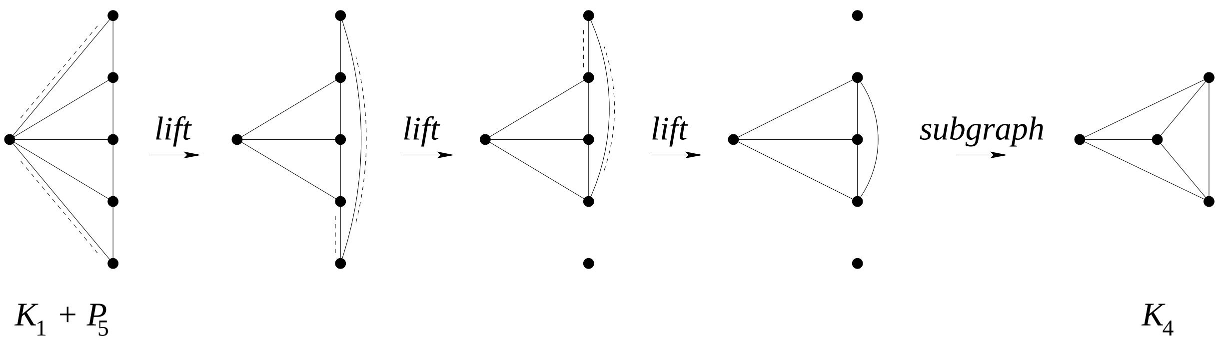


Figure 1: K_4 is immersed in $K_1 + P_5$.

In the next section, we briefly survey some of the history behind the intriguing relationship between graph coloring and various forms of containment. Section 3 contains a few requisite mathematical preliminaries and initial observations. Here and elsewhere the central parameter is t, the number of colors in use. In Section 4, we motivate our work with a foundational conjecture, observe that it is true for $t \leq 4$, and set up the framework for proofs based on minimal counterexamples. The main body of our work is contained in the subsequent two sections. In Section 5, we prove a number of connectivity results for arbitrary $t \geq 5$. In Section 6, we examine the $t = 5$ case in much greater detail. With a final section we summarize our efforts and discuss possible avenues for future research on this general subject.

2 Historical Context

The chromatic number of G, denoted by $\chi(G)$, is the minimum number of colors required by G in any proper coloring of its vertices. Of course it is well known that determining $\chi(G)$ is $\mathcal{NP}$-hard. It is tempting to try to associate $\chi(G)$ with some sort of clique contained within G. After all, if G contains K_t as a subgraph[1], then it is easy to show that G can be colored with no fewer than t colors. To see that the presence of a K_t subgraph is not necessary, however, one needs only to observe that C_5, the cycle of order five, requires three colors yet does not contain K_3 as a subgraph.

[1] Of course what we really mean is that G contains a subgraph isomorphic to K_t. We shall henceforth adopt this slight abuse of terminology, dropping the cumbersome reference to isomorphism as long as it creates no confusion or ambiguity.

Nevertheless, perhaps some weaker form of K_t is present. One possibility is topological containment, in which taking subgraphs is augmented with removing subdivisions. An edge is subdivided when it is replaced by a path formed from two edges and an internal vertex of degree two; subdivision removal reverses this operation. For example, C_5 contains K_3 topologically. Sometime in the 1940s Hajós conjectured that if $\chi(G) \geq t$, then G must contain a topological K_t [12]. The conjecture is trivially true for $t \leq 3$. In 1952 Dirac proved it true for $t = 4$ [5]. It was not until Catlin's work in 1979 that Hajós' conjecture was finally settled, and negatively, with a family of counterexamples for $t \geq 7$ [4]. Ironically, one such counterexample is the 15-vertex graph defined by the crossproduct of C_5 and K_3. It requires eight colors but contains no topological K_8. Subsequently, Erdős and Fajtlowicz were able to prove the rather surprising result that almost all graphs are counterexamples [7]. Thus Hajós' conjecture remains open only for $t \in \{5, 6\}$.

Another possibility is the minor order, for which the allowable operations are taking subgraphs and contracting edges. The minor order is a generalization of the topological order, because subdivision removal is just a special case of edge contraction. Hadwiger conjectured in 1943 that, if $\chi(G) \geq t$, then G must contain a K_t minor [11]. This conjecture equates to Hajós' conjecture for $t \leq 4$. Wagner proved in 1964 that, for $t = 5$, it is equivalent to the four color theorem [31]. In 1993 Robertson, Seymour and Thomas proved it true for $t = 6$ [25]. Whether Hadwiger's conjecture holds true in general, however, has thus far not been decided. This is in spite of decades of research, hordes of supporting evidence and a multitude of results on many of its variants and restrictions [1, 6, 15, 28, 30, 32]. Even the celebrated Graph Minor Theorem [24] appears to shed no particular light on this question. As of this writing, a resolution of Hadwiger's conjecture seems distant.

In this paper we focus instead on the immersion order. Immersion containment is quite distinct from topological and minor containment. Recalling Figure 1, for example, we observe that K_4 is contained in $K_1 + P_5$ in neither the topological nor the minor order. Like the minor order, however, the immersion order is a generalization of the topological order. This is because subdivision removal is just a special case of lifting pairs of edges. Previous investigations into the immersion order have generally been conducted from a purely algorithmic standpoint. We refer the reader to [2, 8, 9, 10, 18] for examples and applications. In contrast, here we mainly consider structural issues. We establish compelling connections between graph coloring and the immersion order, and conjecture that K_t is immersed in any graph requiring t or more colors.

3 Preliminaries

We mainly restrict our attention to finite, simple undirected graphs (multiple edges and loops that may arise from lifting are irrelevant to coloring). G is said to be *t-vertex-connected* if at least t vertex-disjoint paths connect every pair of its vertices. A *vertex cutset* is a set of vertices whose removal breaks G into two or more nonempty connected components. The cardinality of a smallest vertex cutset in G is equal to the largest t for which G is t-vertex-connected[2] (unless G is a complete

[2]Min-max conditions such as this are generally termed "Menger characterizations," after the pioneering work of Menger [22].

graph, which can have no vertex cutset). G is said to be t*-edge-connected* if at least t edge-disjoint paths connect every pair of its vertices. An *edge cutset* is a set of edges whose removal breaks G into two or more nonempty connected components. The cardinality of a smallest edge cutset in G is equal to the largest t for which G is t-edge-connected.

If $\chi(G) \leq t$, then G is said to be t*-colorable*. If $\chi(G) = t$, then G is said to be t*-chromatic*. If $\chi(G) = t$ and $\chi(H) < t$ for every proper subgraph H of G, then G is said to be t*-color-critical*. A t-coloring of G is realized by a map c from the vertices of G to the set $\{1, 2, .., t\}$ so that, if G contains the edge uv, then $c(u) \neq c(v)$. Given such a map, c_{ij} is used to denote the subgraph induced by the vertex set $\{u : c(u) \in \{i, j\}\}$. A path contained within c_{ij} is termed a *Kempe chain* [33], so-named in honor of the foundational work done on them by Kempe in [16]. (Ironically, the main result in [16] was a purported proof of the Four Color Theorem that, like so many others, turned out to be fatally flawed.) Of course c_{ij} need not be connected, and so for any $u \in c_{ij}$ we employ $c_{ij}(u)$ to denote the set $\{v : v$ resides in the same connected component of c_{ij} as does $u\}$. Such sets have useful properties.

Observation 1 *If $\{i, j\} \neq \{k, l\}$, then c_{ij} and c_{kl} are edge disjoint.*

Although the immersion order is traditionally defined in terms of taking subgraphs and lifting pairs of edges, Kempe chains and Observation 1 make it helpful for us to utilize as well the following alternate characterization: H is immersed in G if and only if there exists an injection from the vertices of H to the vertices of G for which the images of adjacent elements of H are connected in G by edge-disjoint paths. Under such an injection, an image vertex is called a *corner* of H in G; all image vertices and their associated paths are collectively called a *model* of H in G.

We use $\delta(G)$ to denote the smallest degree found among the vertices of G. We use $N(u)$ to denote the neighborhood of u. Suppose u has degree $t - 2$ or less in a t-chromatic graph G. Then $G-u$ must also be t-chromatic. Otherwise $G-u$ could be colored with $t-1$ colors, and u assigned one of the $t - 1$ colors unused within $N(u)$.

Observation 2 *If G is t-color-critical, then $\delta(G) \geq t - 1$.*

It is sometimes advantageous to select, restrict or manipulate colorings. For example, if G is t-chromatic but $G - u$ is only $(t - 1)$-chromatic, then it is possible to consider only colorings in which u is assigned a unique color.

Observation 3 *If G is t-color-critical, then for any vertex u there exists a coloring c in which $c(u) = 1$ and $c(v) \neq 1$ for every vertex $v \in G - u$.*

We use $\Delta(G)$ to denote the largest degree found among the vertices of G. When $\Delta(G) = t$, G can be (t+1)-colored by a simple greedy algorithm that assigns to each vertex the smallest color not already assigned to any of its neighbors. Thus, $\chi(G) \leq \Delta(G) + 1$. This upper bound is tight for cliques and odd cycles. A tighter upper bound was proved by Brooks for other graphs.

Theorem 1 *[3] If G is connected but neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

4 Motivation

Given the various connections between graph coloring, degrees and connectivity, and in turn the connections between connectivity and the immersion order, we seek to determine just how $\chi(G)$ is related to immersion containment. Our efforts to date prompt us to set the stage for this with the following conjecture[3].

Conjecture *If $\chi(G) \geq t$, then K_t is immersed in G.*

This speculation motivates our work in the sequel. There we shall present what we believe is compelling preliminary evidence in its support. Our conjecture, like Hadwiger's, is trivially true for $t \leq 4$. This is because the immersion order generalizes the topological order, for which Hajós' conjecture is long known to hold when $t \leq 4$. We address the $t = 5$ case in some detail, and come to within one edge of proving the conjecture holds for it. We also discuss obstacles to the complete resolution of this case. The general case quite naturally appears to be much more difficult. Accordingly, our efforts there focus mainly on global connectivity issues and make fewer specific inroads. Completely settling our conjecture is well beyond the scope of this paper. If history is any guide, a final resolution may be a formidable task indeed.

Before proceeding, we introduce a notion of immersion-criticality and show how it relates to the possible existence of counterexamples.

Definition *G is* t-immersion-critical *if $\chi(G) = t$ and $\chi(H) < t$ whenever H is properly immersed in G.*

Because $\chi(K_t) = t$, any counterexample must either be t-immersion-critical or have properly immersed within it another t-immersion-critical counterexample. Similarly, any t-immersion-critical graph distinct from K_t must be a counterexample. Thus our conjecture is equivalent to the statement that K_t is the only t-immersion-critical graph for every t. Although we have thus far fallen short of establishing this one way or the other, we can show that there are at most a finite number of them. To do this, we rely on properties of well-quasi-orders and immersion order obstruction sets. We refer the reader unfamiliar with these concepts to [8, 9, 17].

Theorem 2 *There are finitely many t-immersion-critical graphs for each fixed t.*

Proof. Consider the family of graphs $F = \{G : \chi(G) < t$ and $\chi(H) < t$ for every $H \leq_i G\}$. Then, by definition, F is closed in the immersion order. Because graphs are well-quasi-ordered by the immersion relation, it follows that F's obstruction set is finite. This set contains precisely the t-immersion-critical graphs. ■

[3] A superficially similar conjecture has been made by Lescure and Meyniel [27]. Although sometimes called "the immersion conjecture," the notion of containment employed there is not the immersion order. In particular, it does not allow a vertex to serve in dual roles both as a corner and as an intermediate node on a path connecting two other corners. This duality is critical. We refer the reader again to Figure 1.

5 Properties of t-Immersion-Critical Graphs

Graph connectivity has long been a central feature of attempts to settle Hadwiger's conjecture. G is said to be *t-minor-critical*[4] if $\chi(G) = t$ and $\chi(H) < t$ whenever H is a proper minor of G. K_t is of course both $(t-1)$-vertex-connected and $(t-1)$-edge-connected. Thus, if any t-minor-critical graph is not as strongly connected, then Hadwiger's conjecture is false for all $t' \geq t$. So suppose G denotes a t-minor-critical graph other than K_t (in which case the conjecture fails). Some 35 years ago [19], Mader showed that G must be at least 7-vertex-connected whenever $t \geq 7$. This provides evidence in support of the conjecture for $t \in \{7, 8\}$. A few years later [28], Toft proved that G must also be t-edge-connected. This provides additional supporting evidence for all t. Very recently, Kawarabayashi has shown that G must be at least $\lceil \frac{t}{3} \rceil$-vertex-connected as well [14]. Following this approach, we study both the vertex and edge connectivity of t-immersion-critical graphs. We assume $t \geq 5$ unless stated otherwise. Kempe chains play a pivotal role in our investigation.

5.1 Vertex Connectivity

Because they are t-color-critical, it is easy to see that t-immersion-critical graphs are 2-vertex-connected [1]. We now establish that they must in fact be at least 4-vertex-connected. Our work linking coloring to the immersion order begins in earnest with Lemma 4. First, however, we present something of an introduction with three easy but useful lemmas about cutsets, paths and coloring. Lemmas 1 and 2 are probably well known, although they may not be formulated anywhere else in precisely the same way we state them in this treatment. Lemma 2, which we dub *The Patching Lemma*, is especially helpful. Lemma 3 is certainly well known, and mentioned in a variety of sources (see, for example, [13, 30, 32]). We include the proofs of these lemmas here both for completeness and, more importantly, to illustrate and clarify their utility in subsequent results.

Lemma 1 *Let S denote a minimum-cardinality vertex cutset in a 2-vertex-connected graph G, and let C denote a connected component of $G \backslash S$. Then any two elements of S must be connected by a path whose interior vertices lie completely within C.*

Proof. Because G is 2-vertex-connected, $|S| \geq 2$. Let a and b denote any two distinct elements of S. It must be that a is adjacent to some vertex u in C, since otherwise S is not minimal ($S \backslash \{a\}$ defines a vertex cutset of cardinality $|S|-1$). Similarly, it must be that b is adjacent to some vertex v in C. If $u = v$, then we are done. If $u \neq v$, then the connectedness of C ensures that there is a subpath P from u to v lying completely within C. Thus $au \cup P \cup vb$ is the desired path. ■

Two colorings are said to be *equivalent* if the partitions induced by their respective color classes are identical.

Lemma 2 (The Patching Lemma) *Let S denote a vertex cutset of G, and let G_1 and G_2 denote a pair of induced subgraphs for which $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. If G_1 and G_2 admit t-colorings whose restrictions to S are equivalent, then G is t-colorable.*

[4] In the literature, this notion has sometimes been termed *t-contraction-critical*.

Proof. Let S, G, G_1 and G_2 be as defined in the statement of the lemma. Let c and d denote the specified t-colorings of G_1 and G_2, respectively. Modify one coloring, say d, by renaming its color classes so that each element of S is assigned the same integer under both colorings. Patched together, c and d now provide a t-coloring of G. ■

The Patching Lemma can be used to establish the following well-known fact.

Lemma 3 *No vertex cutset of a t-color-critical graph can be a clique.*

Proof. Suppose otherwise for some G with cutset S. Let C denote a connected component of $G \backslash S$. Let G_1 denote the subgraph induced by $C \cup S$. Let G_2 denote the subgraph induced by $G \backslash C$. Because G is t-color-critical, G_1 and G_2 must each be $(t-1)$-colorable. And because S is a clique, any $(t-1)$-coloring of G_1 and any $(t-1)$-coloring of G_2 must be equivalent when restricted to S. But now by the Patching Lemma, G must also be $(t-1)$-colorable, and thus not t-color-critical. ■

The preceding lemmas tell us a good deal about the make-up of vertex cutsets, and how they relate to coloring. Armed with this information, we are now able to argue more directly about vertex connectivity and the immersion order. To simplify matters, we shall adopt the following conventions for the remainder of this subsection:

- t is at least five,
- G denotes a t-immersion-critical graph,
- S denotes a minimum-cardinality vertex cutset in G,
- C denotes a connected component of $G \backslash S$,
- G_1 denotes the subgraph induced by $C \cup S$, and
- G_2 denotes $G \backslash C$.

Lemma 4 *Every t-immersion-critical graph is 3-vertex-connected.*

Proof. Suppose otherwise, as witnessed by some G with $S = \{a, b\}$. We know from Lemma 3 that the edge ab is not present in G. Let $i \in \{1, 2\}$. By Lemma 1, there must be a path, P_i, with endpoints a and b, whose vertices lie completely within G_i. Lifting the edges of P_{3-i} to form the single edge ab, and then taking the subgraph induced by the vertices of G_i, produces a graph H_i properly immersed in G. It follows that H_i is $(t-1)$-colorable. Because ab is present in H_i, any such coloring of H_i assigns different colors to a and b. But G_i is a subgraph of H_i. Thus, there are $(t-1)$-colorings of G_1 and G_2 that each assign different colors to a and b. By the Patching Lemma, this ensures a $(t-1)$-coloring of G, a contradiction. ■

Lemma 4 applies to t-topological-critical graphs as well. To see this, note that the two paths defined in the proof are vertex-disjoint except for their endpoints. An analog of Lemma 4 does not hold, however, if the graph is only known to be t-color-critical. Such graphs are guaranteed only to be 2-vertex-connected. A t-color-critical graph that is not 3-vertex-connected can be constructed as follows. Begin with a pair of non-adjacent vertices, u and v, a copy of K_{t-1} and a copy of K_{t-2}. Connect u to every vertex but one in the copy of K_{t-1}. Connect v to the remaining vertex in the copy of K_{t-1}. Now connect both u and v to every vertex in the copy of K_{t-2}. Such a graph is depicted in Figure 2, with $t = 4$. Of course these graphs are not t-immersion-critical by Lemma 4. In the instance shown, for example, K_4 is properly immersed using a model whose corners are u plus the vertices of K_3.

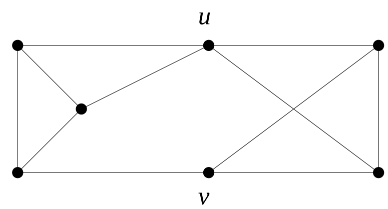


Figure 2: A 4-color-critical graph that is not 3-vertex-connected.

Lemma 5 *If $|S| = 3$, then G_1 and G_2 admit $(t-1)$-colorings that assign more than one color to the elements of S.*

Proof. Let $S = \{u, v, w\}$, and consider the case for G_1. By Lemma 1, there is a path between u and v in G_2. Lifting this path and taking the subgraph induced by the vertices of G_1 produces a graph H properly immersed in G. Because G is t-immersion-critical, and because H contains the edge uv, H must admit a $(t-1)$-coloring that assigns different colors to u and v. As a subgraph of H, G_1 can likewise be colored. A symmetrical argument handles the case for G_2. ■

What we have really just shown is that if G is only 3-vertex-connected, then G_1 admits a $(t-1)$-coloring that assigns different colors to any fixed pair of elements of S. This raises the possibility that a single coloring of G_1 may suffice, simultaneously assigning different colors to all three elements of S. We now show that this cannot happen. It follows that the same must then be true for G_2.

Let a and b denote vertices of G, and let c denote a coloring of G in which $c(a) = i \neq j = c(b)$. If a and b belong to the same connected component of c_{ij}, then they are connected by some Kempe chain P_{ij} contained within c_{ij}. In this event, we say that a and b are *c-chained*.

Lemma 6 *If $|S| = 3$, then neither G_1 nor G_2 admits a $(t-1)$-coloring that assigns three different colors to the elements of S.*

Proof. Suppose otherwise, as witnessed by a $(t-1)$-coloring c of G_1. Let $S = \{u, v, w\}$ and assume, without loss of generality, that $c(u) = 1, c(v) = 2$ and $c(w) = 3$. Let d denote some $(t-1)$-coloring of G_2. By Lemma 5 and the Patching Lemma, we may further assume, again without loss of generality, that d assigns exactly two colors to the elements of S, with $d(u) = d(v)$. If u and v are not c-chained, then we can exchange colors 1 and 2 in $c_{12}(v)$ to produce a $(t-1)$-coloring c' of G_1 that assigns color 1 to both u and v and leaves the color of w set to 3. This means that the restrictions of c' and d to S are equivalent. But now, by the Patching Lemma, G is $(t-1)$-colorable, which is impossible.

Thus it must be that u and v are c-chained by some P_{12} in G_1. Lifting this chain and taking the subgraph induced by the vertices of G_2 produces a graph H properly immersed in G. H contains uv, and so must admit a $(t-1)$-coloring d' that assigns different colors to u and v. G_2 is likewise colored by d'. By the Patching Lemma, d' cannot assign a third color to w. So assume, again without loss of generality, that $d'(w) = d'(u)$. If u and w are not c-chained, then (as in the previous argument) we can construct a $(t-1)$-coloring c'' of G_1 so that the restrictions of c'' and d' to S are equivalent, which is impossible.

Thus it must be that u and w are c-chained by some P_{13} in G_1. Because they are edge disjoint, P_{12} and P_{13} can be lifted simultaneously. Lifting these two chains and taking the subgraph induced by the vertices of G_2 produces a graph H' properly immersed in G. H' contains both uv and uw, and so must admit a $(t-1)$-coloring d'' that assigns different colors to u and v and different colors to u and w. G_2 is likewise colored by d''. By the Patching Lemma, d'' cannot assign three colors to the elements of S. So it must be that $d''(v) = d''(w)$. If v and w are not c-chained, then (as in the previous arguments) we can construct a $(t-1)$-coloring c''' of G_1 so that the restrictions of c''' and d'' to S are equivalent, which is impossible.

Thus it must be that v and w are c-chained by some P_{23} in G_1. Because they are edge disjoint, P_{12}, P_{13} and P_{23} can be lifted simultaneously. Lifting these three chains and taking the subgraph induced by the vertices of G_2 produces a graph H'' properly immersed in G. H'' contains uv, uw and vw, and so must admit a $(t-1)$-coloring d''' that assigns three different colors to S. G_2 is likewise colored by d'''. This means that the restrictions of c and d''' to S are equivalent, which is impossible, contradicting the supposition that c exists. ■

Bolstered by the preceding Lemmas, we are now ready to prove that minimum-cardinality vertex cutsets of t-immersion-critical graphs have at least four elements. The use of Kempe chains in Lemma 6 has been especially effective, so much so that we need only paths not chains in what follows.

Theorem 3 *Every t-immersion-critical graph is 4-vertex-connected.*

Proof. Suppose otherwise, as witnessed by some G with $S = \{u, v, w\}$. Let c and d denote $(t-1)$-colorings of G_1 and G_2, respectively. By Lemmas 5 and 6, we restrict our attention to the case in which both c and d assign exactly two colors to elements of S. Without loss of generality, assume $c(u) = c(v)$ and $d(u) = d(w)$. By Lemma 1, there is a path P_1 in G_1 whose endpoints are u and w. Similarly, there is a path P_2 in G_2 whose endpoints are u and v. Lifting P_i and taking the graph induced by the vertices of G_{3-i} produces a graph H_{3-i} properly immersed in G. H_1 contains uv, and so must admit a $(t-1)$-coloring c' that assigns different colors to u and v. G_1 is likewise colored by c'. By Lemma 6, c' cannot assign a third color to w. Lest the restrictions of c' and d to S be equivalent, it must be that $c'(w) = c'(v)$. H_2 contains uw, and so must admit a $(t-1)$-coloring d' that assigns different colors to u and w. G_2 is likewise colored by d'. By Lemma 6, d' cannot assign a third color to v. But if $d'(v) = d'(u)$, then the restrictions of c and d' to S are equivalent. And if $d'(v) = d'(w)$, then the restrictions of c' and d' to S are equivalent. Thus, under some pair of colorings of G_1 and G_2, the Patching Lemma ensures that G is $(t-1)$-colorable, a contradiction. ■

5.2 Edge Connectivity

Because the immersion order includes the taking of subgraphs, we know that t-immersion-critical graphs are also t-color-critical. From the work of [29] it follows that they are $(t-1)$-edge-connected. We now show that any t-immersion-critical graph other than K_t is in fact t-edge-connected. We begin a pair of well-known observations (see, for example, [32]).

Observation 4 *A minimum-cardinality edge cutset separates a graph into exactly two connected components.*

Observation 5 *If H is obtained by deleting the edge uv from a t-color-critical graph, then H is $(t-1)$-colorable and, under any $(t-1)$-coloring, u and v are assigned the same color.*

The significance of Observation 5 rests with the next lemma, which plays an essential role in our edge-connectivity arguments. This lemma is probably also well known, although it may not be formulated elsewhere in exactly the same way we state it here.

Lemma 7 *Let H be obtained by deleting the edge uv from a t-color-critical graph. Let c denote a $(t-1)$-coloring of H with $c(u) = c(v) = 1$. Then $v \in c_{1i}(u) \forall i \in \{2, 3, \ldots, t-1\}$.*

Proof. Let H and c be defined as stated. Suppose the lemma is false, as witnessed by some i with $v \notin c_{1i}(u)$. Exchanging colors 1 and i in $c_{1i}(u)$ produces c', another $(t-1)$-coloring of H. But then u and v are assigned different colors under c', which is impossible. ■

Aided by this information about color-criticality, we are now able to argue more directly about edge connectivity and the immersion order. We shall adopt the following conventions for the remainder of this subsection:

- t is at least 5,
- G denotes a t-immersion-critical graph,
- S denotes a minimum-cardinality edge cutset in G,
- C_1 and C_2 denote the two connected components of $G \backslash S$,
- S_1 and S_2 denote the endpoints of S contained in C_1 and C_2, respectively,
- uv denotes an element of S, with $u \in S_1$ and $v \in S_2$, and
- H denotes $G \backslash \{uv\}$.

Lemma 8 *If G is not t-edge-connected, then every $(t-1)$-coloring of H assigns either one color to S_1 and all $t-1$ colors to S_2 or vice versa.*

Proof. Suppose G is not t-edge-connected. We know from [29] that S has cardinality $t-1$. Let c denote a $(t-1)$-coloring of H with $c(u) = c(v) = 1$. Lemma 7 ensures that $v \in c_{1i}(u) \forall i \in \{2, 3, .., t-1\}$. Therefore u and v are the endpoints of $t-2$ Kempe chains, where each chain is contained within $c_{1i}(u)$ for some i. By Observation 1, the chains are edge disjoint, and so each contains at least one distinct element of $S' = S \backslash \{uv\}$. Thus there is a one-to-one correspondence between chains and elements of S'. This means that every element of S' has an endpoint assigned color 1 by c. If c assigns only color 1 to S_1, then it must assign all $t-1$ colors to S_2. Similarly, if c assigns all $t-1$ colors to S_1, then it must assign only color 1 to S_2.

The only remaining case to consider occurs if c assigns more than one but fewer than $t-1$ colors to S_1. To show that this cannot happen, we now proceed by contradiction, and suppose S_1 contains vertex w assigned color i, but no vertex assigned color j, where $\{i, j\} \subseteq \{2, 3, .., t-1\}$. It must be that $c_{ij}(w)$ is completely contained within C_1. We exchange colors i and j in $c_{ij}(w)$ to produce c', another $(t-1)$-coloring of H with $c'(u) = c'(v) = 1$. By this construction, c' assigns color j to endpoints of two different elements of S', and accordingly assigns color i to the endpoint of no element of S'. We conclude that $v \notin c'_{1i}(u)$, contradicting Lemma 7. ■

Theorem 4 *Any t-immersion-critical graph other than K_t is t-edge-connected.*

Proof. Suppose otherwise, as witnessed by some G, not isomorphic to K_t, that is only $(t-1)$-edge-connected. We apply Lemma 8 and, without loss of generality, let c denote a $(t-1)$-coloring of H that assigns color 1 to $S_1 \cup \{v\}$. Thus all $t-1$ colors are assigned to S_2, and we index the elements of S_2 by $\{v = v_1, v_2, .., v_{t-1}\}$, where $c(v_i) = i$.

Let i and j denote distinct elements of $\{1, 2, .., t-1\}$. It must be that v_i belongs to $c_{ij}(v_j)$, since otherwise we can exchange colors i and j in $c_{ij}(v_j)$ to produce a $(t-1)$-coloring of H in which the elements of S_2 are assigned $t-2$ colors, thereby contradicting Lemma 8. It follows that v_i and v_j are the endpoints of a Kempe chain contained within $c_{ij}(v_j)$. Moreover, even if i or j is 1, the elements of S_1 are excluded from this chain. Therefore the chain is completely contained within C_2. Because such a chain exists for each pair of vertices in S_2, and because the chains are edge disjoint, K_{t-1} is immersed in C_2 using a model whose corners are the elements of S_2.

Now let i denote an element of $\{2, .., t-1\}$. From the proof of Lemma 8, we know that u and v are the endpoints of a Kempe chain containing v_i. This means that u and v_i are the endpoints of a subchain completely contained within $C_1 \cup S$. Because such a chain exists for each vertex in $S_2 \backslash \{v\}$, because the chains are edge disjoint, and because $uv \in G$, K_t is immersed in G using a model whose corners are $u \cup S_2$. This is the desired contradiction. ■

Corollary 1 *If G is t-immersion-critical and not K_t, then $\delta(G) \geq t$.*

Proof. Immediate from Theorem 4 and the fact that $\delta(G)$ is an upper bound on G's edge connectivity. ■

Corollary 2 *If G is t-color-critical with a vertex u of degree $t-1$, then K_t is immersed in G via a model whose corners are $u \cup N(u)$.*

Proof. Follows from the proof of Theorem 4 by letting S be the set of edges incident on u. ■

6 On Settling the $t = 5$ Case

Let n denote the number of vertices in G. Mader has shown fairly recently that any graph with at least $3n - 5$ edges must contain a topological (and hence an immersed) K_5 [20]. A 5-immersion-critical graph G can therefore contain at most $3n - 6$ edges, and so $\delta(G) \leq \lfloor 2(3n - 6)/n \rfloor = 5$. We know from Corollary 1 that a 5-immersion-critical graph other than K_5 satisfies $\delta(G) \geq 5$.

Observation 6 *If G is 5-immersion-critical and not K_5, then $\delta(G) = 5$.*

Let K_5^- denote the graph obtained by deleting one edge from K_5. Recall that a pair of vertices is said to be c-chained if they are connected by some Kempe chain under coloring c.

Theorem 5 *If $\chi(G) \geq 5$, then K_5^- is immersed in G.*

Proof. Any graph requiring five or more colors has a 5-immersion-critical graph immersed within it. Moreover, if that immersed graph is K_5, then we are done. Thus it suffices to consider only the case in which G itself is 5-immersion-critical and not K_5. By Observation 6, G has a vertex, u, of degree five. By Observation 3, G has a coloring c in which $c(u) = 1$ and $c(v) \neq 1$ for every vertex $v \in G - u$. Let the neighborhood of u be denoted by the set $\{v, w, x, y, z\}$. It must be that c assigns colors 2 through 5 across this collection of five vertices, since otherwise u could be reassigned another color producing a 4-coloring of G. Thus, without loss of generality, we assume that v, w and x are assigned distinct colors, and that y and z are assigned the same color. Every pair of vertices in $\{v, w, x\}$ must be c-chained, else one of them could be recolored. Similarly, every vertex in $\{v, w, x\}$ must be c-chained to at least one vertex in $\{y, z\}$. It follows that some vertex in $\{y, z\}$, say y, is c-chained to at least two elements of $\{v, w, x\}$. Each of the chains so identified is edge-disjoint with the others, as well as with the edges incident on u. Therefore the set $\{u, v, w, x, y\}$ forms the corners of K_5^- model immersed in G. ■

The proof of Theorem 5 uses 5-immersion-criticality only to capitalize on a vertex u with degree exactly five. Color-criticality is then employed to show that some K_5^- model has as corners u and four of its five neighbors. One might ask whether this approach can be used to resolve completely the $t = 5$ case. That is, can one find an entire K_5 model with some five of these six vertices as corners? This turns out not to be possible. To see this, consider the graph fragment illustrated in Figure 3. Only u truly has degree five; the degrees of other vertices are boosted with connections to irrelevant portions of the graph not shown. This fragment requires five colors and contains an immersed K_5. Any K_5 model within this fragment, however, must use as a corner a, which lies outside u's neighborhood. Examples such as this seem to complicate significantly any approach using Kempe chains and edge-disjoint paths. Moreover, we note that this fragment is 5-color-critical but not 5-immersion-critical. This highlights the importance of 5-immersion-criticality in any attempt to prove the $t = 5$ case, and suggests that perhaps it must play a much more central role than it does in our proof of Theorem 5.

On the other hand, it is possible that a resolution of the $t = 5$ case could be based on properties at best indirectly associated with coloring or immersion containment. As a relevant example, we

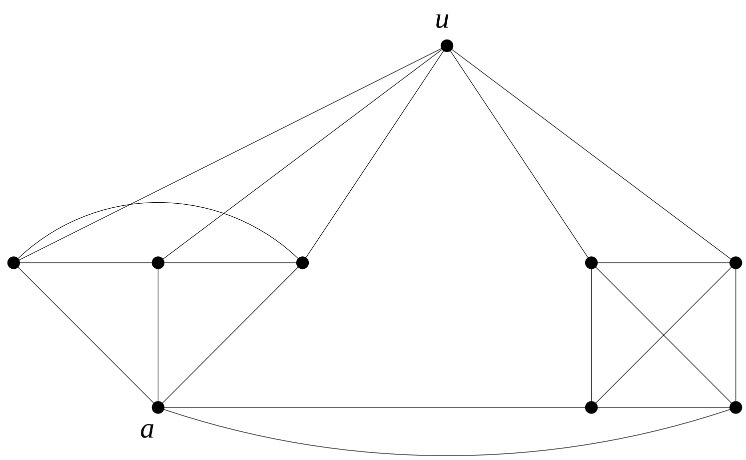


Figure 3: A 5-chromatic graph fragment for which $u \cup N(u)$ contains at most four corners of any K_5 model.

note that the $t = 4$ case of Hajós' conjecture follows simply from the fact that $\delta(G) \geq 3$ for any 4-color-critical graph G. We suspect that every simple 5-edge-connected graph contains K_5 in the immersion order. If this suspicion is true, then of course the $t = 5$ case of our conjecture is settled, because 5-immersion-critical graphs are 5-edge-connected. Incidentally, it is easy to see that our suspicion may be justified only as long as we restrict our attention to simple graphs. See Figure 4.

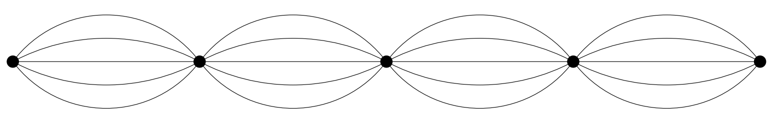


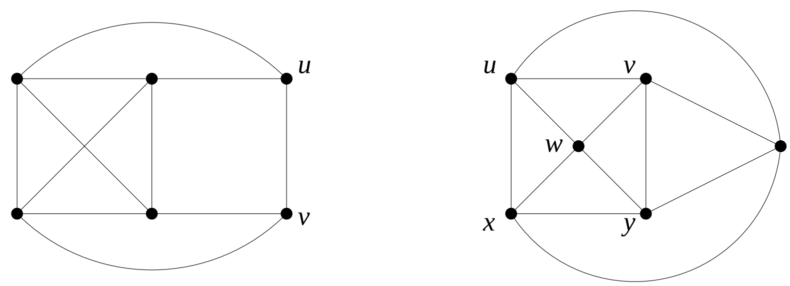
Figure 4: A 5-edge-connected multigraph of order five with no immersed K_5.

7 Potential Applications

Our conjecture, like others of its ilk, is primarily theoretical in nature. Nevertheless, there may be the potential for practical significance in its resolution. As an example, consider our conjecture alongside Hadwiger's conjecture. Together, they may serve as the basis of an efficient algorithmic strategy for estimating $\chi(G)$, where G is an arbitrary input graph. This is because, at least in principle, both the immersion and the minor order have polynomial-time order tests for any fixed graph H. See [9] and [23], respectively. Thus, for such an H, we can test whether $H \leq_i G$ as well as whether $H \leq_m G$ in time that is polynomial in the order of G. Accordingly, for our purposes, we fix H at K_t for some t.

To illustrate, set $t = 5$. If we test and find that K_5 is absent from G in both the immersion and the minor orders, and if either of the two conjectures is true, then four or fewer colors are required to color G. Moreover, it would seem that the two orders are rather orthogonal. See Figure 5. In Figure 5(a), K_5 is found by contracting the edge uv. In Figure 5(b), K_5 is found by lifting uw and wy to form uy and lifting vw and wx to form vx. It is hoped, therefore, that when used together these two tests would provide a better "coloring filter" than when either is used alone. We note also that practical order tests have already been developed for the case $t \leq 4$ in the immersion order [2], and $t \leq 5$ in the minor order [21].

Of course this sort of test provides an upper bound only. If K_5 is present in G, it may signify nothing at all. As an extreme example, observe that if G is constructed by subdividing every edge of a copy of K_5, then two colors suffice for G. Yet G contains K_5 in both orders.



(a) *(b)*

Figure 5: Graphs containing K_5 in the minor but not the immersion order, and vice versa.

8 Summary and Directions for Future Research

We have explored the relationship between graph coloring and the immersion order, and have conjectured that if $\chi(G) \geq t$, then G contains an immersed K_t. We have concentrated mainly on the general case. We have shown that, for each fixed value of t, there can be only a finite number of t-immersion-critical graphs. Our most important results are that t-immersion-critical graphs other than K_t must, if any exist, be both 4-vertex-connected and t-edge-connected.

We have produced supporting evidence specific to the $t = 5$ case. We note that previous work on Hajós' conjecture provides additional supporting evidence for both the $t = 5$ and $t = 6$ cases. If our conjecture is true in these cases, then it has no effect on Hajós' conjecture. This is because a t-chromatic graph may contain an immersed K_t with or without containing a topological K_t. On the other hand, if our conjecture is false for either case, then it means that Hajós' conjecture is also false for that case. This is because a t-chromatic graph without an immersed K_t must also be without a topological K_t. This would be quite a revelation, given that Hajós' conjecture for $t \in \{5, 6\}$ has remained open for roughly 60 years. We are reasonably confident that the $t = 5$

case will be settled in the not-too-distant future, and in the positive, although it may require a great deal of effort. The outlook for resolving the $t = 6$ case looks much murkier.

The general case seems rather foreboding. Perhaps this view is unfairly influenced, however, by knowledge of the long-standing difficulty of settling Hadwiger's conjecture. Observe that Kempe chains are not vertex disjoint. Yet the minor order is inherently dependent on vertex-disjoint paths[5]. In this we sense room for optimism: the immersion order is concerned only with edge-disjoint paths, and Kempe chains are indeed edge disjoint.

Although we have shown that a minimal counterexample must be t-edge-connected, we do not know whether it must be even $(t-1)$-vertex-connected. But we observe the following: if a minimal counterexample is only k-vertex-connected for some $k < (t-1)$, then it cannot be *minimally*-vertex-connected in the sense of [1]. That is, it must contain a proper subgraph that is also k-vertex-connected. Whether this singular property can be exploited to help settle our conjecture is as yet unclear.

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