The Strong Perfect Graph Theorem

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Abstract

A graph G is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest complete subgraph of H, and G is *Berge* if no induced subgraph of G is an odd cycle of length at least 5 or the complement of one. The "strong perfect graph conjecture" (Berge, 1961) asserts that a graph is perfect if and only if it is Berge.

A stronger conjecture was made recently by Conforti and Cornuejols - that every Berge graph either falls into one of four basic classes, or it has a kind of separation that cannot occur in a minimal imperfect graph.

In this paper we prove both these conjectures.

1 Introduction

We begin with definitions of some of the terms we use which may be nonstandard. All graphs in this paper are finite and simple. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\overline{G}$ just when they are not adjacent in G. A *hole* of G is an induced subgraph of G which is a cycle of length > 3. An *antihole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$. A graph G is *Berge* if every hole and antihole of G has even length.

A *clique* in G is a subset X of $V(G)$ so that every two members of X are adjacent. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H equals the size of the largest clique of H. In 1961 Claude Berge [**?**] proposed the so-called *Strong Perfect Graph Conjecture*, the main theorem of this paper:

1.1 *A graph is Berge if and only if it is perfect.*

It is easy to see that every perfect graph is Berge, and so to prove (1.1) it remains to prove the converse. This has received a great deal of attention over the past 40 years, but so far has resisted solution. Most of the previous approaches on (1.1) fall into two classes: proving that the theorem holds for graphs with some particular graph excluded as an induced subgraph (there are a number of these for different subgraphs, but such an approach obviously cannot do the whole thing), and using linear programming methods to investigate the structure of a minimal counterexample. (There are rich conections with linear and integer programming - see [**?**] for example.)

Our approach is different. Recently, Conforti and Cornuejols [**?**] conjectured that every Berge graph either falls into one of four well-understood classes, or it admits one of several kinds of decomposition. They pointed out that if this could be proved, and if also it could be shown that no minimal counterexample to (1.1) admits any such decomposition, then 1.1 would follow (for certainly no minimal counterexample to 1.1 can fall into the four basic classes). We have been able to prove both these things (except we need a fifth class).

Before we can be more precise we need more definitions. If $X \subseteq V(G)$ we denote the subgraph of G induced on X by $G|X$. The *line graph* $L(G)$ of a graph G has vertex set the set $E(G)$ of edges of G, and $e, f \in E(G)$ are adjacent in $L(G)$ if they share an end in G.

We need one other class of graphs, defined as follows. Let H be a bipartite graph, with bipartition (A, B). For each vertex $v \in V(H)$ take two new vertices s_v, t_v, and make a graph G with $V(G) = \{s_v, t_v : v \in V(H)\}$. The edges of G are as follows:

- for $v \in V(H)$, s_v is adjacent to t_v if $v \in A$, and s_v is nonadjacent to t_v if $v \in B$
- for distinct $u, v \in A$, there are no edges between s_u, t_u and s_v, t_v
- for distinct $u, v \in B$, both s_u, t_u are adjacent to both s_v, t_v
- for $u \in A$ and $v \in B$, there are exactly two edges joining one of s_u, t_u to one of s_v, t_v; if $uv \in E(H)$ then $s_u s_v$ and $t_u t_v$ are edges of G, and otherwise $s_u t_v$ and $s_v t_u$ are edges of G.

We call such a graph G a *bicograph* (no relation to Bill Cook, G is made by doubling the vertices of a cograph). Let us say a graph G is *basic* if either G or $\overline{G}$ is bipartite or is the line graph of a bipartite graph, or is a bicograph. (Note that if G is a bicograph then so is $\overline{G}$.) It is easy to see that all basic graphs are perfect.

A *2-join* in G is a partition (X_1, X_2) of $V(G)$ so that there exist disjoint nonempty $A_i, B_i \subseteq X_i (i = 1, 2)$ satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2, and every vertex of B_1 is adjacent to every vertex of B_2,
- there are no other edges between X_1 and X_2,
- for $i = 1, 2$, every component of $G|X_i$ meets both A_i and B_i, and

- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is a path joining the members of A_i and B_i, then it has length ≥ 3.

If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex in X is adjacent to every vertex in Y; and we say X is *co-complete* to Y if there are no edges between X and Y. An *M-join* in G is a partition of $V(G)$ into six nonempty sets, (A, B, C, D, E, F), so that:

- every vertex in A has a neighbour in B and a nonneighbour in B, and vice versa
- the pairs $(C, A), (A, F), (F, B), (B, D)$ are complete, and
- the pairs $(D, A), (A, E), (E, B), (B, C)$ are co-complete.

A set $X \subseteq V(G)$ is *connected* if $G|X$ is connected, or if X is empty; and *co-connected* if $\overline{G}|X$ is connected. A *skew partition* in G is a partition (A, B) of $V(G)$ so that A is not connected and B is not co-connected. A *path* in G is an *induced* subgraph of G which is non-null, connected and in which every vertex has degree ≤ 2 (this definition is highly nonstandard, and we apologise, but it avoids writing "induced" about 2000 times), and an *antipath* is an induced subgraph whose complement is a path. The *length* of a path is the number of edges in it (and the length of an antipath is the number of edges in its complement). We therefore recognize paths and antipaths of length 0. If P is a path of length at least 2, P^* denotes the set of internal vertices of P, called the *interior* of P; and similarly for antipaths. A skew partition (A, B) is *even* if

- every path of length ≥ 2 with ends in A and with interior in B is even, and
- every antipath of length ≥ 2 with ends in B and with interior in A is even.

We shall prove the following, a form of which was conjectured by [**?**]:

1.2 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a 2-join, or G admits an M-join, or G admits an even skew partition.*

Proof of 1.1, assuming 1.2 and 4.8.
Suppose that 1.1 is false, and let G be a counterexample with $|V(G)|$ as small as possible. Since every perfect graph is Berge, it follows that G is Berge and not perfect. Every basic graph is perfect, and so G is not basic. It is shown in [] that G does not admit a 2-join. Since Lovasz [**?**] showed that the complement of a perfect graph is perfect, it follows that $\overline{G}$ is also a counterexample to 1.1 of minimum size, and therefore $\overline{G}$ also does not admit a 2-join. It is shown in [] that G does not admit an M-join, and we shall prove in 4.8 that G does not admit an even skew partition. It follows that G violates 1.2, and therefore there is no such graph G. This proves 1.1. ∎

All nontrivial bicographs admit skew partitions, so if we delete "even" from 1.2 then we no longer need to consider bicographs as basic - four basic classes suffice. Unfortunately, nontrivial bicographs do not admit even skew partitions, and general skew partitions are not good enough for the application to 1.1, so we have to do it the way we did.

How can we prove a theorem of the form of 1.2? There are several other theorems of this kind in graph theory - eg [**?**], [**?**],[**?**],[**?**] and others. All these theorems say that "every graph (or matroid) not containing an object of type X either falls into one of a few basic classes or has a decomposition". And for each of these theorems, the technique of the proof is the same: we find a small X-free graph H (*X-free* means not containing an object of type X) which does not fall into any of the basic classes; check that it has a decomposition of the kind it is supposed to have; and show that this decomposition extends to a decomposition of every bigger X-free graph containing H. That proves that the theorem is true for all X-free graphs that contain H; so now we focus on the X-free graphs that do not contain H. Repeat as often as necessary (with different small graphs in place of H) until the graphs that remain are sufficiently restricted that they can be handled by some other means.

This same technique also works for decomposing Berge graphs. We will in fact make a sequence of types of graph $H_1, H_2, ..$ so that for each $i > 0$, if G is Berge and contains an induced subgraph of type H_i, and does not contain any induced subgraph of type $H_1, .., H_{i-1}$, then G has a decomposition (or, in some cases, G is basic). So, the first type of graph H_1 is line graphs of bipartite subdivisions of large (bigger than $K_{3,3}$) 3-connected graphs; the second type is line graphs of bipartite subdivisions of $K_{3,3}$; the third is line graphs of certain bipartite subdivisions of K_4; and so on. In the end it remains to understand the Berge graphs not containing an induced subgraph of any of these types; and we find it is possible to prove directly that such graphs satisfy 1.2.

The paper is organized as follows. In section 2 we explain some extensions of a lemma of Roussel and Rubio [**?**] which we found to be immensely useful.

2 The Roussel-Rubio lemma

There is a beautiful and very powerful theorem of [**?**] which we use many times throughout the paper. (We proved it independently, in joint work with Carsten Thomassen, but Roussel and Rubio found it several months earlier.) Its main use is to show that in some respects, the common neighbours of a co-connected set of vertices (in a Berge graph) act like or almost like the neighbours of a single vertex.

If $X \subseteq V(G)$ and $v \in V(G)$, we say v is *X-complete* if v is adjacent to every vertex in X (and consequently $v \notin X$). Let P be a path in G (we remind the reader that this means P is an induced subgraph which is a path), of length ≥ 2, and let the vertices of P be $p_1, \dots, p_n$ in order. A *leap* for P *(in G)* is a pair of vertices a, b of G so that there are exactly six edges of G between a, b and $V(P)$, namely $ap_1, ap_2, ap_n, bp_1, bp_{n-1}, bp_n$. If C is an induced subgraph of G, we sometime write C for $V(C)$ when there is no risk of ambiguity.

The Roussel-Rubio lemma (slightly reformulated for convenience) is the following.

2.1 *Let G be Berge, and let P be a path in G with odd length. Let $X \subseteq V(G)$ be co-connected, so that both ends of P are X-complete. Then either:*

1. *there is an edge of P so that both its ends are X-complete, or*
2. *P has length ≥ 5 and X contains a leap for P, or*
3. *P has length 3 and there is an odd antipath joining the internal vertices of P with interior in X.*

This has a number of corollaries that again we shall need throughout the paper, and in this section we prove some of them.

2.2 *Let G be Berge, and let P be a path in G of odd length. Let $X \subseteq V(G)$ be co-connected, so that both ends of P are X-complete, and there is no edge of P so that both its ends are X-complete. Then every X-complete vertex has a neighbour in P^*.*

Proof. Let v be X-complete. Certainly P has length > 1, since its ends are X-complete and therefore nonadjacent. Suppose first it has length > 3. Then by 2.1, X contains a leap, and so there is a path Q with ends in X and with $Q^* = P^*$. Then v is adjacent to both ends of Q, and since $G|(V(Q) \cup \{v\})$ is not an odd hole, it follows that v has a neighbour in $Q^* = P^*$, as required. Now suppose P has length 3, and let its vertices be $p_1, \dots, p_4$ in order. By 2.1, there is an odd antipath Q between p_2 and p_3 with interior in X. Since Q cannot be completed to an odd antihole via p_3-v-p_2, it follows that v is adjacent to one of p_2, p_3, as required. ∎

Here is another little lemma that gets used enough that it is worth stating separately.

2.3 *Let G be Berge, and let P be a path or hole in G. Let $X \subseteq V(G)$ be co-connected, so that at least three vertices of P are X-complete. Let Q be a path of P (and hence of G) with both ends X-complete. Then the number of edges of Q with both ends X-complete has the same parity as the length of Q. In particular, if P is a hole, then either there are an even number of X-complete edges in P, or there are exactly two Y-complete vertices and they are adjacent.*

Proof. The second assertion follows from the first. For the first, we use induction on the length of Q. If some internal vertex of Q is X-complete then the result follows from the inductive hypothesis, so we may assume not. If Q has length 1 or even then the theorem holds, so we may assume its length is ≥ 3 and odd. By hypothesis there is an X-complete vertex v say of P that is not an end of Q, and therefore does not belong to Q; and since P is a path or hole, it follows that v has no neighbour in Q^*, contrary to 2.2. This proves 2.3. ▮

If P is a path of G between a and b say, and v is a vertex of G not in P but with a neighbour in P, then there is a unique path between v and a with vertex set in $V(P) \cup \{v\}$, and we call this path the *resultant* path between v and a (in contexts where the dependence on P is clear). A *triangle* in G is a set of three vertices, mutually adjacent. We say a vertex v can be *linked* onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:

- the three paths P_1, P_2, P_3 are mutually vertex-disjoint
- for $i = 1, 2, 3$ a_i is an end of P_i
- for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$
- v has a neighbour in each of P_1, P_2 and P_3.

The following is well-known and quite useful:

2.4 *Let G be Berge, and suppose v can be linked onto a triangle $\{a_1, a_2, a_3\}$. Then v is adjacent to at least two of a_1, a_2, a_3.*

Proof. Let v be linked via paths P_1, P_2, P_3. For $1 \leq i \leq 3$, v has a neighbour in P_i; let Q_i be the resultant path from v to a_i. At least two of Q_1, Q_2, Q_3 have lengths of the same parity, say Q_1, Q_2; and since $G|(V(Q_1) \cup V(Q_2))$ is not an odd hole, it is a cycle of length 3, and the claim follows. ▮

A variant of 2.2 is sometimes useful, the following:

2.5 *Let G be Berge, and let P be a path in G of odd length. Let its vertices be $p_1, \ldots, p_n$ in order. Let $X \subseteq V(G)$ be co-connected, so that both ends of P are X-complete, and there is no edge of P so that both its ends are X-complete. Let v be X-complete. Then either v is adjacent to one of p_1,p_2, or the only neighbour of v in P^* is p_{n-1}.*

Proof. By 2.2, v has a neighbour in P^*, and we may assume that p_{n-1} is not its only such neighbour, so v has a neighbour in $\{p_2, \ldots, p_{n-2}\}$. If P has length ≤ 3 then the result follows, so we may assume its length is at least 5. By 2.1, there is a leap a, b for P in X; so there is a path with vertices $a, p_2, \ldots, p_{n-1}, b$ in order. Now $\{p_1, p_2, a\}$ is a triangle, and v can be linked onto it via the three paths b-p_1, $P \setminus \{p_1, p_{n-1}, p_n\}$, a; and so v has two neighbours in the triangle, by 2.4, and the claim follows. ▮

Let A, B be disjoint subsets of $V(G)$. We say the pair (A, B) is *balanced* if there is no odd path between nonadjacent vertices in B with interior in A, and there is no odd antipath between adjacent vertices in A with interior in B.

2.6 *If G is Berge and $A, B \subseteq V(G)$ are disjoint, and $v \in V(G) \setminus (A \cup B)$, and v is complete to B and co-complete to A, then (A, B) is balanced.*

The proof is clear.

2.7 *Let (A, B) be balanced in a Berge graph G. Let $C \subseteq V(G) \setminus (A \cup B)$. Then :*

1. *if A is connected and every vertex in B has a neighbour in A, and there are no edges between A and C, then (C, B) is balanced*

2. *if B is co-connected and no vertex in A is B-complete, and B is complete to C, then (A, C) is balanced.*

Proof. The first statement follows from the second by taking complements, so it suffices to prove the second. Suppose $u, v \in A$ are adjacent and joined by an odd antipath P with interior in C. Since B is co-connected and u, v both have non-neighbours in B, they are also joined by an antipath Q with interior in B, which is even since (A, B) is balanced. But then u-P-v-Q-u is an odd antihole, a contradiction. Now suppose there are nonadjacent $u, v \in C$, joined by an odd path P with interior in A. Then P has length ≥ 5, since otherwise its vertices could be reordered to be an odd antipath of the kind we already handled. The ends of P are B-complete, and no internal vertex is B-complete, and so B contains a leap for P by 2.1; and hence there is an odd path with ends in B and interior in A, which is impossible since (A, B) is balanced. This proves 2.7. ∎

We already said what we mean by linking a vertex onto a triangle, but now we do the same for a co-connected set. We say a co-connected set X can be *linked* onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:

- the three paths P_1, P_2, P_3 are mutually vertex-disjoint
- for $i = 1, 2, 3$ a_i is an end of P_i
- for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$
- each of P_1, P_2 and P_3 contains an X-complete vertex.

There is a corresponding extension of 2.4, the following:

2.8 *Let G be Berge, let X be a co-connected set, and suppose X can be linked onto a triangle $\{a_1, a_2, a_3\}$ via P_1, P_2, P_3. For $i = 1, 2, 3$ let P_i have ends a_i and b_i, and let b_i be the unique vertex of P_i that is X-complete. Then either at least two of P_1, P_2, P_3 have length 0 (and hence two of a_1, a_2, a_3 are X-complete) or one of P_1, P_2, P_3 has length 0 and the other two have length 1 (say P_3 has length 0); and in this case, every X-complete vertex in G is adjacent to one of a_1,a_2.*

Proof. Some two of P_1, P_2, P_3 have lengths of the same parity, say P_1 and P_2. Hence the path $Q = b_1$-P_1-a_1-a_2-P_2-b_2 (with the obvious meaning - we shall feel free to specify paths by whatever notation is most convenient) is odd, and its ends are X-complete, and none of its internal vertices are X-complete. If Q has length 1 then the theorem holds, so we assume it has length ≥ 3. By 2.2, every X-complete vertex has a neighbour in Q^*, and since b_3 is X-complete, it follows that $b_3 = a_3$. Hence we may assume both P_1 and P_2 have length ≥ 1 for otherwise the claim holds. Suppose that Q has length 3. Then P_1 and P_2 have length 1, and the claim holds again. So we may assume (for a contradiction) that Q has length ≥ 5, and from the symmetry we may assume P_1 has length ≥ 2. Since b_3 is not adjacent to the end b_1 of Q or to its neighbour in Q, and yet it has at least two neighbours in Q^* (namely a_1 and a_2), this contradicts 2.5. This proves 2.8. ∎

There is an "open-ended" version of 2.1, the following.

2.9 *Let G be Berge, and let P be a path in G of length $t > 0$, with ends a, b, and let r, s be integers so that $r + s + t$ is odd. Let $X \subseteq V(G)$ be co-connected, so that every $x \in X$ has a neighbour in P, and the resultant path from x to a has parity r, and the resultant path from x to b has parity s. Assume that $(V(P), X)$ is balanced. Then there is an edge of P with both ends X-complete.*

Proof. We may assume X is non-null. By deleting any vertices not in $X \cup V(P)$, we may assume that every vertex is in this set. Let us add a new vertex a' to G adjacent only to a; the new graph is still Berge, and still satisfies the hypotheses of the theorem (replacing P by the path a'-a-P-b and r by $r + 1$), and if the theorem holds for the new graph then it holds for G. So by repeated use of this method (adding new vertices to either end of the path) we may assume that r and s are both even (and hence P is odd), and moreover P has length ≥ 3, and that a and b both have no neighbours in X. Let P have vertices $p_1, \ldots, p_n$ in order, where $p_1 = a$

and $p_n = b$. Add two new vertices p_0 and p_{n+1} to G, forming a new graph G', where the sets of neighbours of p_0,p_{n+1} are $X \cup \{p_1\}$ and $X \cup \{p_n\}$ respectively. Let P' be the path p_0-p_1-$\cdots$-p_n-p_{n+1} of G'. Then P' has odd length ≥ 5, and its ends are X-complete. If G' is Berge, then since p_1 and p_n have no neighbours in X, there is no leap in X, so from 2.1 there is an edge of P' with both ends X-complete; and since both its ends have neighbours in X it follows that it is an edge of P, as required. So we may assume that G' is not Berge. Suppose there is an odd hole C in G'. Certainly it uses at least one of the two new vertices p_0,p_{n+1}, so we may assume it uses p_0. If it also uses p_{n+1}, then it contains at most one vertex in X (since p_0,p_{n+1} are X-complete) and the remainder of C is therefore a connected subgraph of P' containing both its ends, and hence equals P', which is impossible (since every vertex in X has a neighbour in P). So p_{n+1} is not in C. Let the two neighbours of p_0 in C be x, y; so $x, y \in X \cup \{p_1\}$, and we may assume $x \in X$. If also $y \in X$ then $C \setminus p_0$ is an odd path with ends in X and interior in $V(P)$, which is impossible since $(V(P), X)$ is balanced. Hence we may assume $y = p_1$.

Then C is a hole of $G'|((V(P') \cup \{x\})$ using both x and p_0; but this hole is unique, and is even from the hypothesis of the theorem, a contradiction. Now assume there is an odd antihole in G', say D, and since we may assume G' has no odd hole it follows that D has length ≥ 7. Consequently all vertices of G' in D have degree ≥ 4, and therefore p_1, p_n are not in D. Again we may assume D uses p_0. Suppose it also uses p_{n+1}. Then since some vertex of D is adjacent to p_0 and not to p_{n+1} it follows that p_1 is in D, a contradiction. So p_{n+1} is not in D. Now D contains exactly two nonneighbours of p_0 (since it is an antihole), and they are adjacent, and both belong to P^*. All the other vertices of D are neighbours of p_0 and therefore belong to X. So $D \setminus p_0$ is an odd antipath with ends in $V(P)$ and with interior in X, which is impossible since $(V(P), X)$ is balanced. This proves 2.9. ∎

As we said earlier, the main use of 2.1 is to show that the common neighbours of a co-connected set behave in some respects like the neighbours of a single vertex. From this point of view, 2.1 itself tells us something about when there can be an odd "pseudohole", in which one "vertex" is actually a co-connected set. We also need a version of this when there are two such vertices, the following.

2.10 *Let G be Berge, and let P be a path in G with even length > 0, with vertices $p_1, \ldots, p_n$ in order. Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be disjoint co-connected sets, so that p_1 is the unique X-complete vertex of P and p_n is the unique Y-complete vertex of P. Let X be complete to Y. Then either:*

1. *P has length ≥ 4 and there are nonadjacent $x_1, x_2 \in X$ so that x_1-p_2-$\cdots$-p_n-x_2 is a path, or*
2. *P has length ≥ 4 and there are nonadjacent $y_1, y_2 \in X$ so that y_1-p_1-$\cdots$-p_{n-1}-y_2 is a path, or*
3. *P has length 2 and there is an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y, and exactly one of Q,R has odd length. In each case, either $(V(P \setminus p_1), X)$ or $V(P \setminus p_n), Y)$ is not balanced.*

Proof. It follows from the hypotheses that X,Y and $V(P)$ are mutually disjoint. If P has length 2, choose an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y. Then p_2-Q-p_3-p_1-R-p_2 is an antihole, and so exactly one of Q,R has odd length and the theorem holds. So we may assume P has length ≥ 4. We may assume that $V(G) = V(P) \cup X \cup Y$, by deleting any other vertices. Let G' be obtained from $G \setminus Y$ by adding a new vertex y with neighbour set $X \cup \{p_n\}$. Let P' be the path p_1-$\cdots$-p_n-y of G'. Then P' has odd length ≥ 5. If G' is Berge then by 2.1 there is a leap for P' in X, and the result follows. So we may assume G' is not Berge. Assume first that there is an odd hole C of length ≥ 7 in G'. It necessarily uses y, and the neighbours of y in C are Y-complete, and no other vertices of $C \setminus y$ are Y-complete. Hence there is an odd path Q in $G \setminus Y$ of length ≥ 5, with both ends Y-complete and no internal vertices Y-complete. So the ends of Q belong to $X \cup \{p_n\}$ and its interior to $V(P) \setminus p_n$. By 2.1 Y contains a leap for Q; so there is an odd path R of length ≥ 5 with ends (y_1, y_2 say) in Y and with interior in $V(P) \setminus p_n$. Since R cannot be completed to a hole via y_2-p_n-y_1 it follows that p_n has a neighbour in R^*, and so p_{n-1} belongs to R. If also p_1 belongs to R then the theorem holds, so we may assume it does not. Since R is odd and P is even it follows that p_2 also does not belong to R, and so p_1 has no neighbour in R^*; yet the ends of R are X-complete and its internal vertices are not, contrary to 2.2. This completes the case when there is an odd hole in G' of length ≥ 7. Since an odd hole of length 5 is also an odd antihole, we may assume that there is an odd antihole in G', say D. Again D must use y, and uses exactly two nonneighbours of y; so in G there is an odd antipath Q between adjacent vertices of $P \setminus p_n$ (say u and v), and with interior in $X \cup \{p_n\}$. Since u and v are not Y-complete, they are also joined by an antipath R with interior in Y, and R must also be odd since its union with Q is an antihole. Since R cannot be completed to an antihole via v-p_n-u it follows that p_n is adjacent to one of u,v, and hence we may assume that $u = p_{n-2}$ and $v = p_{n-1}$. Since P has length ≥ 4 it follows that u,v are also joined by an antipath with interior in X, say S, and again S is odd since its union with R is an antihole. But S can be completed to an antihole via v-p_1-u, a contradiction. This proves 2.10. ∎

Next we need a version of 2.1 for holes, as follows. Let C be a hole in G, and let $e = uv$ be an edge of it. A *leap* for C (*in* G, *at* uv) is a leap for the path $C \setminus e$ in $G \setminus e$. A *hat* for C (*in* G, *at* uv) is a vertex of G adjacent to u and v and to no other vertex of C.

2.11 *Let G be Berge, let C be a hole in G with length > 4, and let $e = uv$ be an edge of C. Let $X \subseteq V(G)$ be co-connected, so that u, v are X-complete and no other vertex of C is X-complete. Then either X contains a hat for C at uv, or X contains a leap for C at uv.*

Proof. Let the vertices of C be $p_1, \ldots, p_n$ in order, where $u = p_1$ and $v = p_n$. Let $G_1 = G|(V(C) \cup X)$, and let $G_2 = G_1 \setminus e$. If G_2 is Berge, then from 2.1 applied to the path $C \setminus e$ in G_2 it follows that X contains a leap for C at uv. So we may assume that G_2 is not Berge. Consequently it has an odd hole or antihole D say, and since D is not an odd hole or antihole in G_1 it must use both p_1 and p_n. Suppose first that D is an odd hole. Since every vertex in X is adjacent to both p_1 and p_n it follows that at most one vertex of X is in D; and since $G_2 \setminus X$ has no cycles, there is exactly one vertex of X in D, say x. Hence $D \setminus x$ is a path of $G_2 \setminus X$ between p_1 and p_n, and so $D \setminus x = C \setminus e$; and since D is a hole of G_2 it follows that x has no neighbours in $\{p_2, \ldots, p_{n-1}\}$, and therefore is a hat as required. Next assume that D is an antihole. Since it uses both p_1 and p_n, and they are nonadjacent in G_2, it follows that they are consecutive in D, so the vertices of D can be numbered $d_1, \ldots, d_m$ in order, where $d_1 = p_1$ and $d_m = p_n$, and therefore $m \geq 5$. Consequently, both d_2 and d_{m-1} are not in X, since they are not complete to $\{p_1, p_n\}$, and therefore all of d_1, d_2, d_{m-1}, d_m are vertices of C. Yet $d_1 d_{m-1}, d_{m-1} d_2, d_2 d_m$ are all edges of G_1, which is impossible since $n \geq 6$. This proves 2.11. ∎

There is an analogous version of 2.10, as follows.

2.12 *Let G be Berge, and let P be a path in G with even length ≥ 4, with vertices $p_1, \ldots, p_n$ in order. Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be co-connected sets, so that p_1 is the unique X-complete vertex of P, and p_1, p_n are the only Y-complete vertices of P. Let X be complete to Y. Then either:*

1. *there exists $x \in X$ non-adjacent to all of $p_2, \ldots, p_n$, or*

2. *there are nonadjacent $x_1, x_2 \in X$ so that x_1-p_2-$\cdots$-p_n-x_2 is a path.*

Proof. The proof is similar to that of 2.10. We may assume $V(G) = V(P) \cup X \cup Y$. Let G' be obtained from $G \setminus Y$ by adding a new vertex y with neighbour set $X \cup \{p_1, p_n\}$. If G' is Berge then the result follows from 2.11, so we may assume G' is not Berge. Assume first that there is an odd hole C of length ≥ 7 in G'. Hence there is an odd path Q in $G \setminus Y$ of length ≥ 5, with both ends Y-complete and no internal vertices Y-complete. So the ends of Q belong to $X \cup \{p_1, p_n\}$ and its interior to $V(P^*)$. By 2.1 Y contains a leap for Q; so there is an odd path R of length ≥ 5 with ends (y_1, y_2 say) in Y and with interior in $V(P^*)$. Since R is odd and R^* is a subpath of the even path P^*, it follows that not both p_2 and p_{n-1} belong to R; but then R can be completed to an odd hole via one of y_2-p_n-y_1 , y_2-p_1-y_1, a contradiction. This completes the case when there is an odd hole in G' of length ≥ 7, so now we may assume that there is an odd antihole in G', say D. Again D must use y, and uses exactly two nonneighbours of y; so in G there is an odd antipath Q between adjacent vertices of P^* (say u and v), and with interior in $X \cup \{p_n\}$. Since u and v are not Y-complete, they are also joined by an antipath R with interior in Y, and R must also be odd since its union with Q is an antihole. Since one of p_1,p_n is nonadjacent to both of u, v, we may complete R to an odd antihole via one of u-p_1-v,u-p_n-v, a contradiction. This proves 2.12. ∎

3 Paths and antipaths meeting

Another class of applications of 2.1 is to the situation when a big path or hole meets a big antipath or antihole. In this section we prove a collection of useful lemmas of this type. First, a neat application of 2.1, not particularly useful, but easy and surprising.

3.1 *Let G be Berge, let C be a hole in G, and D an antihole in G, both of length ≥ 8. Then $|C \cap D| \leq 3$.*

Proof. It is easy to see that $|C \cap D| \leq 4$, without using that G is Berge. Suppose that $|C \cap D| = 4$; then $C \cap D$ is the vertex set of a 3-edge path. Let C have vertices $p_1, \dots, p_m$ in order, and D have vertices $q_1, \dots, q_n$ in order, where $m, n \geq 8$ and $p_1 = q_2, p_2 = q_4, p_3 = q_1, p_4 = q_3$. Let P be the path p_4-p_5-$\cdots$-p_m-p_1, and Q the antipath q_4-q_5-$\cdots$-q_n-q_1. Let X be the interior of Q. Then p_1 and p_4 are X-complete (since D is an antihole), and P is a path with length odd and ≥ 5 between these two vertices. If some vertex p_i say in the interior of P is X-complete, then since p_i is nonadjacent to both p_2 and p_3 we can complete Q to an odd antihole via q_1-p_i-q_4, a contradiction. So by 2.1 X contains a leap for P; so there exists i with $5 \leq i < n$ and a path P' joining q_i and q_{i+1} with the same interior as P. Since $n \geq 8$, either $i > 5$ or $i + 1 < n$ and from the symmetry we may assume the first. But then P' can be completed to an odd hole via q_{i+1}-p_2-q_i, a contradiction. This proves 3.1. ∎

The next two lemmas are results of the same kind:

3.2 *Let p_1-$\cdots$-p_m be a path in a Berge graph G. Let $2 \leq s \leq m-2$, and let p_s-q_1-$\cdots$-q_n-p_{s+1} be an antipath, where where $n \geq 2$. Assume that p_1, p_m are both adjacent to all of $q_1, \dots, q_n$. Then either:*

- *n is even and $m = 4$, or*
- *n is odd, and the only nonedges between $\{p_{s-2}, p_{s-1}, p_s, p_{s+1}, p_{s+2}\}$ and $\{q_1, \dots, q_n\}$ are $p_{s-1}q_n, p_sq_1, p_{s+1}q_n$, or*
- *n is odd, and the only nonedges between $\{p_{s-1}, p_s, p_{s+1}, p_{s+2}, p_{s+3}\}$ and $\{q_1, \dots, q_n\}$ are $p_sq_1, p_{s+1}q_n, p_{s+2}q_1$.*

Proof. Let $Q = \{q1, \dots, q_n\}$. Suppose first that n is even; then every Q-complete vertex w is adjacent to one of p_s, p_{s+1}, for otherwise w-p_s-q_1-$\cdots$-q_n-p_{s+1}-w would be an odd antihole. Since p_1 and p_m are Q-complete it follows that $m = 4$, and the theorem holds. So we may assume that n is odd.

Now choose r with $1 \leq r \leq s$ maximum so that p_r is Q-complete, and choose t with $s+1 \leq t \leq m$ minimum so that p_t is Q-complete. Certainly $t - r \geq 3$, since p_s, p_{s+1} are not Q-complete. Suppose that $t - r$ is odd. Then the path p_r-p_{r+1}-$\cdots$-p_t is odd, and its ends are Q-complete, and its internal vertices are not. If it has length ≥ 5 then by 2.1, Q contains a leap, which is impossible since all vertices in Q are adjacent to one of p_s, p_{s+1} (since $n \geq 2$). So it has length 3, and hence $r = s-1$ and $t = s+2$; but then p_t-p_s-q_1-$\cdots$-q_n-p_{s+1}-p_r-p_t is an odd antihole, a contradiction. This proves that $t - r$ is even. We may therefore assume that $t - (s + 1)$ is even and $s - r$ is odd. Suppose that $s > r + 1$. Then p_r-$\cdots$-p_s is an odd path of length ≥ 3, between $(Q \setminus q_1)$-complete vertices, and the $(Q \setminus q_1)$-complete vertex p_t has no neighbour in its interior, so by 2.2, it contains another $(Q \setminus q_1)$-complete vertex , say p_i where $r < i < s$. But then p_i is nonadjacent to q_1, from the maximality of r, and so p_i-q_1-$\cdots$-q_n-p_{s+1}-p_i is an odd antihole, a contradiction. So $r = s - 1$.

Now $t - s$ is odd, and so p_s-p_{s+1}-$\cdots$-p_t is an odd path between $(Q \setminus q_1)$-complete vertices, and the $(Q \setminus q_1)$-complete vertex p_{s-1} has no neighbours in its interior. By 2.2, there exists i with $s + 1 \leq i \leq t$ such that p_i, p_{i+1} are both $(Q \setminus q_1)$-complete. Since p_{s+1} is not $(Q \setminus q_1)$-complete it follows that $i \geq s + 2$. Now there is no $(Q \setminus q_1)$-complete vertex p_j with $s + 2 < j < t$; for from the minimality of t it would be nonadjacent to q_1, and then p_j-q_1-$\cdots$-q_n-p_{s+1}-p_j would be an odd antihole. Since both p_i and p_{i+1} are $(Q \setminus q_1)$-complete, it follows that $i = s + 2$ and $i + 1 = t = s + 3$. But then the theorem holds. This proves 3.2. ∎

3.3 *Let G be Berge, let C be a hole in G of length ≥ 6, with vertices $p_1, \ldots, p_m$ in order, and let Q be an antipath with vertices $p_1, q_1, \ldots, q_n, p_2$, with length ≥ 4 and even. Let $z \in V(G)$, complete to $V(Q)$ and with no neighbours among $p_3, \ldots, p_m$. There is at most one vertex in $\{p_3, \ldots, p_m\}$ complete to either $\{q_1, \ldots, q_{n-1}\}$ or $\{q_2, \ldots, q_n\}$, and any such vertex is one of p_3, p_m.*

Proof. It follows that none of $q_1, \ldots, q_n$ belong to C, since they are all adjacent to z. Let $X = \{q_1, \ldots, q_n\}$, and let Y_1, Y_2 be the sets of vertices in $\{p_3, \ldots, p_m\}$ complete to $X \setminus q_n$,$X \setminus q_1$ respectively.

(1) *$Y_1 \subseteq Y_2 \cup \{p_m\}$, and $Y_2 \subseteq Y_1 \cup \{p_3\}$.*

For suppose some $p_i \in Y_1$, and is not in Y_2; then since the odd antipath $Q \setminus p_2$ cannot be completed to an odd antihole via q_n-p_i-p_1, it follows that $i = m$. This proves (1).

(2) *If $Y_1 \not\subseteq \{p_m\}$ then $p_3 \in Y_1 \cap Y_2$, and if $Y_2 \not\subseteq \{p_3\}$ then $p_m \in Y_1 \cap Y_2$.*

For assume $Y_1 \not\subseteq \{p_m\}$, and choose i with $3 \leq i \leq m-1$ minimum so that $p_i \in Y_1$. By (1), $p_i \in Y_2$, so we may assume $i > 3$, for otherwise the claim holds. If i is odd, then the path p_2-p_3-$\cdots$-p_i is odd and between $X \setminus q_n$-complete vertices, and no internal vertex is $X \setminus q_n$-complete, and yet the $X \setminus q_n$-complete vertex z does not have a neighbour in its interior, contrary to 2.2. So i is even. The path p_i-$\cdots$-p_m-p_1 is therefore odd, and has length ≥ 3, and its ends are $X \setminus q_1$-complete, and the $X \setminus q_1$-complete vertex z does not have a neighbour in its interior; so by 2.2 some vertex v of its interior is in Y_2, and therefore in $Y_1 \cap Y_2$ by (1). But the path z-$\cdots$-p_i is odd, and between X-complete vertices, and has no more such vertices in its interior, and v has no neighbour in its interior, contrary to 2.2. This proves (2).

Now not both p_3, p_m are in $Y_1 \cap Y_2$, for otherwise Q could be completed to an odd antihole via p_2-p_m-p_3-p_1. Hence we may assume $p_3 \notin Y_1 \cap Y_2$, and so from (2), $Y_1 \subseteq \{p_m\}$. By (1), $Y_2 \subseteq \{p_3\} \cup Y_1$, and so $Y_1 \cup Y_2 \subseteq \{p_3, p_m\}$. We may therefore assume that $Y_1 \cup Y_2 = \{p_3, p_m\}$, for otherwise the theorem holds. In particular, $p_3 \in Y_2$. If also $p_m \in Y_2$, then p_3-p_4-$\cdots$-p_m is an odd path between $X \setminus q_1$-complete vertices, and none of its internal vertices are $X \setminus q_1$-complete, and yet the $X \setminus q_1$-complete vertex z does not have a neighbour in its interior, contrary to 2.2. So $p_m \notin Y_2$, and so $p_m \in Y_1$; but then p_3-q_1-q_2-$\cdots$-q_n-p_m-p_3 is an odd antihole, a contradiction. This proves 3.3. ∎

4 Skew partitions

A maximal connected subset of a nonempty set $A \subseteq V(G)$ is called a *component* of A, and a maximal co-connected subset is called a *co-component* of A. Let us say a skew partition A, B of G is *loose* if either some vertex in B has no neighbour in some component of A, or some vertex in A is complete to some co-component of B.

In this section we investigate what skew partitions look like in a "minimum imperfect graph", a Berge graph which is the smallest counterexample to 1.1; and in particular, we shall show that no skew partition in such a graph can be either even (defined in the first section) or loose. So, in order to prove 1.1, it would be enough to prove a weaker form of our main theorem 1.2, in which "admits an even skew partition" is repaced by "admits an even or loose skew partition". In a way this would be easier, because many of the skew partitions we find later in the paper are loose. But we might as well prove the stronger form, because as we shall show below, any Berge graph admitting a loose skew partition also admits an even one.

4.1 *Let G be Berge, and suppose that G admits a skew partition (A, B) so that either some component of A or some co-component of B has only one vertex. Then G admits an even skew partition.*

Proof. Let $A_1, \ldots, A_m$ be the components of A, and $B_1, \ldots, B_n$ the co-components of B. So the sets $A_1, \ldots, A_m$ are pairwise disjoint, all non-empty, and partition A, and $m \geq 2$; and similarly for B. By taking complements if necessary we may assume that $|A_1| = 1$, $A_1 = \{a_1\}$ say. Let N be the set of vertices of G adjacent to a_1; so $N \subseteq B$. Assume first that N is not co-connected. Then $(V(G) \setminus N, N)$ is a skew partition of G, and it is easy to check that it is even, as required. So we may assume that N is co-connected. Consequently N is a subset of one of $B_1, \dots, B_n$, say B_1. Choose $b_2 \in B_2$. Then $N' = N \cup \{b_2\}$ is not connected, and so $(V(G) \setminus N', N')$ is a skew partition of G, and once again it is easily checked to be even. This proves 4.1. ∎

4.2 *If G is Berge, and admits a loose skew partition, then it admits an even skew partition.*

Proof. Let (A, B) be a loose skew partition of G. By taking complements if necessary, we may assume that some vertex in B has no neighbour in some component of A. With G fixed, let us choose the skew partition (A, B) and a component A_1 of A and a co-component B_1 of B with $|B| - 2|B_1|$ minimum, such that some vertex in B_1 (say b_1) has no neighbour in A_1. (We call this property the "optimality" of (A, B).) Let the other components of A be $A_2, \dots, A_m$, and the other co-components of B be $B_2, \dots, B_n$. By 4.1 we may assume that no $|A_i|$ or $|B_j| = 1$, and in this case we shall show that the skew partition (A, B) is even.

(1) *For $2 \leq j \leq n$, no vertex in A is B_j-complete and not B_1-complete, and every vertex in $B \setminus B_1$ has a neighbour in A_1.*

For the first claim, assume some vertex $v \in A$ is B_2-complete and not B_1-complete, say. Let $A'_1 = A_1$ if $v \notin A_1$, and let A'_1 be a maximal connected subset of $A_1 \setminus v$ otherwise. (So A'_1 is nonempty since we assumed $|A_1| \geq 2$.) Let $A' = A \setminus v$ and $B' = B \cup \{v\}$; then B_2 is still a co-component of B', so (A', B') is a skew partition, violating the optimality of (A, B) (for since v is not B_1-complete, there is a co-component of B' including $B \cup \{v\}$). For the second claim, assume that some vertex $v \in B_2$ say has no neighbour in A_1. Then since $B_2 \geq 2$, it follows that $(A \cup \{v\}, B \setminus v)$ is a skew partition of G, again violating the optimality of (A, B). This proves (1).

By 2.6, the pair (A_1, B_j) is balanced, for $2 \leq j \leq n$, since b_1 is complete to B_j and has no neighbours in A_1. By (1) and 2.7.1, it follows that (A_i, B_j) is balanced for $2 \leq i \leq m$ and $2 \leq j \leq n$. It remains to check all the pairs (A_i, B_1). Let $1 \leq i \leq m$, and let A'_i be the set of vertices in A_i that are not B_1-complete. By (1), no vertex in A'_i is B_2-complete, and (A'_i, B_2) is balanced, and hence by 2.7.2, so is (A'_i, B_1), and consequently so is (A_i, B_1). This proves that (A, B) is even, and so completes the proof of 4.2. ∎

4.3 *Let (A, B) be a skew partition of a Berge graph G. If either:*

- *there exist $u, v \in B$ joined by an odd path with interior in A, and joined by an even path with interior in A, or*
- *there exist $u, v \in A$ joined by an odd antipath with interior in B, and joined by an even antipath with interior in B,*

then (A, B) is loose and therefore G admits an even skew partition.

Proof. By taking complements we may assume that the first case of the theorem applies. Let $A_1, \dots, A_m$ be the components of A, and $B_1, \dots, B_n$ the co-components of B. Since $u, v \in B$, and they are joined by an even path, they are therefore nonadjacent, and so belong to the same B_j, say B_1. There is an even path P_1 and an odd path P_2 joining u, v, both with interior in A. We may assume that P_1 has interior in A_1. Since the union of P_1 and P_2 is not a hole, it follows that P_2 also has interior in A_1. If u, v are joined by a path with interior in A_2, then its union with one of P_1,P_2 would be an odd hole, a contradiction; so there is no such path. Hence one of u, v has no neighbours in A_2, and hence (A, B) is loose, and the theorem follows from 4.2. This proves 4.3. ∎

Let (A, B) be a skew partition of G, and let $A_1, \ldots, A_m$ be the components of A, and $B_1, \ldots, B_n$ the co-components of B. For $1 \leq i \leq m$ and $1 \leq j \leq n$, we say (i, j) is a *path pair* if there is an odd path in G with ends nonadjacent vertices of B_j and with interior in A_i; and (i, j) is an *antipath pair* if there is an odd antipath in G with ends adjacent vertices of A_i and with interior in B_j.

4.4 *Let (A, B) be a skew partition of a Berge graph G, and let $A_1, \ldots, A_m$ be the components of A, and $B_1, \ldots, B_n$ the co-components of B. Then either:*

- *(A, B) is loose or even, or*
- *(i, j) is a path pair for all i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$, or*
- *(i, j) is an antipath pair for all i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. We may assume (A, B) is not loose and not even.

(1) *If for some i, j there is an odd path of length ≥ 5 with ends in B_j and interior in A_i, then the theorem holds.*

For assume there is such a path for $i = j = 1$ say. Let this path, P_1 say, have vertices b_1-p_1-p_2-$\ldots$-p_n-b'_1, where $b_1, b'_1 \in B_1$ and $p_1, \ldots, p_n \in A_1$. Let $2 \leq j \leq n$. Then P_1 is an odd path of length ≥ 5 between common neighbours of B_j, and no internal vertex of it is B_j-complete since (A, B) is not loose. By 2.1, B_j contains a leap; so there exist nonadjacent $b_j, b'_j \in B_j$ so that b_j-p_1-p_2-$\ldots$-p_n-b'_j is a path. Hence $(1, j)$ is a path pair. Now let $2 \leq i \leq m$ and $1 \leq j \leq n$. Since (A, B) is not loose, b_j and b'_j both have neighbours in A_i, and so there is a path P_2 say joining them with interior in A_i; it is odd by 4.3, and so (i, j) is a path pair. This proves (1).

From (1) we may assume that for all i, j, every odd path of length > 1 with ends in B_j and interior in A_i has length 3; and similarly every odd antipath of length > 1 with ends in A_i and interior in B_j has length 3. Consequently, every path pair is also an antipath pair (because a path of length 3 can be reordered to be an antipath of length 3). We may assume that $(1, 1)$ is a path pair, and so there exist $b_1, b'_1 \in B_1$ and $a_1, a'_1 \in A_1$ so that b_1-a_1-a'_1-b'_1 is a path P_1 say. Let $2 \leq ilem$. Since b_1 and b'_1 both have neighbours in A_i, they are joined by a path with interior in A_i, odd by 4.3 ; and so by (1) it has length 3. Hence there exist $a_i, a'_i \in A_i$ so that b_1-a_i-a'_i-b'_1 is a path. By the same argument in the complement, it follows that for all $1 \leq i \leq m$ and $2 \leq j \leq n$, there exist $b_j, b'_j \in B_j$ so that b_j-a_i-a'_i-b'_j is a path. So every pair (i, j) is both a path and antipath pair. This proves 4.4. ∎

We can reformulate the previous result in a form that is easier to apply, as follows.

4.5 *Let G be Berge. Suppose that there is a partition of $V(G)$ into four nonempty sets X, Y, L, R, such that there are no edges between L and R, and X is complete to Y. If either:*

- *some vertex in $X \cup Y$ has no neighbours in L or no neighbours in R, or*
- *some vertex in $L \cup R$ is complete to X or complete to Y, or*
- *(L, Y) is balanced*

then G admits an even skew partition.

Proof. Certainly $L \cup R, x \cup Y)$ is a skew partition, so by 4.2 we may assume it is not loose, and therefore neither of the first two alternative hypotheses holds. So we assume the third hypothesis holds. Let $A_1, \ldots, A_m$ be the components of $L \cup R$, and let $B_1, \ldots, B_n$ be the co-components of $X \cup Y$. Since X, Y, L, R are all nonempty we may assume that $A_1 \subseteq L$, and $B_1 \subseteq X$. By hypothesis, $(1, 1)$ is not a path or antipath pair, and so by 4.4 the skew partition is even. This proves 4.5. ∎

In the main proof there will be several occasions when we need to show that a given skew partition is either loose or even, and some of them can be handled by the following. Let (A, B) be a skew partition of G. We say that a co-connected subset W of B is a *kernel* for the skew partition if some component of A contains no W-complete vertex.

4.6 *Let (A, B) be a skew partition of a Berge graph G, and let W be a kernel for it. Let A_1 be a component of A, and suppose that (A_1, W) is balanced. Then G admits an even skew partition.*

Proof. We may assume (A, B) is not loose. Let the components of A be $A_1, \ldots, A_m$, and the co-components of B be $B_1, \ldots, B_n$.

(1) *(A_i, W) is balanced for $1 \leq i \leq m$.*

For this is true if $i = 1$, so assume $i > 1$. From 4.3 there is no odd path between nonadjacent vertices of W with interior in A_i. Suppose there is an odd antipath Q of length > 1, with ends in A_i and interior in W. Then it has length ≥ 5, for otherwise it can be reordered to be an odd path that we have already shown impossible. Now the ends of Q have no neighbours in the connected set A_1, and its internal vertices all have neighbours in A_1; and so by 2.1 in the complement, there is a leap in the complement; that is, there is an antipath with ends in A_1 and with the same interior as Q, which is impossible. This proves (1).

Since W is co-connected, we may assume that $W \subseteq B_1$. Since (1) restores the symmetry between $A_1, \ldots, A_m$, we may assume that there is no W-complete vertex in A_1. By 4.4 we may assume $(1, 2)$ is a path or antipath pair. Suppose first that it is an antipath pair. Then there is an odd antipath Q_1 of length ≥ 3 with ends in A_1 and interior in B_2. Since its ends both have nonneighbours in W, its ends are also joined by an antipath Q_2 with interior in W, odd by 4.3, contrary to (1). So there is no such Q_1. Hence there is an odd path P with ends in B_2 and interior in A_1, necessarily of length ≥ 5 (since we already did the antipath case). Since the interior of P contains no W-complete vertex, 2.1 implies that W contains a leap; and so there is a path with ends in W with the same interior as P, a contradiction. This proves 4.6. ∎

One can refine 4.6 a little more, as follows.

4.7 *Let (A, B) be a skew partition of a Berge graph G, and let W be a kernel for it. Let A_1 be a component of A, and suppose that*

- *every pair of nonadjacent vertices of W with neighbours in A_1 are joined by an even path with interior in A_1*
- *every pair of adjacent vertices of A_1 with nonneighbours in W are joined by an even antipath with interior in W.*

Then G admits an even skew partition.

Proof. This is immediate from 4.6 and 4.3. ∎

By a *minimum imperfect graph* we mean a *Berge* graph G, not perfect, with $|V(G)|$ minimum. Now let us investigate skew partitions in a minimum imperfect graph.

4.8 *Let (A, B) be a skew partition in a minimum imperfect graph G, and let $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ be defined as usual. For all i with $1 \leq i \leq m$ there exists j with $1 \leq j \leq n$ such that (i, j) is a path or antipath pair, and for all j with $1 \leq j \leq n$ there exists i with $1 \leq i \leq m$ such that (i, j) is a path or antipath pair.*

Proof. The first statement is equivalent to the second by taking complements, since $\overline{G}$ also satisfies the hypotheses of the theorem and (B, A) is a skew partition in it. It therefore suffices to prove the second statement, and we may assume $j = 1$. Let G' be the graph obtained from G by adding a new vertex z with neighbour set B_1.

(1) *We may assume G' is Berge.*

For suppose it is not. Then in G' there is an odd hole or antihole using z. Suppose first that there is an odd hole, C say. Then the neighbours of z in C (say x, y) belong to B_1, and no other vertex of B_1 is in C. For $2 \leq j \leq n$ no vertex of B_j is in C since it would be adjacent to x, y and C would have length 4; so $C \setminus z$ is an odd path of G, with ends in B_1 and with interior in A. Since the interior of this odd path is connected, it is a subset of one of $A_1, \ldots, A_m$, say A_i; but then $(i, 1)$ is a path pair and the theorem holds. So we may assume there is no such C. Now assume there is an odd antihole D in G', again using z. Then exactly two vertices of $D \setminus z$ are nonadjacent to z, so all the others belong to B_1. Hence in G there is an odd antipath Q of length ≥ 3, with ends $x, y \notin B_1$ and with interior in B_1. Since both x and y have nonneighbours in the interior of Q it follows that $x, y \notin B$; and since they are adjacent they both belong to A_i for some i. But then $(i, 1)$ is an antipath pair. This proves (1).

For a subset X of $V(G)$, we denote the size of the largest clique in X by $\kappa(X)$. Let $\kappa(B_1) = s$, and $\kappa(A \cup B) = t$. Since G is minimum imperfect it cannot be t-coloured.

(2) *For $1 \leq i \leq m$ there is a subset $C_i \subseteq A_i$ so that $\kappa(C_i \cup B_1) = s$ and*

$$\kappa((A_i \setminus C_i) \cup (B \setminus B_1)) \leq t - s.$$

For let $H = G'|(B \cup A_i \cup \{z\})$; then H is Berge, by (1). Now by [?], there are at least two vertices of G not in H (all the vertices in $A \setminus A_i$), and since H has only one new vertex it follows that $|V(H)| < |V(G)|$. From the minimality of G we deduce that H is perfect. Now a theorem of Lovasz [?] shows that replicating a vertex of a perfect graph makes another perfect graph; so if we replace z by a set Z of $t - s$ vertices all complete to B_1 and to each other, and with no other neighbours in $A_i \cup B$, then the graph we make is perfect. From the construction, the largest clique in this graph has size $\leq t$, and so it is t-colourable. Since Z is a clique of size $t - s$, we may assume that colours $1, \ldots, s$ do not occur in Z, and colours $s + 1, \ldots, t$ do. Since B_1 is complete to Z, colours $s + 1, \ldots, t$ do not occur in B_1, and so only colours $1, \ldots, s$ occur in B_1; and since $\kappa(B_1) = s$, all these colours do occur in B_1. Since B_1 is complete to $B \setminus B_1$, none of colours $1, \ldots, s$ occur in $B \setminus B_1$. Let C_i be the set of vertices $v \in A_i$ with colours $1, \ldots, s$. Then $C_i \cup B_1$ has been coloured using only s colours, and so $\kappa(C_i \cup B_1) = s$; and the remainder of $H \setminus z$ has been coloured using only colours $s + 1, \ldots, t$, and so

$$\kappa((A_i \setminus C_i) \cup (B \setminus B_1)) \leq t - s.$$

This proves (2).

Now let $C = C_1 \cup \cdots \cup C_m$ and $D = V(G) \setminus C$. Since there are no edges between different A_i's, it follows from (2) that $\kappa(C) = s$, and similarly $\kappa(D) \leq t - s$. Since $|C|, |D| < |V(G)$ it follows that $G|C, G|D$ are both perfect; so they are s-colourable and $(t - s)$-colourable, respectively. But then G is t-colourable, a contradiction. This proves 4.8. ∎

4.9 *Let G be a minimum imperfect graph. Then G admits no even skew partition, and consequently no skew partition of G is loose.*

Proof. The first claim follows from 4.8, and the second from 4.2. This proves 4.9. ∎

At this stage we are not ready to prove that G admits no skew partition at all; that will be shown later in the paper. But let us sketch the route. We will need three special graphs:

- A *prism* means a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, where each P_i has ends a_i, b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(P_i)$ and $V(P_j)$ are a_ia_j and b_ib_j. The prism is *long* if at least one of the three paths has length > 1.

- A *double diamond* means the graph with eight vertices $a_1, \dots, a_4, b_1, \dots, b_4$ in which every two a_i's are adjacent except a_3a_4, every two b_i's are adjacent except b_3b_4, and a_ib_i is an edge for $1 \le i \le 4$.
- The third graph is just $L(K_{3,3} \setminus e)$, the line graph of the graph obtained from $L(K_{3,3})$ by deleting one edge.

Note that the second and third graphs in this list are isomorphic to their complements. We shall eventually prove that every Berge graph containing as an induced subgraph either a long prism or a double diamond or $L(K_{3,3} \setminus e)$ must satisfy the conclusion of 1.2, and consequently cannot be a minimum imperfect graph (and therefore nor is its complement). So once that is established, it will follow from the next theorem that no minimum imperfect graph admits a skew partition.

4.10 *If G is Begre, and admits a skew partition, then either G admits an even skew partition, or one of $G, \overline{G}$ contains as an induced subgraph either a long prism, or a double diamond, or $L(K_{3,3} \setminus e)$.*

Proof. Let (A, B) be a skew partition in G, which is not loose by 4.2. Let $A_1, \dots, A_m, B_1, \dots, B_n$ be as before. Suppose first that for some path pair (i, j) there is an odd path P of length ≥ 5 with ends in B_j and with interior in A_i; and we may assume $i = j = 1$. Let the vertices of P be $p_1, p_2, \dots, p_n$ in order. Now the ends of P are B_2-complete, and its internal vertices are not, since the skew partition is not loose; so by 2.1, B_2 contains a leap x, y, where x is adjacent to p_2. But then the subgraph induced on $V(P) \cup \{x, y\}$ is a long prism, as required. So we may assume that no such path has length ≥ 5; and similarly no odd antipath with ends in some A_i and interior in some B_j has length ≥ 5. So every path pair is also an antipath pair and vice versa (because all the corresponding odd paths and antipaths have length 3 and so each is both a path and an antipath). We may therefore assume that $(1, 1)$ is a path pair, and that there exist nonadjacent $b_1, b_1' \in B_1$ and adjacent $a_1, a_1' \in A_1$ so that b_1-a_1-a_1'-b_1' is a path. Since the skew partition is not loose, a_1, a_1' both have non-neighbours in B_2, and hence are joined by an antipath with interior in B_2; this antipath is odd, since its union with b_1, b_1' induces an antihole, and since all such antipaths have length 3 it follows that there exist nonadjacent $b_2, b_2' \in B_2$ so that b_2-a_1-a_1'-b_2' is a path. Now b_1, b_1' both have neighbours in A_2, since the skew partition is not loose, and hence are joined by a path with interior in A_2, and it is odd as usual, and hence has length 3; so there exist adjacent $a_2, a_2' \in A_2$ so that b_1-a_2-a_2'-b_1' is a path. Since b_2-b_1-a_2-a_2'-b_1'-b_2 is not an odd hole, b_2 is adjacent to one of a_2, a_2', and similarly so is b_2'. But b_2, b_2' have no common neighbour in A_2, for if $v \in A_2$ were adjacent to them both then v-b_2-a_1-a_1'-b_2'-v would be an odd hole. So there are exactly two edges between $\{a_2, a_2'\}$ and $\{b_2, b_2'\}$, forming an induced 2-edge matching. There are two possible pairings; in one case the subgraph induced on these eight vertices is a double diamond, and in the other it is $L(K_{3,3} \setminus e)$. This proves 4.10. ∎

5 Small attachments to a line graph

We come now to the first of the major steps of the proof. Suppose that G is Berge, and contains as an induced subgraph a substantial line graph $L(H)$. Then in general, G itself can only be basic by being a line graph, so 1.2 would imply that either G is a line graph, or it has a decomposition in accordance with 1.2. Proving a result of this kind is our first main goal, but exactly how it goes depends on what we mean by "substantial". To make the theorem as powerful as possible, we need to weaken what we mean by "substantial" as much as we can; but when $L(H)$ gets very small, all sorts of bad things start to happen. One is that the theorem is not true any more. For instance, when $H = K_{3,3}$ or $K_{3,3} \setminus e$, then $L(H)$ is not only a line graph but also the complement of a line graph (indeed, it is isomorphic to its own complement). So $L(H)$ can live happily inside bigger graphs that are complements of line graphs, without inducing any kind of decomposition. The best we can hope for, when $L(H)$ is so small, is therefore to prove that either G is a line graph or the complement of a line graph, or has a decomposition of the kind we like. This works for $L(K_{3,3})$, but for $L(K_{3,3} \setminus e)$ the situation is even worse, because this graph is basic in *three* ways - it is a line graph, the complement of a line graph, and a bicograph. So for Berge graphs that contain $L(K_{3,3} \setminus e)$, the best we can hope is that either G

is a line graph or the complement of one or a bicograph, or it has a decomposition. And that turns out to be true, but it also explains why the small cases will be something of a headache, as the reader will see.

Our route through these complications is as follows. If H is a subdivision of K_4, we say it is "squared" if there is some cycle C in K_4 of length 4 such that the corresponding cycle of H still has length 4 (so the four edges of C were not subdivided at all in producing H). First we assume that G contains some $L(H)$ where H is a subdivision of a 3-connected graph, such that if H is a subdivision of K_4 then it is not squared. Then everything works properly - we can show that either G is a line graph, or G has a decomposition, or $H = K_{3,3}$ and $\overline{G}$ is a line graph. Next we shall show that if G does not contain any such $L(H)$, and it does contain some $L(H)$ where H is a squared subdivision of K_4, then either G is a bicograph or it has a decomposition. The third step is, now assume that neither G nor its complement contains the line graph of any bipartite subdivision of K_4, and that G does contain a long prism (even or odd, though the odd case is much more difficult); then it has a decomposition (except for one graph which is basic).

The goal of the next few sections is therefore to prove the following two theorems (the bicographs and long prisms come later).

5.1 *Let G be Berge, and assume it contains an induced subgraph isomorphic to the line graph of a nonsquared bipartite subdivision of K_4. Then either G is a line graph, or G admits a 2-join, or G admits an even skew partition.*

5.2 *Let G be Berge, and assume it contains an induced subgraph isomorphic to $L(K3,3)$. Then either one of $G, \overline{G}$ is a line graph, or one of $G, \overline{G}$ admits a 2-join, or G admits an even skew partition.*

The proof is roughly as follows. We choose a 3-connected graph J, as large as possible so that G contains $L(H)$ for some bipartite subdivision H of J (and when $H = K_{3,3}$, we also assume that passing to the complement will not give us a better choice of J). Now we investigate how the remainder of G can attach onto $L(H)$. The edges of J correspond to edge-disjoint paths of H, which in turn become vertex-disjoint paths of $L(H)$, which we call "rungs" (we will do the definitions properly later). One thing we find is that the remainder of G can contain alternative rungs - paths that could replace one of the rungs in $L(H)$ to give a new $L(H')$, for some other bipartite subdivision H' of the same graph J. We find it advantageous to assemble all these alternative rungs in one "strip", for each edge of J, and to maximize the union of these strips (being careful that there are no unexpected edges of G between strips). Each strip corresponds to an edge of J, and runs between two sets of vertices (called "potatoes" for now) that correspond to vertices of J. Let the union of the strips be Z say. Again we ask, how does the remainder of G attach onto this "generalized line graph" Z? This turns out to be quite pretty. There are only two kinds of vertices in the remainder of G, vertices with very few neighbours in Z, and vertices with a lot of neighbours. For the first kind, all their neighbours lie either in one of the strips, or in one of the potatoes; and we can show that for any connected set of these "small" vertices, the union of their neighbours in Z has the same property (they all lie in one strip or in one potato). For the second kind of vertex, they have so many neighbours in Z that all their *non-neighbours* in any one potato lie inside one strip incident with the potato; and the same is true for the union of the nonneighbours of any co-connected set of these "big" vertices. In other words, every co-connected set of these big vertices has a great many common neighbours in Z, so many that they separate all the strips from one another, and that is where we find skew partitions. If there are no big vertices, then we find that G either admits a 2-join, or G is a line graph.

First, we assume that G contains $L(H)$, and we shall study how the remaining vertices of G attach to $L(H)$. Any such vertex has a set of neighbours in $V(L(H))$, that we want to investigate; but this set is more conveniently thought of as a subset of $E(H)$, and we begin with some lemmas about subsets of edges of a graph H. (Our lemmas also apply to the common neighbours of a co-connected set.) Our goal in this section is to examine how individual vertices attach to $L(H)$, and how connected sets of small vertices attach. In the next section we think about co-connected sets of big vertices.

In this section and the next few, we have to pay for our convention that "path" means "induced path", because here we need paths in the conventional sense. So to compound the confusion, let us use a different word for them. A *track* P is a non-null connected graph in which every vertex has degree ≤ 2 ; and its *length* is the number of edges in it. (Its ends and internal vertices are defined in the natural way.) A *track* in a graph H means a subgraph of H (not necessarily induced) which is a track. Note that there is a correspondence between the tracks (with at least one edge) in a graph H and the paths in $L(H)$; the edge-set of a track becomes the vertex-set of a path, and vice versa. And two tracks are vertex-disjoint if and only if the corresponding paths are vertex-disjoint and there is no edge of $L(H)$ between them. However, *the parity changes*; a track in H and the corresponding path in $L(H)$ have lengths of opposite parity.

A *branch-vertex* of a graph H means a vertex with degree ≥ 3; and a *branch* of H means a maximal track P in H such that no internal vertex of P is a branch-vertex. *Subdividing* an edge uv means deleting the edge uv, adding a new vertex w, and adding two new edges uw and wv. Starting with a graph J, the effect of repeatedly subdividing edges is to replace each edge of J by a track joining the same pair of vertices, where these tracks are disjoint except for their ends. We call the graph we obtain a *subdivision* of J. Note that $V(J) \subseteq V(H)$. Let J be a 3-connected graph. (We use the convention that a k-connected graph must have $> k$ vertices.) If H is a subdivision of J then $V(J)$ is the set of branch-vertices of H, and the branches of H are in 1-1 correspondence with the edges of J. We say H is *cyclically 3-connected* if it is a subdivision of some 3-connected graph J.

Let us mention in passing that the second result 5.2 above can be reformulated in the following, perhaps more informative, way:

5.3 *Let G be Berge, and assume it contains an induced subgraph isomorphic to $L(K3,3)$. Then either:*

- *$G = L(K3,3)$, or*
- *one of $G, \overline{G}$ contains an induced subgraph isomorphic to the line graph of a nonsquared bipartite subdivision of K_4, or*
- *one of $G, \overline{G}$ admits a 2-join, or G admits an even skew partition.*

Proof. of 5.3, assuming 5.2.
Since we shall not need the result, we merely sketch the proof. From 5.2 we may assume that G is a line graph, $G = L(H)$ say. If H is not cyclically 3-connected, then $G = L(H)$ admits a 2-join and we are done, so we assume that G is cyclically 3-connected. It follows easily that H is bipartite, and we may assume that every subgraph of H isomorphic to a subdivision of K_4 is squared. But the only cyclically 3-connected graph with this property that is not itself a subdivision of K_4 is $K_{3,3}$, and so $G = L(K_{3,3})$. This proves 5.3. ∎

We observe:

5.4 *Let H be cyclically 3-connected, and let C, D be subgraphs with $C \cup D = H$, $|V(C \cap D)| \leq 2$, and $V(C), V(D) \neq V(H)$. Then one of C, D is contained in a branch of H.*

The proof is clear.

5.5 *Let c_1, c_2 be nonadjacent vertices of a graph H, so that $H \setminus \{c_1, c_2\}$ is connected. For $i = 1, 2$, let the edges incident with c_i be partitioned into two sets A_i, B_i, where A_1, A_2 are both nonempty and at least one of B_1, B_2 is nonempty. Assume that for every edge $uv \in A_1 \cup A_2$, $H \setminus \{u, v\}$ is connected, and that no vertex of $V(H)$ is incident with all edges in $A_1 \cup A_2$. Then one of the following holds:*

1. *there is a track in H with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2, or*
2. *there is a track in H with first edge in A_2, second edge in B_2 (and hence second vertex c_2), last vertex c_1 and last edge in A_1.*

Proof. For $i = 1, 2$ let X_i be the set of ends (different from c_i) of edges in A_i, and define Y_i similarly for B_i. So by hypothesis, $|X_1 \cup X_2| >= 2$, and we may assume Y_1 is nonempty. Choose $x_1 \in X_1$ so that $X_2 \not\subseteq \{x_1\}$ (this is possible since $|X_1 \cup X_2| >= 2$). Both Y_1 and X_2 meet the connected graph $H \setminus \{c_1, x_1\}$, and so there is a track in $H \setminus \{c_1, x_1\}$ from Y_1 to $X_2 \cup Y_2$, say P, with vertices $p_1, \ldots, p_n$ say. We may assume that $p_1 \in Y_1$, and no other p_i is in Y_1; and $p_n \in X_2 \cup Y_2$, and no other p_i is in $X_2 \cup Y_2$. In particular it follows that $c_2 \notin V(P)$. Since $x_1 \notin V(P)$ we may assume that $p_n \notin X_2$ (for otherwise the theorem holds), so $p_n \in Y_2$. If any vertex of X_1 is in P then again the theorem holds (since X_2 is nonempty and none of its vertices are in P), so we may assume that P is disjoint from $X_1 \cup X_2$. Since $H \setminus \{c_1, c_2\}$ is connected, there is a minimal track Q in $H \setminus \{c_1, c_2\}$ from $X_1 \cup X_2$ to $V(P)$, and we may assume that only its first vertex (q say) is in $X_1 \cup X_2$. If $q \in X_1 \setminus X_2$, choose $x \in X_2$; if $q \in X_2 \setminus X_1$ choose $x \in X_1$; and if $q \in X_1 \cap X_2$ choose $x \in X_1 \cup X_2$ different from q. Thus we may assume that $q \in X_1$ and there exists $x \in X_2$ different from q and hence not in Q. So $P \cup Q$ contains a path from q to B_2 not containing x, and hence the theorem holds. This proves 5.5. ▮

If v is a vertex of H, the set of edges of H incident with v is denoted by $\delta(v)$ or $\delta_H(v)$. Let H be bipartite and cyclically 3-connected, and let X be some set. We say that X *saturates* $L(H)$ if for every branchvertex v of H, at most one edge of $\delta_H(v)$ is not in X (or equivalently, for every K_3 subgraph of $L(H)$, at least two of its vertices are in X). When H is connected and bipartite, we speak of vertices having the same or different *biparity* depending whether every track between them is even or odd respectively. Two edges of G are *disjoint* if they have no end in common.

5.6 *Let H be bipartite and cyclically 3-connected. Let $X \subseteq E(H)$, satisfying:*
(a) for every track P of H of length $>= 4$ and even, with both end-edges in X and with no internal edge in X, every edge in X has an end in the interior of P
(b) There do not exist three tracks of H with an end (b say) in common and otherwise vertex-disjoint, such that each contains an edge in X, and at least two of the three edges of the tracks incident with b do not belong to X.
Then either:

1. *X saturates $L(H)$, or*
2. *there is a branch B of H so that every edge in X has an end in B, or*
3. *$|X| = 2$ and there is a track P in H of even length ≥ 4 with both end-edges in X and no internal edge in X, such that there is a branchvertex of H in P not incident with either end-edge of P, or*
4. *$|X| = 4$, and the edges in X form a 4-cycle whose four vertices are all branch-vertices, or*
5. *there are two vertices c_1, c_2 of H, of different biparity and not in the same branch of H, so that $X = \delta(c_1) \cup \delta(c_2)$.*

Proof.

(1) *There do not exist a connected subgraph T of $H \setminus X$ and three mutually disjoint edges $x_1, x_2, x_3 \in X$ so that each x_i has at least one end in T.*

For suppose such T, x_1, x_2, x_3 exist. We may assume T is a maximal connected subgraph of $H \setminus X$. Let x_i have ends a_i, b_i ($i = 1, 2, 3$), where a_1, a_2, a_3 have the same biparity. Make a graph K with vertex set $a_1, a_2, a_3, b_1, b_2, b_3$, where we say two vertices of K are adjacent if there is a track in T joining them not using any other vertex of K. Since T is connected and meets all of x_1, x_2, x_3 it follows that there is a component of K containing an end of each of these three edges. Now if a_1a_2 is an edge of K, then the corresponding track in T is even, and hypothesis (a) is contradicted. So the only possible edges in K join some a_i to some b_j. Also, if say a_3 is adjacent in K to both b_1 and b_2, then hypothesis (b) is contradicted. Since there is a component of K containing an end of each of x_1, x_2, x_3, we may assume that $a_1b_3, b_2a_3, a_3b_3 \in E(K)$, and the only other possible edges of K are a_1b_1, a_2b_2, a_2b_1. In particular, there are no more edges of K incident with a_3 or b_3. Let the tracks in T corresponding to $a_1b_3, b_2a_3, a_3b_3 \in E(K)$ be P_1, P_2, P_3 respectively. Since P_3 joins the adjacent vertices a_3, b_3 and does not use the edge x_3, it follows that P_3 has nonempty interior. Choose a maximal connected subgraph S of T including the interior of P_3 and not containing either of a_3, b_3. Since there are no more edges of K incident with a_3 or b_3, it follows that none of a_1, b_1, a_2, b_2 is in $V(S)$, and therefore S is vertex-disjoint from P_1 and P_2 as well. Consequently the only edges of T between $V(S) \cup \{a_3, b_3\}$ and the remainder of H are incident with a_3 or b_3. Since H is cyclically 3-connected and a_3, b_3 are adjacent, it follows that $H \setminus \{a_3, b_3\}$ is connected, and therefore there is an edge sv of H such that $s \in V(S)$ and $v \in V(H) \setminus (V(S) \cup \{a_3, b_3\})$. Since T is maximal, it follows that $sv \in X$; and from the symmetry we may assume $v \notin \{a_1, b_1\}$. Choose a minimal track in S between s and the interior of P_3; then it can be extended via a subpath of P_3 and via sv to become a track P_4 in H, of length ≥ 2, from v to a_3, using none of a_1, b_1, b_3, and with only its first edge in X. But then the tracks P_1, P_4, and the one-edge track made by x_3, violate hypotheses (b). This proves (1).

Now we may assume that X does not saturate $L(H)$, and so there is a branch-vertex of H incident with ≥ 2 edges not in X. Hence there is a connected subgraph A of $H \setminus X$, containing a branch-vertex and at least two edges incident with it. Choose such a subgraph A maximal. It follows that A is not contained in any branch of H. By (1), there is no 3-edge matching among the edges in X that meet A; and since this set of edges makes a bipartite subgraph, it follows from Konig's theorem that there are two vertices c_1, c_2 so that every edge in X with an end in A is incident with one of c_1, c_2.

(2) *We may assume that every edge in X is incident with one of c_1, c_2.*

For suppose not; then there is an edge in X vertex-disjoint from $V(A) \cup \{c_1, c_2\}$. $B = H \setminus V(A)$. From the maximality of A, every edge of H between $V(A)$ and $V(B)$ belongs to X and therefore is incident with one of c_1, c_2, and so there are two subgraphs C, D of H with $V(C) = V(A) \cup \{c_1, c_2\}$, $C \cup D = H$, $V(C \cap D) = \{c_1, c_2\}$, $A \subseteq C$ and $B \subseteq D$. In particular, $V(C), V(D) \neq V(G)$. Since H is cyclically 3-connected it follows from 5.4 that one of C,D is contained in a branch of H. Now C is not, because it contains A (and we already saw that A is not contained in a branch); so D is contained in a branch. In particular, this branch contains c_1 and c_2, and also meets all edges in X with no end in $V(A)$, and therefore meets all edges in X; but then statement 2 of the theorem holds. This proves (2).

(3) *We may assume that $|X| \geq 3$.*

For if $|X| \leq 1$ then statement 2 of the theorem holds; suppose $|X| = 2$, and $X = \{a_1b_1, a_2b_2\}$ say, where a_1, a_2 have the same biparity. If any branch of H meets both these edges then the theorem holds, so we may assume not, and in particular these four vertices are distinct. Since H is cyclically 3-connected it follows from 5.4 that $H \setminus \{b_1, b_2\}$ is connected, so there is a track P of H between b_1, b_2, with even length ≥ 4, with first edge b_1a_1 and last edge a_2b_2. Since a_1, a_2 do not belong to the same branch of H, there is a branchvertex of H in P not incident with either end-edge of P; and so statement 3 of the theorem holds. This proves (3).

We may assume that c_1, c_2 do not belong to the same branch, for otherwise statement 2 of the theorem holds; and consequently c_1, c_2 are nonadjacent, and $H \setminus \{c_1, c_2\}$ is connected, by 5.4.

Now assume that c_1, c_2 have the same biparity. Since $|X| \geq 3$, we may assume there are at least two edges in X incident with c_1, say c_1a_1, c_1a_2. If there is an edge c_2a_3 incident with c_2 where $a_3 \neq a_1, a_2$, take a minimal track in $H \setminus \{c_1, c_2\}$ between a_3 and one of a_1, a_2; it violates hypothesis (a) of the theorem. So the only possible edges in X incident with c_2 are c_2a_1 and c_2a_2. If both are present, then by exchanging c_1 and c_2 it follows that there are no more edges incident with c_1, and so either statement 2 or 4 of the theorem holds. If exactly one is present, say c_2a_1, then the branch of H containing c_1a_1 satisfies statment 2 of the theorem. If none are present, then any branch containing c_1 satisfies statement 2. This completes the proof if c_1, c_2 have the same biparity.

Now assume that c_1, c_2 have different biparity. For $i = 1, 2$ let $A_i = \delta(c_i) \cap X$, and let $B_i = \delta(c_i) \setminus A_i$. We may assume that A_1, A_2 are nonempty. Since c_1, c_2 have different biparity, no vertex is incident with all the edges in $A_1 \cup A_2$. If both B_1, B_2 are empty, then statement 5 of the theorem holds, so we may assume at least one of them is nonempty. By 5.5, we may assume there is a track in H with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2. By choosing such a track as short as possible, it follows that only one edge in A_2 meets its interior. By hypothesis (b), all edges in X meet its interior, and hence in particular $|A_2| = 1$. But then we can replace c_2 by the other end of the edge in A_2, and will be in the "same parity" case that we have already done. This proves 5.6. ∎

Now we apply what we just proved to the neighbours of a single vertex. If G,J are graphs, we say that J *appears* in G if there is a bipartite subdivision H of J so that $L(H)$ is isomorphic to an induced subgraph of G. We call $L(H)$ an *appearance* of J in G. Note that if $L(H)$ is isomorphic to some induced subgraph K of G, there is another subdivision H' isomorphic to H, made from H by replacing each edge of H by the corresponding vertex of K; and now $L(H') = K$ (rather than just being isomorphic to it). So whenever it is convenient we shall assume that the isomorphism between $L(H)$ and K is just equality, without further explanation. Note in particular that $E(H) = V(K)$, and so some vertices of G are edges of H.

An appearance $L(H)$ of J in G is *degenerate* if either $J = K_4$ and H is a squared subdivision of K_4, or $J = H = K_{3,3}$, and otherwise it is *nondegenerate*. So all appearances of any graph $J \neq K_4, K_{3,3}$ are nondegenerate. If J is 3-connected, we say a graph J' is a *J-enlargement* if J' is 3-connected, and has a proper subgraph which is isomorphic to a subdivision of J. We remind the reader that we are currently trying to prove two statements, 5.1 and 5.2.

To do so we shall prove the following:

5.7 *Let G be Berge. Let J be a 3-connected graph, such that either:*

- *there is a nondegenerate appearance of J in G, and there is no J-enlargement with a nondegenerate appearance in G, or*
- *$J = K_{3,3}$, there is an appearance of J in G, and no J-enlargement appears in either G or $\overline{G}$.*

Then either G is a line graph, or G admits a 2-join or an even skew partition.

The proof of this will take several sections; but let us see now that 5.7 implies 5.1 and 5.2.

Proof. of 5.1, assuming 5.7.

Let G be Berge, and assume it contains a subgraph isomorphic to the line graph of a nonsquared bipartite subdivision of K_4, that is, there is a nondegenerate appearance of K_4 in G. Choose a 3-connected graph J with maximal (under J-enlargement) so that there is a nondegenerate appearance of J in G; then the hypotheses of 5.7 are satisfied, and the claim follows from 5.7. This proves 5.1. ∎

Proof. of 5.2, assuming 5.7.

Let G be Berge, and assume it contains a subgraph isomorphic to $L(K3,3)$. Choose a 3-connected graph J maximal (in the sense of J-enlargement) so that there is an appearance of J in G. If there is a nondegenerate appearance of J in G, then the hypotheses of 5.7 hold, and the claim follows from 5.7. So we may assume that every appearance of J in G is degenerate, and in particular $J = K_{3,3}$ or K_4. Since there is an appearance of $K_{3,3}$ in G and $K_{3,3}$ is a K_4-enlargement it follows that $J = K_{3,3}$. If there is a J-enlargement which appears in $\overline{G}$, choose it maximal; then the claim follows by applying 5.7 to $\overline{G}$. So we may assume that there is no J-enlargement that appears in $\overline{G}$. But then again the claim follows from 5.7. This proves 5.2. ∎

5.8 *Let G be Berge. Let J be a 3-connected graph, and let $L(H)$ be an appearance of J in G. Let $y \in V(G) \setminus V(L(H))$, and let X be the set of vertices of $L(H)$ that are adjacent to y in G. Then either:*

1. *X saturates $L(H)$, or*
2. *there is a branch-vertex v of H with $X \subseteq \delta_H(v)$, or*
3. *there is a branch B of H with $X \subseteq E(B)$, or*

4. there is a branch B of H with ends b_1, b_2 say, so that $X \setminus E(B) = \delta_H(b_1) \setminus E(B)$, or

5. there is a branch B of H of odd length with ends b_1, b_2 say, so that $X \setminus E(B) = (\delta_H(b_1) \cup \delta_H(b_2)) \setminus E(B)$, or

6. there are two vertices c_1, c_2 of H, of different biparity and not in the same branch of H, so that $X = \delta(c_1) \cup \delta(c_2)$.

In particular, either statements 1 or 6 hold, or there are at most two branch-vertices of H incident with more than one edge in X; and exactly two only if statement 5 holds.

Proof.

The second assertion (the final sentence) follows from the first, because if statements 2,3 or 4 hold then there is at most one branch-vertex incident with more than one edge in X; while if B, b_1, b_2 are as in statement 5, then since B is odd, it follows that b_1, b_2 have no common neighbour, and so no branch-vertex different from b_1, b_2 is incident with more than one edge in X. So it remains to prove the first assertion.

(1) *Every track of H with both end-edges in X and no internal edge in X has length odd or 2.*

For since X is the set of neighbours in $L(H)$ of a single vertex, it follows that every path in $L(H)$ with both ends in X and with no interior vertex in X has length even or 1, and this proves (1).

In particular, hypotheses (a) and (b) of 5.6 are satisfied. Hence one of statements 1-5 of 5.6 applies. If 5.6.1 applies then statement 1 of the theorem holds. 5.6.3 and 5.6.4 cannot apply, by (1), and if 5.6.5 holds then statement 6 of the theorem holds. So we may assume that 5.6.2 applies. Let C be a branch of H meeting all the edges in X, and let c_1, c_2 be the ends of C. For $i = 1, 2$ let A_i be the set of edges in X that are incident with c_i and are not in C; and let B_i be the set of edges in $\delta(c_i)$ that are not in B and not in X. If one of A_1, B_1 is empty and also A_2 is empty, then statement 3 or 4 of the theorem holds as required. Suppose that if B_1, B_2 are both empty. Choose $a_1 \in A_1$ and $a_2 \in A_2$, disjoint, and let T be a track of H from c_1 to c_2 with end-edges a_1 and a_2. Then no internal edge of T is in X, and its end-edges are in X, and so it cannot be even; so c_1,c_2 have different biparity, and therefore C is odd. But then statement 5 of the theorem holds. So we may assume that A_1, B_1 are both nonempty.

(2) *We may assume A_2 is nonempty.*

For assume A_2 is empty. If c_1 meets every edges in X then statement 4 of the theorem holds, so we may assume not; and hence some edge of $C \setminus c_1$ is in X. Let P be a minimal subtrack of C containing c_2 and some edge in X. Choose an edge $c_1a_1 \in A_1$. Since $H \setminus \{c_1, a_1\}$ is connected, there is a track Q from B_1 to c_2 in this graph, and we can extend it to a track from a_1 to c_2 with second vertex c_1, first edge in A_1 and second edge in B_1. By combining the latter with P we deduce from (1) that the lengths of P and Q have the same parity. On the other hand, there is a track R in $H \setminus c_1$ between a_1 and c_2; and by extending it via a_1c_1, combining it with P and applying (1) we deduce that the lengths of R and P have opposite parity. But since H is bipartite, the lengths of Q and R have the same parity, a contradiction. This proves (2).

(3) *We may assume C has even length.*

For assume it is odd, and so c_1, c_2 have different biparity. Consequently no vertex is incident with all the edges in $A_1 \cup A_2$, and hence we can apply 5.5 to the graph H' obtained from H by deleting the internal vertices and edges of C. We deduce that (without loss of generality) there is a track in H' with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2. But since c_1, c_2 have different biparity, this track has even length ≥ 4, contrary to (1). This proves (3).

Assume next that for $i = 1, 2$ there are edges $c_ia_i \in A_i$ disjoint from each other. There is a track in $H \setminus \{c_1, c_2\}$ between a_1 and a_2, and it is even since c_1, c_2 have the same biparity, and it has length ≥ 2. By extending it via the edges c_1a_1 and c_2a_2 we obtain a track violating (4). So there do not exist such edges. Hence there is a vertex $a \in V(H)$ so that $A_i = \{c_ia\}$ for $i = 1, 2$. Now there is only one branch of H incident with c_1 and c_2, since J is simple, so a is not in the interior of a branch, and so it is a branch-vertex. Choose a branch-vertex b of H different from a, c_1, c_2, and choose three paths P_1, P_2, P_3 between b and a, c_1, c_2 respectively, pairwise disjoint except for b. So P_1 and P_2 have lengths of the same parity, and P_3 has length of different parity. We may assume that there is an edge in X not incident with a, for otherwise statement 3 of the theorem holds, so for $i = 1, 2$ there is a minimal subtrack Q_i of B containing c_i and an edge in X. If $Q_1 = C$ then (since C has even length) $P_1 \cup P_2$ is the interior of an even track with end-edges in X and no internal edges in X, contrary to (1). So c_2 is not a vertex of Q_1, and similarly c_1 is not in Q_2. From the track Q_1-c_1-P_1-b-P_2-c_2-a and (1) it follows that Q_1 is even; and from the track Q_1-c_1-P_1-b-P_3-a-c_2 and (1) it follows that Q_1 is odd, a contradiction. This proves 5.8. ∎

We recall that H is a subdivision of J, and $L(H)$ is an induced subgraph of G. For each vertex v of J, we denote the set of edges of H incident with v by N_v, and for each edge uv of J, we denote the set of edges of the branch of H between u and v by R_{uv}. So each N_v and each R_{uv} is a subset of $V(L(H))$. We say a subset X of $V(L(H))$ is *local* (with respect to $L(H)$) if either $X \subseteq N_v$ for some vertex v of J, or $X \subseteq R_{uv}$ for some edge uv of J. In general, if K is an induced subgraph of G, and $F \subseteq V(G)$ is a connected set or subgraph disjoint from $V(K)$, a vertex in $V(K)$ is an *attachment* of F if it has a neighbour in F.

5.9 *Let G be Berge. Let J be a 3-connected graph, let $L(H)$ be an appearance of J in G, and let F be a connected set of vertices, disjoint from $V(L(H))$, such that the set of attachments of F in $L(H)$ is not local. Assume that for every $v \in F$ the set of neighbours of v in $L(H)$ does not saturate $L(H)$. Then there is a path P of G with $V(P) \subseteq F$ and with ends p_1 and p_2, so that either:*

1. *there are vertices c_1, c_2 of H, not in the same branch of H, so that for $i = 1, 2$ p_i is complete in G to $\delta_H(c_i)$, and there are no other edges between $V(P)$ and $V(L(H))$, or*
2. *there is an edge b_1b_2 of J, with the following properties (for $i = 1, 2$, r_i denotes the unique vertex in $N_{b_i} \cap R_{b_1b_2}$):*
   - *all attachments of P in $L(H)$ belong to $N_{b_1} \cup N_{b_2} \cup R_{b_1b_2}$, and*
   - *p_1 is adjacent in G to all vertices in $N_{b_1} \setminus r_1$, and no other vertex of P has any neighbours in $N_{b_1} \setminus r_1$, and*
   - *no vertex of P except p_2 has any neighbours in $N_{b_2} \cup R_{b_1b_2} \setminus r_1$, and either:*
     (a) p_2 has no neighbours in $N_{b_2} \setminus r_2$, and has a neighbour in $R_{b_1b_2} \setminus r_1$, or
     (b) p_2 is adjacent to all vertices in $N_{b_2} \setminus r_2$, and to no vertex of $R_{b_1b_2}$ except possibly r_2, and P has the same parity as $R_{b_1b_2}$, or
     (c) $p_1 = p_2$ is adjacent to all vertices in $N_{b_2} \setminus r_2$, and $R_{b_1b_2}$ is even.

Proof. We may assume F is minimal so that its set of attachments is not local. Let X be the set of attachments of F in $L(H)$. Suppose first that $|F| = 1$, $F = \{z\}$ say. Apply 5.8 to z. Now 5.8.1 is false since by hypothesis X does not saturate $L(H)$, and 5.8.2, and 5.8.3 are false since X is not local. So one of 5.8.4-6 holds, and the claim follows. Consequently we may assume that $|F| \geq 2$.

(1) *There exist two attachments x_1,x_2 of F so that $\{x_1, x_2\}$ is not local.*

For $X \subseteq E(H)$. If there exists $x_1 \in X$ not incident in H with a branch-vertex, and in some branch B, choose any $x_2 \in X$ not in B, then $\{x_1, x_2\}$ is not local. So we may assume that every edge in X is incident with a branch-vertex of H. Choose $x_1 \in X$, in some branch B_1 of H, and incident with a branch-vertex b_1. There exists $x_2 \in X$ not incident with b_1, and we may assume that $x_2 \in E(B_1)$, for otherwise $\{x_1, x_2\}$ is not local. Hence x_2 is incident with the other end b_2 say of B_1. There exists $x_3 \in X$ not belonging to $E(B)$, and it cannot share an end both with x_1 and with x_2, so we may assume x_3 is not incident with b_1. But then $\{x_1, x_3\}$ is not local, as required. This proves (1).

From the minimality of F, it follows that F is minimal such that x_1 and x_2 are both attachments of F, and so (since x_1 and x_2 are nonadjacent), F is the interior of a path with vertices $x_1, p_1, \dots, p_n, x_2$ in order. Let X_1 be the attachments in $L(H)$ of $F \setminus p_n$, and let X_2 be the attachments of $F \setminus p_1$. From the minimality of F, X_1 and X_2 are both local.

(2) *If there is an edge uv of J so that $X_1 \subseteq N_u$ and $X_2 \subseteq R_{uv}$ then the theorem holds.*

For let the ends of R_{uv} be r_1, r_2 where $r_1 \in N_u$. Since X is not local, it follows that p_1 has a neighbour in $N_u \setminus r_1$ and p_n has a neighbour in $R_{uv} \setminus r_1$. If p_1 is adjacent to every vertex in $N_u \setminus r_1$ then statement 2.a of the theorem holds, so we may assume p_1 has a neighbour s_1 and a nonneighbour s_2 in $N_u \setminus r_1$. Choose $w \in V(J)$ so that $s_1 \in R_{uw}$. Let Q be the path between r_2 and s_1 with interior in $F \cup R_{uv} \setminus r_1$. Now H is a subdivision of a 3-connected graph, so if we delete all edges of H incident with u except s_2, the graph we produce is still connected. Consequently there is a track of H from u to v with first edge s_2; and hence there is a path S_2 of $L(H)$ from s_2 to r_2, vertex-disjoint from $R_{uv} \cup N_u$ except for its ends. Indeed, if we delete from H both the vertex w and all edges incident with u except s_1, the graph remains connected; so there is a path S_1 of $L(H)$ between s_1 and r_2, vertex-disjoint from $R_{uv} \cup N_u \cup R_{uw} \cup N_w$ except for its ends. Now S_1 and S_2 have the same parityi since H is bipartite. Yet S_1 can be completed via r_2-Q-s_1 and S_2 can be completed via r_2-Q-s_1-s_2, a contradiction. This proves (2).

(3) *If there are nonadjacent vertices $v_1, v_2 \in V(J)$ so that $X_i \subseteq N_{v_i}$ for $i = 1, 2$, then the theorem holds.*

Let A_1 be the set of vertices in N_{v_1} adjacent to p_1, and $B_1 = N_{v_1} \setminus A_1$; and let A_2 be the set of vertices in N_{v_2} adjacent to p_n, and $B_2 = N_{v_2} \setminus A_2$. So $X = A_1 \cup A_2$. If both B_1 and B_2 are empty then statement 1 of the theorem holds, so we may assume that at least one of B_1,B_2 is nonempty. Certainly A_1 and A_2 are both nonempty, so there is a track in H from v_1 to v_2 with end-edges in A_1 and A_2 respectively. Hence there is a path S_1 in $L(H)$ from A_1 to A_2, vertex-disjoint from $N_{v_1} \cup N_{v_2}$ except for its ends. Since $X = A_1 \cup A_2$ is not local, there is no $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$. Hence we can apply 5.5, and we deduce (possibly after exchanging v_1 and v_2) that there is a path S_2 in $L(H)$ with first vertex in A_1, second vertex in B_1, last vertex in A_2, and otherwise disjoint from $N_{v_1} \cup N_{v_2}$. Since H is bipartite, S_1 and S_2 have opposite parity; but they can both be completed via F, a contradiction. This proves (3).

(4) *If there are adjacent vertices $v_1, v_2 \in V(J)$ so that $X_i \subseteq N_{v_i}$ for $i = 1, 2$, then the theorem holds.*

For $i = 1, 2$ let r_i be the end of $R_{v_1 v_2}$ in N_{v_i}. Let A_1 be the set of vertices in $N_{v_1} \setminus r_1$ adjacent to p_1, and $B_1 = N_{v_1} \setminus (A_1 \cup r_1)$; and define A_2, B_2 similarly. Then $X \subseteq A_1 \cup A_2 \cup \{r_1, r_2\}$. Now A_1 and A_2 are both nonempty. Suppose that both B_1 and B_2 are empty. Then there is a cycle in J of length ≥ 4 using the edge $v_1 v_2$, and so there is a path in $L(H)$ of length ≥ 2 from A_1 to A_2 with no internal vertex in $N_{v_1} \cup V(R_{v_1 v_2}) \cup N_{v_2}$. The union of this path with $R_{v_1 v_2}$ induces a hole, and so does its union with F, and therefore these two paths have lengths of the same parity. Consequently either statement 2.b or 2.c of the theorem holds. So we may assume that at least one of B_1, B_2 is nonempty. There is a path S_1 from A_1 to A_2 with no vertex in $N_{v_1} \cup N_{v_2} \cup R_{v_1 v_2}$ except for its ends. Suppose that there is no vertex $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$. Then we can apply 5.5 to the graph obtained from H by deleting the edges and internal vertices of the branch between v_1 and v_2. We deduce (possibly after exchanging v_1 and v_2) that there is a path S_2 of $L(H)$ with first vertex in A_1, second vertex in B_1, last vertex in A_2, and otherwise disjoint from $N_{v_1} \cup N_{v_2} \cup R_{v_1 v_2}$. Since H is bipartite, S_1 and S_2 have opposite parity; but they can both be completed via F, a contradiction. Consequently there is a vertex $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$. Since H is bipartite, it follows that $R_{v_1 v_2}$ has odd length, and in particular $r_1 \neq r_2$. Since $|N_{v_i} \cap N_w| \leq 1$ (since J is simple) it follows that $|A_i| = 1$, $A_i = \{a_i\}$ say, for $i = 1, 2$. Since X is not local it is not a subset of N_w and so there is a vertex of $R_{v_1 v_2}$ in X. Since $X_i \subseteq N_{v_i}$ for $i = 1, 2$, no internal vertex of $R_{v_1 v_2}$ is in X, so we may assume that $r_1 \in X$. Since $r_1 \notin N_{v_2}$ it follows that $r_1 \notin X_2$, and hence p_1 is the only vertex in F adjacent to r_1. Now the hole p_1-$\cdots$-p_n-a_2-a_1-p_1 is even, and so n is even. If we delete the vertex v_2 and the edge a_1 from H, what remains is still connected, and so contains a track from w to v_1. Hence there is a path T in $L(H)$ from some $a_3 \in N(w)$ to r_1, disjoint from $N_{v_2} \cup a_1$. But T can be completed to a hole via r_1-$R_{v_1v_2}$-r_2-a_2-a_3 and via r_1-p_1-$\cdots$-p_n-a_2-a_3, and these two completions have different parity, a contradiction. This proves (5).

(5) *If $X_1 \cap X_2$ is nonempty, and in particular if one of $p_2, \ldots, p_{n-1}$ has a neighbour in $L(H)$, then the theorem holds.*

For any neighbour in $L(H)$ of one of $p_2, \ldots, p_{n-1}$ belongs to $X_1 \cap X_2$, so assume $x \in X_1 \cap X_2$. Then $x \in R_{v_1v_2}$ for a unique edge v_1v_2 of J, and $x \in N_v$ for at most two $v \in V(J)$, namely v_1 and v_2. Since both X_1 and X_2 are local, each is a subset of one of $N_{v_1}, N_{v_2}, R_{v_1v_2}$, and they are not both subsets of the same one. So we may assume that $X_1 \subseteq N_{v_1}$. Hence either $X_2 \subseteq N_{v_2}$ or $X_2 \subseteq R_{v_1v_2}$, and therefore the theorem holds by (5) or (2). This proves (5).

(6) *If there is a vertex u and an edge v_1v_2 of J so that $X_1 \subseteq N_u$ and $X_2 \subseteq R_{v_1v_2}$ then the theorem holds.*

For by (2) we may assume u is different from v_1 and v_2. Choose a cycle C_1 of H using the branch between v_1 and v_2 and not using u; and choose a minimal track S in $H \setminus \{v_1, v_2\}$ between u and the interior of S. Let the ends of S be u and w say. Hence in $L(H)$ there are three vertex-disjoint paths, from N_{v_1}, N_{v_2},N_u respectively to N_w, and there are no edges between them except in the triangle T formed by their ends in N_w. If p_n has a unique neighbour (say r) in $R_{v_1v_2}$, then r can be linked onto the triangle T, contrary to 2.4. If p_n has two nonadjacent neighbours in $R_{v_1v_2}$, then p_n can be linked onto the triangle T, contrary to 2.4. So p_n has exactly two neighbours in $R_{v_1v_2}$, and they are adjacent. If p_1 is adjacent to all of N_u, then statement 1 of the theorem holds, so we may assume that p_1 has a neighbour and a non-neighbour in N_u. Let A be the neighbours of p_1 in N_u and $B = N_u \setminus A$. In H there is a cycle C_2 using the branch between v_1 and v_2, and using an edge in A and an edge in B. (To see this, divide u into two adjacent vertices, one incident with the edges in A and the other with those in B, and use Menger's theorem to deduce that there are two vertex-disjoint paths between these two vertices and $\{v_1, v_2\}$.) Hence in G, there is a path between N_{v_1} and N_{v_2} using a unique edge of $N(u)$, and that edge is between a vertex $a \in A$ say and some vertex in B. Hence a can be linked onto the triangle formed by p_n and its two neighbours in $R_{v_1v_2}$, a contradiction.

(7) *If there are edges u_1v_1 and u_2v_2 of J with $X_i \subseteq R_{u_iv_i}$ for $i = 1, 2$, then the theorem holds.*

For in this case it follows that the edges u_1v_1 and u_2v_2 are different, and hence we may assume that v_2 is different from u_1 and v_1, and v_1 is different from u_2 and v_2; possibly $u_1 = u_2$. If p_1 has exactly two neighbours in $R_{u_1v_1}$ and they are adjacent, and also p_n has exactly two neighbours in $R_{u_2v_2}$ and they are adjacent, then statement 1 of the theorem holds; so we may assume that p_1 has either only one neighbour, or two nonadjacent neighbours, in $R_{u_1v_1}$. There is a cycle in H using the branch between u_1 and v_1, and using u_2 and not v_2 (since $J \setminus v_2$ is 2-connected). There correspond two paths in $L(H)$, say P and Q, from N_{u_1} and N_{v_1} respectively to N_{u_2}, disjoint from each other, and there is a third path R say from p_1 to N_u via F and a subpath of $R_{u_2v_2}$. There are no edges between these paths except within the triangle T formed by their ends in N_u. If p_1 has only one neighbour $r \in R_{u_1v_1}$, then we may assume that r is in the interior of $R_{u_1v_1}$, by (6), and so r can be linked onto T, contrary to 2.4. If p_1 has two nonadjacent neighbours in $R_{u_1v_1}$, then p_1 can be linked onto T, again a contradiction. This proves (7).

But (2)-(7) cover all the possibilities for the local sets X_1 and X_2, and so this proves 5.9. ∎

6 Big attachments to a line graph

In this section we study co-connected sets of "big" vertices, and their common neighbours in $L(H)$. Conveniently we can again apply 5.6, because of the following.

6.1 *Let G be Berge, and let $L(H)$ be an appearance in G of a 3-connected graph J. let Y be a co-connected set of vertices in $V(G) \setminus V(L(H))$, and let X be the set of Y-complete vertices in $L(H)$. Then either:*

- *$J = K_{3,3}$ or K_4, and $L(H)$ is degenenerate, and there is a J-enlargement that appears in $\overline{G}$, or*
- *X satisfies hypotheses (a) and (b) of 5.6.*

Proof. That X satisfies hypothesis (a) of 5.6 is immediate from 2.2, so it remains to prove that if hypothesis (b) is false then the first alternative of the theorem holds. We begin with:

(1) *If $x_1, x_2, x_3 \subseteq X$ is a matching in H, and P is an even track of H with end-edges x_1 and x_2 and with no edge of X in its interior, then P has length 4 and its middle vertex is incident with x_3.*

Let the vertices of P be $p_1, \dots, p_n$; so $n \geq 4$ and is even, x_1 is p_1p_2, and x_2 is $p_{n-1}p_n$. Since x_1, x_2, x_3 is a matching, it follows that x_3 is not incident with any of p_1, p_2, p_{n-1}, p_n. By 2.2, x_3 in incident with some vertex of P, and hence shares an end with at least two internal edges of P. By 2.5, x_3 shares an end with one of the first two edges of P, and with one of the last two. It is therefore incident with p_2 and with p_{n-2}. Since H is bipartite and n is even, it follows that $p_2 = p_{n-2}$ and so $n = 4$. This proves (1).

Now assume that hypothesis (b) of 5.6 fails to hold; and we may therefore assume that Y is minimal such that it is co-connected and hypothesis (b) is false. Then there are three tracks T_1, T_2, T_3 of H with an end (v say) in common and otherwise vertex-disjoint, such that each contains an edge in X, and at least two of the three edges of the tracks incident with v do not belong to X. Let T_i be from a_i to v for $i = 1, 2, 3$. We may assume that the only edge of each T_i in X is the edge (x_i, say) incident with a_i. Now two of T_1, T_2, T_3 have lengths of the same parity, say T_1, T_2; and yet x_1, x_2, x_3 is a matching in H, since at most one of these edges is incident with v. Hence from (1), T_1 and T_2 both have length 2, and T_3 has length 1. For $i = 1, 2$, let u_i be the middle vertex of T_i, let e_i be the edge vu_i of H, and let A_i be the set of all $a \in V(H)$ so that $au_i \in E(H) \cap X$. Let C be the set of all $c \in V(H)$ so that $cv \in E(H) \cap X$. Since H is bipartite it follows that C is disjoint from $A_1 \cup A_2$. Let Q be an antipath of G between e_1 and e_2, with interior in Y. Since Q can be completed to an antihole via e_2-x_1-x_2-e_1 it follows that Q is odd. From the minimality of Y it follows that $Y = V(Q^*)$. Now $A_1 \cap A_2$ is empty; for if there exists $a \in A_1 \cap A_2$, then $au_1 = v_1$ and $au_2 = v_2$ are vertices of G in X, and Q could be completed to an odd antihole via e_2-v_1-x_3-v_2-e_1, a contradiction. So A_1, A_2, C are mutually disjoint. Note also that every edge of H in X is incident with one of v, u_1, u_2, by 2.2.

(2) *We may assume there is no track in $H \setminus \{u_1, u_2, v\}$ between A_1 and A_2.*

For assume T is a minimal such track. It is even (since all the vertices in $A_1 \cup A_2$ have the same biparity), and no internal vertex is in $A_1 \cup A_2$; and none of its edges are in X, since all edges in X are incident with one of v, u_1, u_2. Hence by (1) applied to the track formed by T, x_1 and x_2, it follows that T has length 2 and its middle vertex is c (for every $c \in C$.) So $C = \{c\}$, say, where ca_1 and ca_2 are edges. Also, every edge in X is incident with one of the vertices of T, and so for $i = 1, 2$, $|A_i| = 1$, and $A_i = \{a_i\}$. Suppose that $|V(H)| \geq 7$, and let Z be any component of $H \setminus \{v, u_1, u_2, a_1, a_2, c\}$. At most one of u_1, u_2, c has a neighbour in Z, for otherwise since they have the same biparity there would be an even track in H between with end-edges two of x_1, x_2, x_3 disjoint from the third, contrary to (1). Similarly at most one of a_1, a_2, v has neighbours in Z. Since H is cyclically 3-connected, there are two nonadjacent vertices among v, u_1, u_2, a_1, a_2, c with neighbours in Z, so from the symmetry we may assume that u_1 and a_2 are the two vertices of these six with neighbours in Z. Let S be a track between u_1 and a_2 with interior in Z; then c-v-u_1-S-a_2-u_2 is an even path with end-edges in X and no internal edge in X, of length ≥ 6, contrary to (1). So there is no such Z, and hence $|V(H)| = 6$, and the only other possible edges of H are a_1u_2 and a_2u_1. Since H is cyclically 3-connected, at least one of these is present, so we may assume a_1u_2 is an edge. Thus $J = K_{3,3}$ or K_4, and $L(H)$ is degenerate. We claim that also there is an appearance in $\overline{G}$ of a 3-connected graph with more edges than J. For recall that $Y = V(Q^*)$, where Q is an odd antipath between e_1 and e_2. Let the end-vertices of Q^* be y_1 and y_2, where y_i is nonadjacent to e_i. For $i = 1, 2$, let $Y_i = Y \setminus y_i$, and let X_i be the set of Y_i-complete vertices in $V(H)$. From the minimality of Y it follows that each Y_i satisfies hypothesis (b) of 5.6, and so by 5.6, X_i saturates $L(H)$ (for the other possibilities listed in 5.6 cannot hold). Since $e_1 \in X_1 \setminus X_2$ and $e_2 \in X_2 \setminus X_1$, and $X_1 \cap X_2 = X$, it follows that every edge of H is in one of X_1, X_2, and every branch-vertex of H is incident with exactly two edges in X_1 and two in X_2. So $a_1u_2 \in X_1$, $a_1c \in X_2$, $a_2c \in X_1$, and if the edge u_1a_2 exists it belongs to X_2. But then $\overline{G}|(V(L(H)) \cup Y)$ is an appearance in $\overline{G}$ of a J-enlargement, and so the theorem holds. This proves (2).

From (2), there is a partition (F_1, F_2) of $V(H) \setminus \{u_1, u_2, v\}$ with $A_i \subseteq F_i$ $(i = 1, 2)$ so that there is no edge between F_1 and F_2. We may assume that $C \cap F_2$ is nonempty, so (since $H \setminus \{u_2, v\}$ is connected, by 5.4) there is a track in $H \setminus \{u_2, v\}$ between u_1 and $C \cap F_2$. Since there are no edges between F_1 and F_2, this track does not meet F_1, and so none of its edges are in X. Since u_1 and the vertices in C have the same biparity, this track is even; and it is the interior of a track P say from a_1 to v with first and last edges in X and no other edges in X. From 2.2, the edge cv must have an end in the interior of P, and so c is a vertex of P, for every $c \in C$; and so (since we may assume P is minimal), $|C| = 1$, $C = \{c\}$ say. From (1), we deduce that P has length 4, and every vertex in A_2 is the middle vertex of P, and so $A_2 = \{a_2\}$, and a_2 is adjacent to both u_1 and c. The three tracks a_1-u_1-a_2 , v-c-a_2,u_2-a_2 are another instance of three tracks violating hypothesis (b) of 5.6, and so by (2) applied to these three tracks, we deduce that there is no track in $H \setminus \{c, u_1, a_2\}$ between A_1 and v. Since A_1 is nonempty and is contained in F_1, there is a maximal connected subset $F \subseteq F_1$ with nonempty intersection with A_1. So the only vertices of $L(H)$ not in F which might have neighbours in F are u_1, u_2, v. On the other hand, neither u_2 nor v has a neighbour in F, since there is no track in $H \setminus c, u_1, a_2$ between A_1 and v. So u_1 is the only such vertex, contradicting that H is cyclically 3-connected. This proves 6.1. ▮

The remainder of this section is concerned with analyzing the five possible outcomes of 5.6. Let us say a vertex $v \in V(G) \setminus V(L(H))$ is *big* (with respect to $L(H)$) if the set of its neighbours in $L(H)$ saturates $L(H)$. An appearance $L(H)$ of J in G is *overshadowed* if there is a branch B of H with odd length ≥ 3, with ends b_1, b_2, so that some vertex of G is nonadjacent in G to at most one vertex in $\delta(b_1)$ and at most one in $\delta(b_2)$.

6.2 *Let G be Berge, and let $L(H)$ be an appearance in G of a 3-connected graph J. let Y be a co-connected set of big vertices in $V(G) \setminus V(L(H))$, and let X be the set of Y-complete vertices in $L(H)$. Assume that X does not saturate $L(H)$. Then $J = K3, 3$ or K_4. Moreover:*

- *If $J = K_{3,3}$, then either:*
  - *there is an overshadowed appearance of J in G, or*
  - *$L(H)$ is degenerate and there is an overshadowed appearance of J in $\overline{G}$, or*
  - *$L(H)$ is degenerate and there is a J-enlargement that appears in $\overline{G}$.*
- *If $J = K_4$ and $L(H)$ is nondegenerate then there is an overshadowed appearance of J in G.*
- *If $J = K_4$ and $L(H)$ is degenerate, let $V(J) = \{1, 2, 3, 4\}$, and for $1 \leq i < j \leq 4$ let $B_{ij} = B_{ji}$ be the branch of J joining i and j; let the end-edges of B_{ij} be r_{ij} (incident with i) and r_{ji} (incident with j); and let R_{ij} R_{ij} be the path $L(B_{ij})$ in G (so r_{ij} and r_{ji} are the end-vertices of this path). Let $R_{1,3}, R_{1,4}, R_{2,3}, R_{2,4}$ have length 0. Then either:*
  1. *there is an overshadowed appearance of J in G, or*
  2. *there is a J-enlargement that appears in $\overline{G}$, or*

3. *(up to symmetry) there exist nonadjacent $y, y' \in Y$ so that the neighbours of y in $L(H)$ are $r_{1,2}, r_{1,4}, r_{3,2}, r_{3,4}, r_{2,4}$ and possibly $r_{1,3}$, and the neighbours of y' are $r_{2,1}, r_{2,3}, r_{4,1}, r_{4,3}, r_{1,3}$ and possibly $r_{2,4}$, or*
4. *(up to symmetry) $R_{1,2}, R_{3,4}$ have length 1, and there is an antipath Q in Y of odd length ≥ 3, with ends y, y' , so that y is nonadjacent to $r_{2,1}, r_{4,3}$ and possibly $r_{1,3}$, and y' is nonadjacent to $r_{3,4}, r_{1,2}$ and possibly $r_{2,4}$, and there are no other nonedges between $V(H)$ and $V(Q)$, or*
5. *(up to symmetry) $R_{1,2}, R_{3,4}$ have length 1, and there is an antipath Q in Y of even length ≥ 2, with ends y, y', so that y is nonadjacent to $r_{1,3}$, and y' is nonadjacent to $r_{1,2}$ and $r_{3,4}$, and there are no other nonedges between $V(Q)$ and $V(L(H) \setminus r_{2,4})$.*

Proof. Suppose not; then we may assume that Y is minimal such that it is co-connected and its common neighbours do not saturate $L(H)$. Choose two vertices of $L(H)$, both incident in H with the same branch-vertex of H, and both not in X. Then there is an antipath joining them with interior in Y, and the common neighbours of the interior of this antipath do not saturate $L(H)$. From the minimality of Y it follows that this antipath contains all vertices in Y. Consequently, Y is the vertex set of an antipath Q say, with ends y_1,y_2 say. From the hypothesis, $|Y| \geq 2$, since the neighbours of any vertex in Y saturate $L(H)$, so y_1, y_2 are distinct. Now for $i = 1, 2$, $Y \setminus y_i$ ($= Y_i$ say) is co-connected; let X_i be the set of Y_i-complete vertices in $L(H)$. From the minimality of Y, both X_1 and X_2 saturate $L(H)$.

(1) *For every branch-vertex b of H, X contains all edges of H incident with b except at most two; and if there are two such edges incident with b not in X, then one is in $X_1 \setminus X_2$ and the other in $X_2 \setminus X_1$.*

For both X_1 and X_2 saturate $L(H)$. Therefore, X_1 contains at least all except one of the edges of H incident with b, and so does X_2. Since $X_1 \cap X_2 = X$ this proves (1).

Now by 6.1, we may assume that X satisfies one of the five alternatives of 5.6, for otherwise the theorem holds. Let us examine them. It is convenient to postpone the case $J = K_4$ until later.

(2) *If $J \neq K_4$ then the theorem holds.*

For if 5.6.1 holds we are done. Since H has at least five branch-vertices, and each of them is incident with an edge in X, it follows that 5.6.3 and 5.6.4 do not hold. We may therefore assume that either 5.6.2 or 5.6.5 holds. In the first case, let B be a branch of H with ends b_1, b_2 so that every edge in X has an end in $V(B)$; and in the second case, let b_1, b_2 be vertices of H, of opposite parity and not in the same branch, so that $X = \delta(b_1) \cup \delta(b_2)$, and let B be the subgraph of H consisting just of b_1 and b_2 (and no edges). (Note that in this second case, b_1 and b_2 need not be branch-vertices.) Let $H' = H \setminus V(B)$. By 5.4, H' is connected, but none of its edges are in X. By (1), all vertices of H' have degree ≤ 2 , and so H' is a path or hole in H. If H' is a hole, let the branch-vertices of H that lie in this hole be $p_1, \dots , p_n$ in order, (there are at least three, since $J \neq K_4$); define $p_0 = p_n$, and let B_i the branch of H with ends p_{i-1} and p_i , for $1 \leq i \leq n$. (So H' is the union of $B_1, \dots , B_n$.) If H' is a path, let its ends be p_0 and p_n, and let the branch-vertices of H in its interior be $p_1, \dots , p_{n-1}$, so that $p_0, \dots , p_n$ are in order (so $n \geq 2$); and for $1 \leq i \leq n$ let B_i be the path of H' between p_{i-1} and p_i. In either case, for $1 \leq i \leq n$ let the end-edges of B_i be e_i (incident with p_{i-1}) and f_i (incident with p_i). So for $1 \leq i < n$, one of e_{i-1}, f_i is in X_1 and the other is in X_2; and the same holds for e_1, f_n if H' is a hole. We recall that Q is an antipath in Y between y_1 and y_2; there are two cases depending whether Q is odd or even. Assume first it is even. Then there do not exist two disjoint edges of H', one in X_1 and the other in X_2, for if there were we could complete Q to an odd antihole in G using them. Since there is a branch-vertex in H' different from p_0, p_n, it follows that H' has an edge in X_1 and one in X_2, and since all edges of the first type meet all those of the second type, there are at most two of each. Suppose that there are two of each. Then H' is a hole of length 4, and its four edges are alternately in X_1 and in X_2. If b_1 is not a branch-vertex then it is not adjacent to b_2, and so all its neighbours in H are in H'; while if b_1 is a branch-vertex then it has degree ≥ 3, and at least all except one of its neighbours are in H'. So in either case b_1 has at least twp neighbours in H', and so does b_2. Since H is bipartite, they are each adjacent to exactly two vertices of H', and these two vertices are nonadjacent; and b_1 and b_2 cannot be adjacent to the same pair of vertices of H', since H is cyclically 3-connected. Since H has at least five branch-vertices, one of b_1, b_2 is a branch-vertex, and so B is a branch, and hence both b_1 and b_2 are branch-vertices. All four vertices of H' have degree ≥ 3 in H, and so $J = K_{3,3}$. Now B has odd length. If it has length ≥ 3, then $L(H)$ is overshadowed since any vertex in Q is complete in G to the four vertices of $L(H)$ that in H are the edges joining $V(B)$ and $V(H')$. So we may assume B has length 1; let its edge be b. All branches of H now have length 1, so $L(H)$ is degenerate. Then $\overline{G}|(V(L(H)) \setminus b) \cup V(Q)$ is an appearance of $J = K_{3,3}$ in $\overline{G}$; and this appearance is overshadowed, because Q has even length ≥ 2, and because the vertex b is adjacent in $\overline{G}$ to each of e_1, e_2, e_3, e_4. So in either case the theorem holds. Hence we may assume that exactly one edge of H' is in X_2 say. If H' is a hole, then it contains at least 3 branch-vertices of H, each incident with an edge of H' in X_2, a contradiction. So H' is a path, and by the same argument, $n \leq 3$. Assume that $n = 3$. Then it follows that $e_2 = f_2 \in X_2$, and $f_1, e_3 \in X_1$. Since p_1 and p_2 are branch-vertices they are each adjacent to one of b_1, b_2, and not the same one since H is bipartite. Hence b_1, b_2 have opposite biparity, and hence have no common neighbours in H'. Since they each have at least two neighbours in $V(H')$, it follows that they both have exactly two, and each of p_0, p_1, p_2, p_3 is adjacent to exactly one of b_1, b_2. But then p_0, p_3 have degree 2 in H, and so H has only four branch-vertices, a contradiction. So $n = 2$. Since H has ≥ 5 branch-vertices, it follows that b_1, b_2, p_0, p_2 are all branch-vertices, and so B is a branch, and p_0, p_2 are both adjacent to both b_1, b_2. Since p_1 is adjacent to one of b_1, b_2, it follows that p_0, p_1, p_2 all have the same biparity, and so B_1, B_2 both have even length, and so does B. We may assume that $f_1 \in X_1$ and $e_2 \in X_2$. Consequently, $e_1 \notin X_1$, and $f_2 \notin X_2$; and not both $e_1 \in X_2$, and $f_2 \in X_1$, for all edges in X_1 meet all those in X_2. So we may assume that e_1 is not in $X_1 \cup X_2$. Since both X_1 and X_2 saturate $L(H)$, it follows that the edges p_0b_1 and p_0b_2 are both in both X_1 and X_2, and hence are both in X. Also, p_1 is adjacent to one of b_1, b_2, say b_1, and the edge $p_1b_1 \in X$. So the track b_2-p_0-B_1-p_1-b_1 is even, has length ≥ 4, and both its end-edges are in X and none of its internal vertices are in X. Consequently every edge of H in X has an end in the interior of this track, by 2.2, and in particular no edge incident with p_2 is in X, a contradiction. This completes the proof of (2) when Q is even. Now we prove (2) when Q is odd. For this we shall use repeatedly the fact that if $e_1 \in X_1 \setminus X_2$, and $e_2 \in X_2 \setminus X_1$, and $e_0 \in X$, and e_1 meets e_2, then one of them meets e_0; for otherwise Q could be completed to an odd antihole in G via y_2-e_1-e_0-e_2-y_1. If H' is a hole, then we may choose p_i, p_j nonadjacent; and since one of f_i, e_{i+1} is in X_1 and the other in X_2, and there is an edge in X between p_j and $\{b_1, b_2\}$, this is a contradiction. So H' is a path. Since one of f_1, e_2 is in X_1 and the other in X_2, every edge in X meets one of these two edges, and since for $1 \leq i < n$ there is an edge in X between p_i and $\{b_1, b_2\}$, it follows that $n \leq 3$. Assume that $n = 3$; then there is no edge in X incident with p_3, and so p_3 is not a branch-vertex, and similarly p_0 is not a branch-vertex (by the same argument with p_1 replaced by p_2); but then H has only four branch-vertices, a contradiction. So $n = 2$, and therefore b_1, b_2, p_0, p_2 are all branch-vertices, and B is a branch, and p_0, p_2 are adjacent to both of b_1, b_2. Since there is an edge in X between p_0 and $\{b_1, b_2\}$, which therefore meets either f_1 or e_2, it follows that B_1 has length 1, and similarly so does B_2; but then since H is bipartite, p_1 is nonadjacent to both b_1 and b_2, a contradiction since it is a branch-vertex. This proves (2).

Henceforth we may assume that $J = K_4$. To simplify notation, let $V(J) = \{1, 2, 3, 4\}$, and let the branch of H joining i and j be $B_{i,j} = B_{j,i}$; let its end-edges be $r_{i,j}$ (incident with i) and $r_{j,i}$ (incident with j); and let $R_{i,j} = R_{j,i} = L(B_{i,j})$ (that is, the path in G between $r_{i,j}$ and $r_{j,i}$ formed by the vertices of G in $E(B_{i,j})$). For $k = 1, 2, 3, 4$, we denote by C_k the hole in G induced on the union of the three paths $R_{i,j}$ $(i, j \neq k)$, and we denote by T_k the triangle $\{r_{k,i} : 1 \leq i \leq 4, i \neq k\}$. We observe first that

(3) *If* $r_{3,1}, r_{3,2} \notin X$, *then every vertex of* C_4 *in* X *is adjacent to one of* $r_{3,1}, r_{3,2}$.

For this is trivial if C_4 has length 4, so assume it has length ≥ 6. Suppose that some vertex x of C_4 nonadjacent to $r_{3,1}, r_{3,2}$ belongs to X. Now by (1), $r_{3,4} \in X$, and we may assume $r_{3,i} \in X_i$ for $i = 1, 2$. So $r_{3,1}$-y_1-Q-y_2-$r_{3,2}$ is an antipath, of length ≥ 3. It is even, since it can be completed to an antihole via $r_{3,2}$-x-$r_{3,1}$; and this contradicts 3.3. This proves (3).

(4) *If one of $C_1, \dots, C_4$ contains no member of X then the theorem holds.*

For suppose that no vertex of C_4 say is in X. Then from (1), $r_{1,4}, r_{2,4}, r_{3,4}$ are all in X. Now one of $R_{1,2}, R_{1,3}, R_{2,3}$ has odd length, since C_4 has even length, say $R_{1,2}$. If $r_{1,4}, r_{2,4}$ are not adjacent then $r_{1,4}$-$r_{1,2}$-$R_{1,2}$-$r_{2,1}$-$r_{2,4}$ is an odd path between vertices in X, and none of its internal vertices are in X, and $r_{3,4}$ has no neighbour in its interior, contrary to 2.2. So they are adjacent, that is, $R_{1,4}, R_{2,4}$ both have length 0. Since we may assume that $r_{2,1}, r_{2,3}$ are in $X_1 \setminus X_2$ and $X_2 \setminus X_1$ respectively, it follows that Q can be completed to an antihole via y_2-$r_{2,3}$-$r_{1,4}$-$r_{2,1}$-y_1, and hence Q has even length ≥ 2. By 3.3, since Q is the interior of an even antipath between $r_{1,3}$ and $r_{1,2}$, and $r_{2,1}, r_{2,3}$ are both different from $r_{1,3}, r_{1,2}$, and one of $r_{2,1}$ and $r_{2,3}$ is in $X_1 \setminus X_2$ and the other in $X_2 \setminus X_1$, it follows that C_4 has length 4, and therefore $R_{1,3}, R_{2,3}$ have length 0, and $R_{1,2}$ has length 1. Now $\overline{G}|(T_1 \cup T_2 \cup T_3 \cup Y)$ is an appearance in $\overline{G}$ of K_4; and since Q has even length ≥ 2, it is overshadowed in $\overline{G}$ because of the vertex $r_{4,3}$, and so statement 1 of the theorem holds. This proves (4).

(5) *If one of $C_1, \dots, C_4$ contains at most one member of X then the theorem holds.*

By (4) we may assume at least one vertex of each C_k is in X. We may therefore assume that exactly one vertex (say x) of C_4 say is in X; and so from the symmetry we may assume that x is in $R_{1,2}$. So neither of $r_{3,1}, r_{3,2}$ is in X, and so from 6.1, we may assume that Y cannot be linked onto T_3. Consequently $x = r_{2,1}$ (for otherwise $r_{2,4} \in X$ and we could use the paths $r_{3,4}$, $r_{2,4}$-$r_{2,3}$-$R_{2,3}$-$r_{3,2}$, and a subpath of C_4 from x to $r_{3,2}$ to link Y onto T_3), and similarly $x = r_{1,2}$. So $R_{1,2}$ has length 0. Also, from (3), x is adjacent to one of $r_{3,1}, r_{3,2}$, so we may assume that $R_{2,3}$ has length 0 . Therefore $R_{1,3}$ is odd. Hence $r_{1,2}$-$r_{1,3}$-$R_{1,3}$-$r_{3,1}$-$r_{3,4}$ is an odd path between vertices in X, and its internal vertices are not in X. By 2.2, every vertex in X has a neighbour in $R_{1,3}$. Hence $r_{3,4}$ is the only vertex of C_1 in X; and so (by exchanging C_1 and C_4) we deduce that $R_{3,4}$ has length 0. So $R_{1,4}$ is even, and $R_{2,4}$ is odd, and the latter is the interior of a second odd path between members of X. We claim that $R_{1,4}$ has length 0; for if $X = \{r_{1,2}, r_{3,4}\}$, this follows from the symmetry between $R_{1,4}$ and $R_{2,3}$, while if $|X| > 2$ then since every vertex in X has a neighbour in both $R_{1,3}$ and $R_{2,4}$, the third vertex in X must be both $r_{1,4}$ and $r_{4,1}$, and hence again $R_{1,4}$ has length 0.

Now we recall that there is an antipath Q in Y between y_1 and y_2, with $V(Q) = Y$. Let its vertices be $y_1 = q_1, q_2, \dots, q_k = y_2$ in order. We may assume that $r_{3,1} \in X_1$ and $r_{3,2} \in X_2$; and consequently $r_{2,4} \in X_1$. Assume that $R_{1,3}$ has length > 1 and hence ≥ 3 since it is odd. Then $r_{1,2}$-$r_{1,3}$-$R_{1,3}$-$r_{3,1}$-$r_{2,3}$ is an odd path of length ≥ 5, with ends in X and no internal vertex in X. By 2.1, Y contains a leap. But the only vertex in Y nonadjacent to $r_{3,1}$ is $y_1 = q_1$, and its only nonneighbour in Y is q_2; so q_1, q_2 is the leap. Hence q_2 is adjacent to $R_{1,4}$, since it has two neighbours in T_1. Now the path $r_{1,2}$-$r_{2,4}$-$R_{2,4}$-$r_{4,2}$-$r_{4,3}$ is odd and has length ≥ 3, between common neighbours of $\{q_1, q_2\}$; and these two vertices have no common neighbour in its interior, since we could complete the path q_1-$r_{1,3}$-$R_{1,3}$-$r_{3,1}$-q_2 to an odd hole through any such common neighbour. So $\{q_1, q_2\}$ is also a leap for this second path. But then the only neighbours of q_1 in $L(H)$ are $r_{1,2}, r_{1,3}, r_{4,2}, r_{4,3}, r_{2,3}$ and possibly $r_{1,4}$; and the only neighbours of q_2 are $r_{2,1}, r_{2,4}, r_{3,1}, r_{3,4}, r_{1,4}$ and possibly $r_{2,3}$. But then statement 3 of the theorem holds. So we may assume that $R_{1,3}$ has length 1,and from the symmetry that $R_{2,4}$ has length 1. Choose i with $1 \leq i \leq k$ minimum so that one of $r_{4,2}, r_{1,3}$ is nonadjacent to q_i, and we may assume $r_{1,3}$ is nonadjacent to q_i. Then $r_{3,1}$-q_1-$\dots$-q_i-$r_{1,3}$ is an antipath Q' say, and it has odd length since it can be completed to an antihole via $r_{1,3}$-$r_{3,4}$-$r_{1,2}$-$r_{3,1}$. So i is even. If $i = 2$ then again q_1, q_2 is a leap, for the same path as before, and again statement 3 of the theorem holds. So we may assume $i \geq 4$. Now Q' cannot be completed to an antihole via $r_{1,3}$-$r_{4,2}$-$r_{3,1}$, so $r_{4,2}$ has a nonneighbour in the interior of Q'; and from the minimality of i, its only nonneighbour in the interior of Q' is q_i. Now X_1 saturates $L(H)$, and since $r_{1,3} \notin X_1$ it follows that $r_{1,4} \in X_1$. But then statement 4 of the theorem holds. This proves (5).

(6) *If $r_{3,1}, r_{3,2} \notin X$ then either the theorem holds, or C_4 has length 4 and its other two vertices are in X.*

For by (5), we may assume that at least two vertices of C_4 are in X. Since Y cannot be linked onto T_3, there are exactly two such vertices and they are adjacent. By (3) they are both adjacent to one of $r_{3,1}, r_{3,2}$, and the claim follows. This proves (6).

(7) *If $r_{3,1}, r_{3,2} \notin X$ and $R_{1,2}$ has length > 0 then the theorem holds.*

For then by (6) we may assume that $R_{1,2}$ has length 1, and $R_{1,3}, R_{2,3}$ both have length 0, and $r_{1,2}, r_{2,1}$ are both in X. Since Y cannot be linked onto T_3 it follows that $r_{1,4} \notin X$, and similarly $r_{2,4} \notin X$. Now there are two cases, depending on whether $R_{3,4}$ has positive length or not. Suppose first that it has length ≥ 1. Since $r_{1,3}$ and $r_{1,4}$ are not in X, it follows from (6) applied to C_2 that we may assume C_2 has length 4, and hence $R_{1,4}$ has length 0, and moreover, its two other vertices are in X, that is, $r_{4,3} \in X$. Similarly we may assume $R_{2,4}$ has length 0. We may assume that $r_{1,3} \in X_1$, and therefore $r_{2,3} \in X_2$, $r_{2,4} \in X_1$ and $r_{1,4} \in X_2$. But then $\overline{G}|(V(L(H)) \cup Y$ is an appearance of $K_{3,3}$ in $\overline{G}$, and statement 2 of the theorem holds. So we may assume that $R_{3,4}$ has length 0. By (6) applied to C_2 we may assume that $R_{1,4}$ has length 1, and that $r_{4,1} \in X$, and similarly $R_{2,4}$ has length 1, and $r_{4,2} \in X$. We may assume that $r_{1,3} \in X_1$; but then $r_{2,3} \in X_2$, $r_{1,4} \in X_2$ and $r_{2,4} \in X_1$. Hence Q can be completed to an antihole via y_2-$r_{2,4}$-$r_{1,4}$-y_1 or via y_2-$r_{1,3}$-$r_{4,2}$-$r_{1,4}$-y_2, and one of these gives an odd antihole, a contradiction. This proves (7).

Now, to complete the proof: we may assume that $r_{3,1}, r_{3,2} \notin X$. By (6) and (7) we may assume that $R_{1,2}$ and $R_{2,3}$ have length 0, and $R_{1,3}$ has length 1, and $r_{1,3}, r_{1,2} \in X$. Since Y cannot be linked onto T_3 it follows that $r_{2,4} \notin X$; so by (6) and (7) applied to C_1, we may assume that $R_{3,4}$ has length 0, $R_{2,4}$ has length 1, and $r_{4,2} \in X$. We may assume that $r_{3,1} \in X_1$; but then $r_{2,3} \in X_2$ and $r_{2,4} \in X_1$. Since Q can be completed via y_2-$r_{2,3}$-$r_{4,2}$-$r_{3,1}$-y_1 it follows that Q is even. If $R_{1,4}$ has length 0 then statement 5 of the theorem holds, and otherwise $L(H)$ is overshadowed (since any vertex in Y is adjacent to all of $r_{1,2}, r_{1,3}, r_{4,2}, r_{4,3}$), and so statement 1 of the theorem holds. This proves 6.2. ∎

7 Rung replacement

Before we apply 6.2, let us simplify it a little. We can effectively eliminate the two cases of $L(H)$ being overshadowed. We need a few lemmas.

7.1 *Let c_1, c_2 be adjacent vertices of a 3-connected graph J, and let e, f be edges of J incident with c_1 and different from c_1c_2. There are three tracks of J from c_1 to c_2, pairwise vertex-disjoint except for their ends, and with first edges c_1c_2, e, f respectively.*

Proof. Since J is 3-connected, if we delete from J all edges incident with c_1 except e and f, the graph we make is still 2-connected, and so it has a cycle containing c_1 and c_2. This proves 7.1. ∎

7.2 *Let G be Berge, and let H be a prism which is an induced subgraph of G, with notation as above. Assume P_1, P_2, P_3 all have length > 1. Let Y be a co-connected set of vertices of G, so that every vertex in Y is adjacent to at least two of $\{a_1, a_2, a_3\}$ and to at least two of $\{b_1, b_2, b_3\}$. Then at least two of $\{a_1, a_2, a_3\}$ and at least two of $\{b_1, b_2, b_3\}$ are Y-complete.*

Proof. Suppose not; then there is an antipath with interior in Y, joining two vertices either both in $\{a_1, a_2, a_3\}$ or both in $\{b_1, b_2, b_3\}$. Let Q be the shortest such antipath. We may assume Q joins a_1 and a_2 say. Since every vertex in Y is adjacent to either a_1 or a_2 it follows that Q has length ≥ 3. From the minimality of Q, a_3 is Q-complete, and so is at least one of b_1, b_2, b_3, say b_i. Since Q can be completed to an antihole via a_1-b_i-a_2 it follows that Q is even. From 3.3 applied to the hole formed by $P_1 \cup P_2$ and hat a_3, neither of b_1, b_2 is Q^*-complete, and so there is an antipath between b_1 and b_2 with interior in Q^*. By the minimality of Q, the two antipaths have the same interior; but this again contradicts 3.3. This proves 7.2. ∎

7.3 *Let G be Berge, and for $1 \leq i \leq 3$ let P_i be a path of even length ≥ 2, from a_i to b_i, so that these three paths form a prism with triangles $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let P_1' be a path from a_1' to b_1, so that P_1', P_2, P_3 also form a prism. Let $y \in V(G)$ have at least two neighbours in A and in B. Then it also has at least two neighbours in $\{a_1', a_2, a_3\}$.*

Proof. Suppose not. Let X be the set of neighbours of Y in G. Then $a_1' \notin X$, and $a_1 \in X$, and exactly one of $a_2, a_3 \in X$, say $a_2 \in X$. Also, y cannot be linked onto the triangle $A' = \{a_1', a_2, a_3\}$, by 2.4, and since one of $b_2, b_3 \in X$ it follows that no internal vertex of P_1' is in X. Hence $b_1 \notin X$, for otherwise a_2-a_1'-P_1'-b_1 would be an odd minimal path between members of X of length > 1, impossible since G is Berge. So $b_2, b_3 \in X$. Since a_1-a_3-P_3-b_3 is also not an odd minimal path between members of X, and so there is a member of X in $P_3 \setminus b_3$. But then y can be linked onto A', via a subpath of P_3, the path b_2-b_3-P_3-a_3, and the path a_2, a contradiction. This proves 7.3. ∎

We shall only need the following when $J = K_4$ or $K_{3,3}$, but we might as well prove it in general.

7.4 *Let G be Berge, and let $L(H)$ be an overshadowed appearance of J in G, where J is 3-connected. Then either:*

- *there is a J-enlargement with a nondegenerate appearance in G, or*
- *G admits an even skew partition.*

Proof. For each edge uv of J, let B_{uv} be the branch of H with ends u, v, and let R_{uv} be the path $L(B_{uv})$ of $L(H)$. For each $v \in V(J)$ let N_v be the clique of $L(H)$ with vertex set $\delta_H(v)$. There is a branch B of H with odd length ≥ 3, with ends c_1, c_2, so that some vertex of G is nonadjacent in G to at most one vertex of N_{c_1} and to at most one vertex of N_{c_2}. We say such a vertex v is *B-dominant with respect to $L(H)$.* Let the ends of $R_{c_1c_2}$ (that is, the end-edges of B) be r_1, r_2, where $r_i \in N_{c_i}$. Let Y be a maximal co-connected set of vertices each with at most one non-neighbour in N_{c_1} and at most one non-neighbour in N_{c_2}. We shall prove that Y and some of its common neighbours separate the interior of $R_{c_1c_2}$ from the remainder of $L(H)$ in G, so that will be the skew partition we are looking for. Let X be the set of all Y-complete vertices in G.

(1) *For $i = 1, 2$, at most one vertex of N_{c_i} is not in X.*

For let a_1, a_2 be any two distinct vertices in $N_{c_1} \setminus r_1$; we shall show that at most one of a_1, a_2, r_1 is not in X. By 7.1, there are two paths Q_1,Q_2 of H between c_1 and c_2, so that Q_1,Q_2,B are vertex-disjoint except for their ends, and for $i = 1, 2$, a_i is the first edge of Q_i. Let b_i be the other end-edge of Q_i. Both Q_1 and Q_2 have odd length,since B is odd and H is bipartite; and they have length ≥ 3 since b_1, b_2 are nonadjacent (for they are the ends of a branch of length > 1.) Hence there are two paths P_1,P_2 of $L(H)$ from N_{c_1} to N_{c_2}, so that P_1,P_2,$R_{c_1c_2}$ are vertex-disjoint and form a prism, and P_i is from a_i to b_i. Now B is odd and therefore $R_{c_1c_2}$ is even, and similarly P_1 and P_2 are even. By hypothesis, each member of Y is adjacent to at least two vertices of the triangle $\{a_1, a_2, r_1\}$ and to two vertices of the triangle $\{b_1, b_2, r_2\}$. By 7.2 it follows that X contains at least two members of $\{a_1, a_2, r_1\}$. This proves (1).

Let

$$\begin{aligned}
X_1 &= X \cap (N_{c_1} \cup N_{c_2}) \\
X_2 &= X \cap (V(L(H)) \setminus (N_{c_1} \cup N_{c_2})) \\
X_0 &= X \setminus V(L(H)) \\
S &= V(R_{c_1c_2}) \setminus X_1 \\
T &= (V(L(H)) \setminus V(R_{c_1c_2})) \setminus X_1.
\end{aligned}$$

We observe first that no vertex of S is adjacent to any vertex in T; for such an edge would join two vertices both in N_{b_i} for some i, and therefore both not in X, contradicting (1).

(2) *If $F \subseteq V(G)$ is connected and some vertex of S has a neighbour in F, and so does some vertex of T, then $F \cap (X_0 \cup X_1 \cup Y)$ is nonempty.*

We shall prove this by induction on $|F|$; so, we assume it holds for all smaller choices of F (even for different choices of $L(H)$). We may assume no vertex in F is in $X_0 \cup X_1$. Also, by induction we may assume that $G|F$ is a path with vertices $f_1, \dots, f_n$ say, where f_1 is the only vertex of F with a neighbour in S, and f_n is the only vertex with a neighbour in T. From the minimality of F it also follows that F is disjoint from $L(H)$; for any vertex of F in $L(H)$ would be in S or T, since it is not in X_1, and then we could make F shorter by omitting this vertex. Consequently $F \cap X = \emptyset$. Suppose some vertex in $v \in F$ is big with respect to $L(H)$. Then since $v \notin X$ it follows that v has a nonneighbour in Y, and so $Y \cup v$ is co-connected; the maximality of Y therefore implies that $v \in Y$, and hence $F \cap Y \neq \emptyset$ and the claim holds. So we may assume that no vertex in F is big. On the other hand, the set of attachments of F in $L(H)$ is not local, because it has an attachment in $R_{c_1c_2}$, and its attachments are not all contained in any of $V(R_{c_1c_2})$, N_{b_1},N_{b_2}. Let us apply 5.9. Suppose first that 5.9.1 holds. Then we obtain an appearance $L(H')$ in G of some J-enlargement, with $L(H)$ an induced subgraph of $L(H')$. Since $R_{c_1c_2}$ has even nonzero length, it follows that $L(H)$ is not degenerate, and therefore neither is $L(H')$, and hence the theorem holds. So we may assume that 5.9.2 holds, and there is an edge b_1b_2 of J, and a path P of G with $V(P) \subseteq F$ and with ends p_1 and p_2, with the following properties (for $i = 1, 2$, s_i denotes the unique vertex in $N_{b_i} \cap R_{b_1b_2}$):

- all attachments of P in $L(H)$ belong in H to $N_{b_1} \cup N_{b_2} \cup R_{b_1b_2}$, and
- p_1 is adjacent in G to all vertices in $N_{b_1} \setminus s_1$, and no other vertex of P has any neighbours in $N_{b_1} \setminus s_1$, and
- no vertex of P except p_2 has any neighbours in $N_{b_2} \cup R_{b_1b_2} \setminus s_1$, and either:
  1. p_2 has no neighbours in $N_{b_2} \setminus s_2$, and has a neighbour in $R_{b_1b_2} \setminus s_1$, or
  2. p_2 is adjacent to all vertices in $N_{b_2} \setminus s_2$, and to no vertex of $R_{b_1b_2}$ except possibly s_2, or
  3. $p_1 = p_2$ is adjacent to all vertices in $N_{b_2} \setminus s_2$.

In case 1, let R' be the (unique) path from p_1 to s_2 in $(V(P) \cup V(R_{b_1b_2})) \setminus s_1$, and in cases 2 and 3 let R' be P. So if in $L(H)$ we replace $R_{b_1b_2}$ by R' we obtain another appearance of J in G, say $L(H')$, where H' is obtained from H by replacing the branch $B_{s_1s_2}$ by some new branch B' joining the same two vertices. For each $v \in V(J)$ let N'_v be the clique in $L(H')$ formed by the edges in $\delta_{H'}(v)$. So $N'_v = N_v$ for all vertices v of J except for b_1 and b_2. Let R' be between r'_1 and r'_2, where $r'_i \in N'_{b_i}$ for $i = 1, 2$. Now suppose that $R_{b_1b_2} \neq R_{c_1c_2}$. Then B is still a branch of H', and we claim that every $y \in Y$ is B-dominant with respect to $L(H')$. For let e, f be two edges of J incident with c_1 and different from c_1c_2. By 7.1 there are three tracks of J from c_1 to c_2, vertex-disjoint except for their ends, and one of them is the edge c_1c_2, and the first edges of the other two are e and f. There are three tracks corresponding to these in H, and their line graph is a prism in $L(H)$. There also correspond three tracks in H', yielding a prism in $L(H')$. Since $R_{b_1b_2} \neq R_{c_1c_2}$, it follows that $R_{b_1b_2}$ is incident with at most one of c_1, c_2, so these two prisms are related as in 7.3. Hence by 7.3, since y has two neighbours in both triangles of the first prism, it also has two neighbours in the triangles of the second. This proves that y is B-dominant with respect to $L(H')$. The same argument in the reverse direction shows that Y remains a maximal co-connected set of B-dominant vertices. Since there is a proper subset F' of F with attachments in S and in the new set T' in $V(H')$ corresponding to T (for T' contains all the vertices of $R_{b_1b_2}$ that are in F, and there is at least one such vertex), it follows that we may apply the inductive hypothesis. So F', and hence F, contains a vertex of X. This completes the argument when $R_{b_1b_2} \neq R_{c_1c_2}$, so now we assume that $R_{b_1b_2} = R_{c_1c_2}$. We may therefore assume that $b_i = c_i$ for $i = 1, 2$. Now there were three cases in the definition of R, listed above. Case 3 is impossible, since then the vertex p_1 would be B-dominant with respect to $L(H)$, and therefore would be in either X or Y, a contradiction. Also, case 1 is impossible, by applying 7.3 as before to show that Y remains a maximal co-connected set of B'-dominant vertices, and applying the inductive hypothesis. So case 2 applies; that is, p_2 is adjacent to all vertices in $N_{c_2} \setminus r_2$, and to no vertex of $R_{c_1c_2}$ except possibly r_2. So $N'_{c_i} = (N_{c_i} \setminus \{r_i\}) \cup \{r'_i\}$ for $i = 1, 2$. We recall that in this case $R' = P$, and P is a subpath of the path with vertices $f_1, \dots, f_n$. Choose h with $1 \le h \le n$ minimum so that f_h is a vertex of R'. Since both R' and $G|F$ are paths it follows that f_h is one end of R', say r'_1. (This is without loss of generality, because in this case 2, the symmetry between $b_1 = c_1$ and $b_2 = c_2$ is restored.) We claim that every vertex in $N_{c_1} \setminus r_1$ is in X. For suppose not; then r_1 has a neighbour in T, contrary to the minimality of F. We claim also that every vertex of $N_{c_2} \setminus r_2$ is in X. For if not, then $r_2 \in X$, and by 7.1 there is a prism $R_{c_1c_2}, P_1, P_2$ say, in $L(H)$, where each P_i has an end $a_i \in N_{c_1}$ and an end $b_i \in N_{c_2}$, and $b_2 \notin X$. (Consequently $r_2, b_2 \in X$.) Hence at most one vertex of the triangle $\{r'_2, b_1, b_2\}$ is in X, and some vertex in X (namely a_1) has no neighbour in this triangle, so by 2.8, Y cannot be linked onto this triangle. In particular, no vertex of P_2 is in X except a_2. But then a_2-P_2-b_2-r_2 is an odd path between members of X, and none of its internal vertices are in X, and a_1 has no neighbour in its interior, contrary to 2.2. This proves that every vertex of $N_{c_2} \setminus r_2$ is in X. Consequently all vertices of Y are B'-dominant with respect to $L(H')$. We claim also that Y is still maximal. For suppose not, and let $Y \subset Y'$ for some larger co-connected set Y' of B'-dominant vertices. Since r'_1, r'_2 are not in X, they are certainly not Y'-complete, and since by (1) applied to Y', at most one vertex of N'_{c_i} is not Y'-complete for $i = 1, 2$, it follows that every vertex of $N'_{c_1} \setminus r'_1$ and $N'_{c_2} \setminus r'_2$ are Y'-complete. But then all the members of Y' are B-dominant with respect to $L(H)$, contrary to the maximality of Y. This proves that Y is a maximal co-connected set of B'-dominant vertices with respect to $L(H')$. Hence we can apply induction on F, and the result follows. This proves (2).

It follows from (2) that there is a partition of $V(G) \setminus (X_0 \cup X_1 \cup Y)$ into two sets L and M say, where there is no edge between L and M, and $S \subseteq L$ and $T \subseteq M$. So $(L \cup M, X_0 \cup X_1 \cup Y)$ is a skew partition of G. By 4.2 we may assume it is not loose, and so X_2 is empty; and we shall show it is even.

(3)*For $i = 1, 2$, all vertices of $N_{c_i} \setminus r_i$ belong to X_1.*

For suppose there is a vertex n_1 of $N_{c_1} \setminus r_1$ not in X. Therefore all other vertices of N_{c_1} belong to X, and in particular, $r_1 \in X$. Suppose no other vertex of $R_{c_1c_2}$ is in X; then $r_2 \notin X$, so X includes $N_{c_2} \setminus r_2$. Choose any $n_2 \in N_{c_2} \setminus r_2$, and any $n'_1 \in N_{c_1} \setminus r_1$ different from n_1. Then r_1-$R_{c_1c_2}$-r_2-n_2 is an odd path between Y-complete vertices, and none of its internal vertices are Y-complete, and yet n'_1 has no neighbour in its interior, contrary to 2.2. This proves that some vertex of $R_{c_1c_2}$ different from r_1 is in X; yet X_2 is empty, so the interior of $R_{c_1c_2}$ contains no vertex in X. Consequently $r_2 \in X$. Choose $n_2 \in N_{c_2} \setminus r_2$ so that $N_{c_2} \setminus n_2 \subseteq X$. Since J is 3-connected, there is a track of H from c_1 to c_2 with first edge n_1 and last edge different from n_2. This track is odd since c_1 and c_2 have opposite biparity; and so in G there is an even path, P say, from n_1 to some $n'_2 \in N_{c_2} \setminus n_2$, with no vertex in $N_{c_1} \cup Nc_2$ except its ends. But then r_1-n_1-P-n'_2 is an odd path between Y-complete vertices, no vertex in its interior is Y-complete, and the Y-complete vertex r_2 has no neighbour in its interior, contrary to 2.2. This proves (3).

Let $W = (N_{c_1} \setminus r_1) \cup (N_{c_2} \setminus r_2)$. Then $W \subseteq X_1$ by (3), and since there are no edges between N_{c_1} and N_{c_2}, it follows that W has exactly two components, both cliques. In particular, W is co-connected. Now every W-complete vertex is B-dominant, and so belongs to $X \cup Y$; and hence there are no W-complete vertices in $L \cup M$. Consequently W is a kernel for the skew partition. Suppose $u_1, u_2 \in W$ are nonadjacent and joined by an odd path with interior in L. Then one of u_1, u_2 is in $N_{c_1} \setminus r_1$ and the other in $N_{c_2} \setminus r_2$, and therefore they are joined by a path in $L(H)$ using no more vertices in $N_{c_1} \cup N_{c_2}$, which is even (since H is bipartite). But this contradicts 4.3, and so there are no such u_1, u_2. Finally, suppose there is a pair of vertices of L joined by an odd antipath with interior in W, necessarily of length ≥ 5 (since we already did the odd path case). Then $G|W$ contains an antipath of length 3, which is impossible since its components are cliques. From 4.6 it follows that the skew partition is even. This proves 7.4. ∎

8 Generalized line graphs

As we said earlier, our strategy is to find the biggest line graph in G that we can, and then assemble all the alternative rungs for a given edge of J into a "strip". In this section we make that precise.

Let J be 3-connected, and let G be Berge. A *J-strip system* (S, N) in G means:

- for each edge uv of J, a subset $S_{uv} = S_{vu} \subseteq V(G)$
- for each vertex v of J, a subset $N_v \subseteq V(G)$

satisfying the following conditions (for $uv \in E(J)$, a *uv-rung* means a path R of G with ends s, t say, where $V(R) \subseteq S_{uv}$, and s is the unique vertex of R in N_u, and t is the unique vertex of R in N_v):

- The sets $S_{uv}(uv \in E(J))$ are pairwise disjoint
- For distinct $u, v \in V(J)$, $N_u \cap N_v$ is empty if u,v are nonadjacent, and otherwise $N_u \cap N_v \subseteq S_{uv}$
- For $uv \in E(J)$ and $w \in V(J)$, if $w \neq u, v$ then $S_{uv} \cap N_w = \emptyset$
- For each $u \in V(J)$, $N_u \subseteq \bigcup(S_{uv} : v \in V(J)$ adjacent to $u)$
- For each $uv \in E(J)$, every vertex of S_{uv} is in a uv-rung
- If $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between S_{uv} and S_{wx}
- If $uv, uw \in E(J)$ with $v \neq w$, then $N_u \cap S_{uv}$ is complete to $N_u \cap S_{uw}$, and there are no other edges between S_{uv} and S_{uw}
- For each $uv \in E(J)$ there is a uv-rung so that for every cycle C of J, the sum of the lengths of the uv-rungs for $uv \in E(C)$ has the same parity as $|V(C)|$.

The last axiom looks strange, but we shall show immediately that the same property holds for *every* choice of uv-rungs.

8.1 *If (S, N) is a J-strip system in a Berge graph G, where J is 3-connected. Then for every $uv \in E(J)$, all uv-rungs have lengths of the same parity.*

Proof. Since J is 3-connected, there is a cycle C of J with $|V(C)| \geq 4$ and with $uv \in E(C)$. For each $xy \in E(C)$ different from uv, choose an xy-rung R_{xy}. For every uv-rung R, $G|V(P) \cup V(R)$ is a hole, because it is not a triangle since $|V(C)| \geq 4$, But all these holes have even length, and so all choices of R have lengths of the same parity. This proves 8.1. ∎

For each edge uv of J, choose a uv-rung R_{uv}. It follows from 8.1 and the final axiom above that the subgraph of G induced on the union of the vertex sets of these rungs is a line graph of a bipartite subdivision H of J. For brevity we say that this choice of rungs *forms* $L(H)$.

We need two easy observations:

8.2 *Let (S, N) be a J-strip system in a Berge graph G, where J is 3-connected. If there is an edge uv of J so that some uv-rung has length 0 and another uv-rung has length ≥ 1, then there is an overshadowed appearance of J in G.*

Proof. For each edge ij of J choose an ij-rung R_{ij}, so that R_{uv} has length ≥ 1 and otherwise arbitrarily; and let this choice of rungs form $L(H)$. Let y be the vertex of some uv-rung of length 0. By 8.1, R_{uv} has even length. Let B be the branch of H between u and v, so $E(B) = V(R_{uv})$. Then B is odd and has length ≥ 3 and y is nonadjacent in G to at most one vertex of G in $\delta_H(u)$ and at most one in $\delta_H(v)$. Hence $L(H)$ is overshadowed. This proves 8.2. ∎

A J-strip system is *nondegenerate* if there is some choice of rungs so that the line graph $L(H)$ they form is a nondegenerate appearance of J. 8.2 has the following corollary:

8.3 *Let (S, N) be a nondegenerate J-strip system in a Berge graph G, where J is 3-connected. If there is no overshadowed appearance of J in G, then for every choice of rungs, the line graph they form is a nondegenerate appearance of J in G.*

Proof. Take a choice of rungs $R_{ij}(ij \in E(J))$, forming $L(H)$ say, where $L(H)$ is nondegenerate; and suppose there is another choice, $R'_{ij}(ij \in E(J))$, forming $L(H')$ say, where $L(H')$ is degenerate. Then for some $ij \in E(J)$, R_{ij} has nonzero length and R'_{ij} has length 0. By 8.2 there is an overshadowed appearance of J in G. This proves 8.3, ∎

Given a J-strip system (S, N), we define $V(S, N) = \bigcup(S_{uv} : uv \in E(J))$. Hence every $N_v \subseteq V(S, N)$. If $u, v \in V(J)$ are adjacent, we define $N_{uv} = N_u \cap S_{uv}$. So every vertex of N_u belongs to N_{uv} for exactly one v. Note that N_{uv} is in general different from N_{vu}, but S_{uv} and S_{vu} mean the same thing. We say $X \subseteq V(S, N)$ *saturates* the strip system if for every $u \in V(J)$, there is at most one neighbour v of u in J so that $N_{uv} \not\subseteq X$; and a vertex $y \in V(G) \setminus V(S, N)$ is *big* (with respect to the strip system) if the set of its neighbours in $V(S, N)$ saturates (S, N). We say $X \subseteq V(S, N)$ is *local* (with respect to the strip system) if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$.

8.4 *Let G be Berge, and let J be a 3-connected graph. Let (S, N) be a J-strip system in G, nondegenerate if $J = K_4$. Let $y \in V(G) \setminus V(S, N)$, and let X be the set of neighbours of y. If there is a choice of rungs, forming a line graph $L(H)$, so that X saturates $L(H)$, then either:*

- *X saturates the strip system, or*
- *there is a J-enlargement with a nondegenerate appearance in G, or*
- *$J = K_4$ and there is an overshadowed appearance of J in G.*

Proof. We define the *fork number* of a choice of rungs to be the number of branch-vertices of H incident in H with ≥ 2 edges in $X_y \cap E(H)$, where $L(H)$ is the line graph formed by this choice of rungs. Let us say that a choice of rungs R_{ij} forming a line graph $L(H)$ is *saturated* if X saturates $L(H)$, and in this case its fork number is $|V(J)|$. If every choice of rungs is saturated, then X saturates the strip system as required, so we may therefore assume that there is some choice of rungs that is not saturated. Let this choice of rungs form $L(H)$, and let us apply 5.8 to $L(H)$. Hence 5.8.1 is false; suppose that 5.8.6 holds. Then $G|(V(L(H)) \cup \{y\}) = L(H')$, and $L(H')$ is an appearance in G of a J-enlargement. We may assume that $L(H')$ is degenerate, for otherwise the theorem holds. Hence $J = K_4$ and $L(H)$ is degenerate. Since the strip system is nondegenerate, the result follows from 8.3. So we may assume that one of 5.8.2-5 holds. Hence for any nonsaturated choice of rungs, the fork number is ≤ 2. Now there are two choices of rungs $R_{ij}(ij \in E(J))$ and $R'_{ij}(ij \in E(J))$, so that the first is saturated and the second is not, differing only on one edge of J; say $R_{ij} = R'_{ij}$ for all edges ij of J except the edge 1-2. Since these two choices of rungs differ only on one edge of J, their fork numbers differ by at most 2; and so $|V(J)| = 4$, and so $J = K_4$.

Let $V(J) = \{1, 2, 3, 4\}$, and $R_{ij} \neq R'_{ij}$ only for the edge 1-2. As usual, the ends of each R_{ij} will be denoted by r_{ij} and r_{ji}, and the ends of each R'_{ij} will be denoted by r'_{ij} and r'_{ji}. For $i = 1, 2, 3, 4$, we denote the triangle $\{r_{ij} : j \in \{1, \ldots, 4\} \setminus i\}$ by T_i, and the triangle $\{r'_{ij} : j \in \{1, \ldots, 4\} \setminus i\}$ by T'_i. The line graphs made by R_{ij} and R'_{ij} are $L(H)$ and $L(H')$ respectively. Since X saturates $L(H)$, it has at least two members in each of $T_1, \ldots, T_4$; and since X does not saturate $L(H')$, there is some T'_i containing at most one member of X. Since $T_3 = T'_3$ and $T_4 = T'_4$, we may assume that $|X \cap T_1| = 2$ and $|X \cap T'_1| = 1$; and so $r_{1,2} \in X$, $r'_{1,2} \notin X$, and exactly one of $r_{1,3}, r_{1,4} \in X$, say $r_{1,3} \in X$ and $r_{1,4} \notin X$.

Also, at least two vertices of T_3 and T_4 are in X, so there are at least two branch-vertices of H' incident in H' with more than one edge in X. By 5.8 applied to H', we deduce that 5.8.5 holds, and so there is an edge ij of J so that R'_{ij} is even and $(X \cap V(L(H'))) \setminus V(R'_{ij}) = (T'_i \cup T'_j) \setminus V(R'_{ij})$. In particular, T'_i and T'_j

both contain at least two vertices in X, and so $i, j \geq 2$. Since $r_{1,3} \in X$ it follows that one of $i, j = 3$, say $j = 3$, and $r_{1,3} \in T_3$; so $R_{1,3}$ has length 0. Now there are two cases, $i = 2$ and $i = 4$. Suppose first that $i = 2$. Then $(X \cap V(L(H'))) \setminus V(R_{2,3}) = \{r_{1,3}, r_{3,4}, r_{2,4}, r'_{2,1}\}$, and since at least two vertices of T_4 are in X it follows that $R_{2,4}, R_{3,4}$ both have length 0, a contradiction since $R'_{ij} = R_{2,3}$ is even. So $i = 4$, and hence $X \setminus V(R_{3,4}) = \{r_{3,1}, r_{4,1}, r_{3,2}, r_{4,2}\}$. Since $r_{1,2}$-$r_{1,4}$-$R_{1,4}$-$r_{4,1}$ is a path of length ≥ 2 with both ends in X and no internal vertex in X, it is even and so $R_{1,4}$ is odd. Since $R_{1,3}$ has length 0, it follows that $R_{3,4}$ is even. Hence one of $R_{2,3}, R_{2,4}$ is odd and the other even, so in particular they do not both have length 0, and hence at most one of $r_{2,3}, r_{2,4} \in X$. Since X saturates $L(H)$, it follows that exactly one of $r_{2,3}, r_{2,4} \in X$ (and hence one of $R_{2,3}, R_{2,4}$ has length 0), and also that $r_{2,1} \in X$. Since no vertex of $R'_{1,2}$ is in X, this restores the symmetry between T'_1 and T'_2.

Suppose that $R_{2,3}$ has length 0. Then $R_{2,4}$ and $R_{1,2}$ are odd, and particular $r_{2,1} \neq r_{1,2}$. If $r_{2,1}$ has no neighbour in $R'_{1,2}$, then $r_{2,1}$-$r_{2,4}$-$r'_{2,1}$-$R'_{1,2}$-$r'_{1,2}$-$r_{1,4}$-$R_{1,4}$-$r_{4,1}$ is an odd path with ends in X and no internal vertex in X, a contradiction. So $r_{2,1}$ has a neighbour in $R'_{1,2}$; but then y can be linked onto the triangle T'_1 via $R'_{1,2}$ and $R_{1,4}$, contrary to 2.4. This proves that $R_{2,3}$ has length ≥ 1. Hence $R_{2,3}$ has odd length and $R_{2,4}$ has length 0, and consequently $R_{1,2}$, $R_{3,4}$ have even length and $R_{1,4}$ is odd. If $R_{3,4}$ has positive length then $L(H)$ is overshadowed (because of the vertex y), and so the theorem holds. We may therefore assumethat $R_{3,4}$ has length 0. If $r_{2,1} \neq r_{1,2}$ and $r_{2,1}$ has no neighbour in $R'_{1,2}$, then $r_{2,1}$-$r_{2,4}$-$r'_{2,1}$-$R'_{1,2}$-$r'_{1,2}$-$r_{1,3}$ is an odd path of length > 1 with ends in X and no internal vertex in X, a contradiction; while if $r_{2,1} \neq r_{1,2}$ and $r_{2,1}$ has a neighbour in $R'_{1,2}$, then then y can be linked onto the triangle T'_1 via $R'_{1,2}$ and $R_{1,4}$, contrary to 2.4. So $r_{2,1} = r_{1,2}$. But then $L(H)$ is degenerate. Since the strip system is nondegenerate, it follows from 8.3 that there is an overshadowed appearance of K_4 in G. This proves 8.4. ∎

Let G be Berge. Let J be a 3-connected graph, such that either:

- there is a nondegenerate appearance of J in G, and there is no J-enlargement with a nondegenerate appearance in G, or
- $J = K_{3,3}$, there is an appearance of J in G, and no J-enlargement that appears in either G or $\overline{G}$.

A J-strip system (S, N) in G is *maximal* if there is no J-strip system (S', N') in G such that $V(S, N) \subset V(S', N')$, and $S'_{uv} \cap V(S, N) = S_{uv}$ for every $uv \in E(J)$, and $N_v \subseteq N'_v$ for every $v \in V(J)$. It is *strongly maximal* if in addition either:

- it is nondegenerate, and there is no J-enlargement with a nondegenerate appearance in G, or
- $J = K_{3,3}$, and no J-enlargement appears in either G or $\overline{G}$.

We need to analyze strongly maximal strip systems. For an edge $uv \in E(J)$, we call the set S_{uv} a *strip* of the strip system.

8.5 *Let G be Berge, and let J be a 3-connected graph. Let (S, N) be a strongly maximal J-strip system in G. Let $F \subseteq V(G) \setminus V(S, N)$ be connected and contain no vertices that are big with respect to (S, N). Then either there is an overshadowed appearance of J in G, or the set of the attachments of F in $V(S, N)$ is local.*

Proof. We may assume that there is no overshadowed appearance of J in G. Let X be the set of attachments of F in $V(S, N)$, and we suppose for a contradiction that X is not local. We may assume that F is minimal (connected) with this property.

(1) *For every choice of rungs, forming $L(H)$ say:*

- *for each $y \in F$, the set of neighbours of y does not saturate $L(H)$, and*
- *if $J = K_4$ then $L(H)$ is not degenerate.*

For by hypothesis no $y \in F$ is big with respect to the strip system, and we may assume that no J-enlargement has a nondegenerate appearance in G, and that there is no overshadowed appearance of J in G, so the first claim follows from 8.4. For the second claim, assume $J = K_4$; then by hypothesis, the strip system is not degenerate, and the claim follows from 8.3. This proves (1).

(2) *There is no $v \in V(J)$ so that $X \cap V(S, N) \subseteq \bigcup(S_{uv} : uv \in E(J))$.*

For assume that v is such a vertex. Since X is not local, there exists $x_1 \in X \cap S_{uv} \setminus N_v$ for some edge uv of J. We also may assume that $X \not\subseteq S_{uv}$, and so there exists $x_2 \in X \cap S_{u'v}$ for some edge $u'v$ of J with $u' \neq u$. For $w \in V(J)$, x_1 belongs to N_w only if $w = u$, and x_2 belongs to N_w only if $w \in \{v, u'\}$; and since x_1, x_2 do not belong to the same strip it follows that $\{x_1, x_2\}$ is not local with respect to the strip system. Make a choice of rungs R_{ij} ($ij \in E(J)$) so that $x_1 \in V(R_{uv})$ and $x_2 \in V(R_{u'v})$, forming $L(H)$. Then $\{x_1, x_2\}$ is not local with respect to $L(H)$, so by (1) we can apply 5.9. Now $L(H)$ is nondegenerate if $J = K_4$, by (1), and the strip system is strongly maximal, and consequently 5.9.1 does not hold. Hence 5.9.2.a holds, and there is a branch D of H with an end d so that $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$. Since x_1 and x_2 are disjoint edges in $X \cap E(H)$, they are not both incident with d, and so one of them is in $E(D \setminus d)$. The branch containing x_2 does not meet x_1, so D is the branch between u and v, and $d = v$. Hence x_2 is incident with v in H, and $\delta_H(v) \subseteq X \cup E(D)$. Consequently, for all neighbours $w \neq u$ of v in J, X contains the vertex of R_{vw} that belongs to N_v, and contains no other vertex of R_{vw}. This restores the symmetry between u' and the other neighbours of v different from u; and since it holds for all choices of the rungs R_{vw}, we deduce that $X \setminus S_{uv} = N_v \setminus S_{uv}$. The minimality of F implies that there is a path P with $V(P) = F$, with ends p_1, p_2 so that p_1 is complete to $N_v \setminus N_{vu}$, and no other vertex of P has any neighbours in $N_v \setminus N_{vu}$, and p_2 is adjacent to x_1, and no other vertex of P has any neighbours in $S_{uv} \setminus N_v$. But then we can add p_1 to N_v and F to S_{uv}, contradicting the maximality of (S, N). This proves (2).

Let $K = \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$.

(3) *There are two disjoint edges in K.*

For make a choice of rungs R_{uv} ($uv \in E(J)$) so that $X \cap V(R_{uv}) \neq \emptyset$ for each $uv \in K$, forming $L(H)$. If there are no two disjoint edges in K, then by (1) and 5.9, it follows that either $X \cap V(L(H))$ is local (with respect to $L(H)$) or 5.9.2.a holds, and in either case there is a branch D of H with an end d so that every edge of $X \cap E(H)$ either is in $E(D)$ or is incident with d. In particular, every branch containing an edge of X is incident with d, and so d meets all edges of J in K, contrary to (2). This proves (3).

From (4) it follow that there exists a 2-element subset of X that is not local, and so from the minimality of F it follows that F is the vertex set of a path, say $f_1, \dots, f_n$. Let us say a choice R_{uv} ($uv \in E(J)$) of rungs is *broad* if there are two disjoint edges ij and hk of J so that X meets both R_{ij} and R_{hk}. From (3) there is a broad choice. We denote the ends of R_{uv} by r_{uv} and r_{vu}, where $r_{uv} \in N_u$ and $r_{vu} \in N_v$.

(4) *For every broad choice of rungs R_{uv} ($uv \in E(J)$), there is a unique edge ij of J so that:*

- *for every w different from j and adjacent to i in J, there is a unique edge between $V(R_{iw})$ and F, namely the edge $r_{iw} f_1$*
- *for every w different from i and adjacent to j in J, there is a unique edge between $V(R_{jw})$ and F, namely the edge $r_{jw} f_n$*
- *for every edge uv of J disjoint from ij, there are no edges between $V(R_{uv})$ and F.*

For by (1) we can apply 5.9, and since the choice of rungs is broad, the minimality of F implies that 5.9.2.b or 5.9.2.c holds. Hence there is an edge ij as in (4). Suppose there is another, say $i'j'$. Since $i'j'$ meets all edges of J that share exactly one end with ij, and J is 3-connected, it follows that $J = K_4$ and the two edges

$ij, i'j'$ are disjoint. Moreover, the unique vertex of $R_{ii'}$ in X is both $r_{ii'}$ and $r_{i'i}$, so $R_{ii'}$ has length 0. Similarly $R_{ij'}, R_{ji'}, R_{jj'}$ all have length 0, and so $L(H)$ is degenerate, contrary to (1). This proves (4).

For a given broad choice of rungs, let us call the edge ij as in (4) the *traversal* for the choice.

(5)*Every choice of rungs is broad.*

For from (3), there is a broad choice, and from (4) in any broad choice R_{uv} $(uv \in E(J))$ there are four different edges $a_1b_1, \ldots, a_4b_4$ of J, so that a_1b_1 is disjoint from a_2, and a_3b_3 is disjoint from a_4b_4, and X meets $R_{a_ib_i}$ for $1 \le i \le 4$. Consequently, if we take another choice of rungs, differing from this one on only one edge, then it too is broad. It follows that every choice is broad. This proves (5).

(6) *There are two choices of rungs with different traversals.*

Take a choice of rungs, and let ij be its traversal; and suppose that all other choices of rungs have the same traversal. Let $A_1 = N_i \setminus S_{ij}$, and $A_2 = N_j \setminus S_{ij}$. From (4),(5), and the uniqueness of ij it follows that $X \cap (V(S,N) \setminus S_{ij}) = A_1 \cup A_2$. Hence $n \ge 2$, for if $n = 1$ then we can add f_1 to N_i, N_j and S_{ij}, contrary to the maximality of the strip system. Choose $x_1 \in A_1$ and $x_2 \in A_2$ in disjoint strips. From (4), x_1 is adjacent to exactly one of f_1, f_n, say f_1. For any other vertex $x_3 \in A_2$, let R_{uv} $(uv \in E(J))$ be an optimal choice of rungs forming $L(H)$ say, so that $x_1, x_3 \in V(H)$. From (4) and (5) it follows that f_n is adjacent to x_3; and so f_n is complete to A_2, and similarly f_1 is complete to A_1. From the minimality of F, there are no other edges between F and $A_1 \cup A_2$; but then we can add f_1 to N_i, f_n to N_j, and F to S_{ij}, contrary to the maximality of the strip system. This proves (6).

Let us say a choice R_{uv} $(uv \in E(J))$ is *optimal* if R_{uv} has a vertex in X for all edges uv in K. For any choice of rungs, there is an optimal choice with the same traversal (just replace rungs that miss X by rungs that meets X wherever possible); so (6) implies that there are two optimal choice of rungs with different traversals. Now for any optimal choice of rungs, if hi is its traversal, then by (4) and the optimality of the choice, it follows that K consists precisely of the edges of J with exactly one end in common with hi, together possibly with hi itself. In particular hi meets all edges in K. We may assume that some other edge jk is the traversal for some other optimal choice; and hence (since J is 3-connected) it follows that $J = K_4$ and jk is disjoint from hi, and neither edge is in K. Hence $V(J) = \{h, i, j, k\}$. Now since the strip system is not degenerate, there is one of the four edges hj, hk, ij, ik whose strip contains a rung of nonzero length; some hj-rung R has length > 0 say. From (4) it follows that exactly one vertex of R is in X, one of its ends; say the end in N_h. Let R_{uv} $(uv \in E(J))$ be any choice of rungs such that $R_{hj} = R$. Since the end of R in N_j does not belong to X, it follows from (4) that for each of R_{hk}, R_{ij}, R_{ik}, its unique vertex in X is its end in $N_h \cup N_i$. Since the choice of these rungs was arbitrary, it follows that $X \cap S_{hk} = N_{hk}$, $X \cap S_{ij} = N_{ij}$, and $X \cap S_{ik} = N_{ik}$. If also $X \cap S_{hj} = N_{hj}$ then hi is the traversal for every choice of rungs, contrary to (6), so $X \cap S_{hj} \neq N_{hj}$. It follows that every ij-rung has length 0; for if one, R' say, has length > 0, then its unique vertex in X is its end in N_i, and by exchanging h and i it follows that $X \cap S_{hj} = N_{hj}$, a contradiction. Similarly all hk and ik-rungs have length 0, and therefore all hj-rungs have even length, since G is Berge. From (1), we may assume that f_1 is adjacent to r_{hj} and complete to S_{hk}, and f_n is complete to $S_{ij} \cup S_{ik}$, and there are no other edges between F and $S_{hk} \cup S_{ij} \cup S_{ik} \cup \{r_{hj}\}$. Let R' be an hj-rung such that its vertex in N_h (r'_{hj}, say) is not its unique vertex in X. Consequently, its other end (r'_{jh}) is its unique vertex in X. By the same argument with hi and jk exchanged, it follows that one of f_1, f_n is complete to $S_{ij} \cup \{r'_{jh}\}$ and the other to $S_{hk} \cup S_{ik}$; and hence $n = 1$. But then the path f_1-r_{hj}-R_{hj}-r_{jh}-r_{ji}-f_1 is an odd hole, a contradiction. This proves 8.5. ∎

If G is Berge, let us say G *admits a line graph decomposition* if either G is a line graph, or it admits a 2-join or an even skew partition. We are now ready to prove 5.7, which we restate:

8.6 *Let G be Berge. Let J be a 3-connected graph, such that either:*

- *there is a nondegenerate appearance of J in G, and there is no J-enlargement with a nondegenerate appearance in G, or*

- $J = K_{3,3}$, *there is an appearance of J in G, and no J-enlargement appears in either G or $\overline{G}$.*

Then G admits a line graph decomposition.

Proof. If $J = K_4$ or $K_{3,3}$ and some appearance of J in G or $\overline{G}$ is overshadowed, the theorem follows from 7.4, so we assume not. Choose an appearance $L(H_0)$ of J in G, nondegenerate if possible. From the hypothesis, it follows that if $L(H_0)$ is degenerate, then $J = K_{3,3}$, and there is no nondegenerate appearance of $K_{3,3}$ in G, and no J-enlargement appears in either G or $\overline{G}$. Regard $L(H)$ as a J-strip system in the natural way, and enlarge it to a maximal J-strip system (S, N). If $L(H_0)$ is nondegenerate then so is the strip system, and so the strip system is strongly maximal.

Let Y be the set of vertices in $V(G) \setminus V(S, N)$ that are big with respect to the strip system, and let $Z = V(G) \setminus (V(S, N) \cup Y)$. By 8.5, for each component of Z, its set of attachments in $V(S, N)$ is local.

(1) *If $Y \neq \emptyset$ then G admits an even skew partition.*

For let Y' be a co-component of Y, and let X be the set of all Y'-complete vertices in $V(G)$. For every choice of rungs, forming $L(H)$ say, and for every $y \in Y'$, the set of neighbours of y in $L(H)$ saturates $L(H)$. Since we may assume that $L(H)$ is nondegenerate if $J = K_4$ (because of 8.3), it follows from it follows from 6.2 that X saturates $L(H)$. Since this holds for every choice of rungs, it follows that X saturates the strip system. Let b_1b_2 be an edge of J, chosen if possible so that $S_{b_1b_2} \not\subseteq X$. Now the sets $(N_{b_1v}\colon\ b_1v \in E(J))$ form a partition of N_{b_1} into say m sets, and at least $m-1$ of them are subsets of X. Choose $m-1$ of them that are subsets of X, not using $N_{b_1b_2}$ if possible (that is, if the other $m-1$ sets are all subsets of X), and let their union be X_1. Define $X_2 \subseteq N_{b_2}$ similarly. We note that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$; for if some vertex of $S_{b_1b_2}$ is not in X then this is clear, while if $S_{b_1b_2} \subseteq X$ then $V(S, N) \subseteq X$ from our choice of b_1b_2, and then from the way we chose X_1 it follows that $X_1 \cap S_{b_1b_2} = \emptyset$, and similarly $X_2 \cap S_{b_1b_2} = \emptyset$, and again our claim holds. This proves that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$. Define X_3 to be the set of vertices in $X \cap V(S, N)$ that are not in $X_1 \cup X_2$, and let X_0 be the set of vertices of X that are not in $V(S, N)$. So X_0, X_1, X_2, X_3 are four disjoint subsets of X, with union X. Note that $Y \setminus Y' \subseteq X_0$. Let B be the union of all components of $G \setminus (Y' \cup X_0 \cup X_1 \cup X_2)$ that have nonempty intersection with $V(L(H)) \setminus S_{b_1b_2}$, and let A be the union of all the other components. We claim that B is nonempty; for there is an edge c_1c_2 of J disjoint from b_1b_2, and no vertex of $S_{c_1c_2}$ is in $N_{b_1} \cup N_{b_2} \cup S_{b_1b_2}$, and therefore no vertex of $S_{c_1c_2}$ is in $Y' \cup X_0 \cup X_1 \cup X_2$. Suppose that A is also nonempty. Then $(A \cup B, Y' \cup X_0 \cup X_1 \cup X_2)$ is a skew partition of G. By 4.2 we may assume it is not loose; and so X_3 is empty (since any vertex of X_3 is in $A \cup B$ and yet is complete to Y'). In particular, $X \cap V(L(H)) \subseteq N_{b_1} \cup N_{b_2}$. Since $X \cap V(L(H))$ saturates the strip system, it follows that for every vertex w of J different from b_1, b_2, w has at most one neighbour in J different from b_1, b_2, and w is adjacent in J to both b_1 and b_2, and all wb_1 and wb_2-rungs have length 0. Since J is 3-connected it follows that $J = K_4$; but then the strip system is not nondegenerate, a contradiction. So in this case the theorem holds. We may therefore assume that A is empty. Now we already saw that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$. Since A is empty, it follows that there is a path of G between $S_{b_1b_2}$ and $V(L(H)) \setminus S_{b_1b_2}$, disjoint from $Y' \cup X_0 \cup X_1 \cup X_2$. Choose such a path, minimal. From the choice of X_1 and X_2 this path has a nonempty interior; from its minimality, none of its internal vertices belong to $L(H)$; since all big vertices are in $Y' \cup X_0$, its interior contains no big vertices; by 8.5, the set of attachments of its interior is local; yet its ends are both attachments of its interior, so there exist $u \in S_{b_1b_2}$ and $v \in V(L(H)) \setminus S_{b_1b_2}$, so that $u, v \notin X_1 \cup X_2$, and yet $\{u, v\}$ is local. Now u, v do not lie in the same strip, and therefore there is some N_w containing them both; and the only $w \in V(J)$ with $u \in N_w$ are b_1, b_2, so we may assume that $u, v \in N_{b_1}$. Since they are not in X_1, and not in the same strip, this is impossible. This proves (1).

We may therefore assume that Y is empty.

(2) *If there is a component F of Z so that for some $v \in V(J)$, all attachments of F in $L(H)$ belong to N_v, then G admits an even skew partition.*

For let $F' = V(G) \setminus (F \cup N_v)$; then $F' \neq \emptyset$, and every path in G from F to F' meets N_v. Since N_v is not co-connected, it follows that $F \cup F', N_v)$ is a skew partition. By 4.2 we may assume it is not loose; and we will prove that it is even. Let the neighbours of v in J be $u_1, \ldots, u_k$; then every co-component of N_v is a subset of one of $N_{vu_1}, \ldots, N_{vu_k}$. Choose a neighbour w of u_1 in J different from v, u_2, choose $n_1 \in N_{u_1 w}$, and choose $n_2 \in N_{vu_2}$. Then n_1, n_2 belong to strips $S_{u_1 w}, S_{vu_2}$, where $u_1 w, vu_2$ are disjoint edges of J; and so n_1, n_2 are not adjacent in G. Let $K = \{n_1\} \cup S_{vu_1} \setminus N_{vu_1}$. Then K is connected (since every vertex of S_{vu_1} is in a vu_1-rung and n_1 is complete to $N_{u_1 v}$), every vertex in N_{vu_1} has a neighbour in K (for the same reason), and n_2 is not in K and has no neighbour in K. (For the last claim, n_2 is not in K since it is in only one strip; and it has no neighbour in $S_{vu_1} \setminus N_{vu_1}$ from the definition of a strip system; and it is not adjacent to n_1 as we already saw.) By 2.6, $(K, N_{u_1 v})$ is balanced, and so by 2.7.1, so is (K, F). By 4.5, G admits an even skew partition. This proves (2).

We assume therefore that there are no such components F of Z. Consequently, for every component F of Z, there is an edge $b_1 b_2$ of J so that all the attachments of F in $L(H)$ are in $S_{b_1 b_1}$. If Z is empty and for all $b_1 b_2$ there is only one $b_1 b_2$-rung, then G is a line graph and the theorem holds. So we may assume that there is an edge $b_1 b_2$ of J such that either there is more than one $b_1 b_2$-rung in $S_{b_1 b_2}$ or there is a component F of Z with all its attachments in $S_{b_1 b_2}$. Let A be the union of $S_{b_1 b_2}$ and any components of Z that have an attachment in $S_{b_1 b_2}$ (and which therefore have attachments only in $S_{b_1 b_2}$), and $B = V(G) \setminus A$. Let $A_1 = N_{b_1 b_2}$, $A_2 = N_{b_2 b_1}$, $B_1 = N_{b_1} \setminus N_{b_1 b_2}$, and $B_2 = N_{b_2} \setminus N_{b_2 b_1}$. Then $A_1, A_2 \subseteq A$, and B_1, B_2 are disjoint subsets of B, and for $i = 1, 2$ A_i is complete to B_i, and there are no other edges between A and B. Also $|B_1| \geq 2$, and we chose $b_1 b_2$ so that if A_1, A_2 both have only one vertex then A is not the vertex set of a path joining them. Since we may assume that G does not admit a 2-join, it follows that $A_1 \cap A_2 \neq \emptyset$. Choose $a \in A_1 \cap A_2$. Then a is complete to $B_1 \cup B_2$, and since $|A| \geq 2$, it follows that $((B \setminus (B_1 \cup B_2)) \cup (A \setminus a), B_1 \cup B_2 \cup \{a\})$ is a skew partition of G. Since $\{a\}$ is a co-component of $B_1 \cup B_2 \cup \{a\}$, 4.1 implies that G admits an even skew partition. This proves 5.7. ∎

9 Bicographs

In this section we handle degenerate appearances of K_4. There is another way to view them, not as a line graph but as a set of paths and antipaths with certain properties, as we shall see.

Let P_1, P_2 be paths in a graph G, and let Q_1, Q_2 be antipaths. Suppose that P_1, P_2, Q_1, Q_2 are pairwise disjoint, and we can label the ends of each P_i as a_i, b_i, and label the ends of each Q_j as x_j, y_j, so that:

- P_1, P_2, Q_1, Q_2 all have length ≥ 1
- there are no edges between P_1 and P_2, and Q_1 is complete to Q_2
- for $(i, j) = (1, 1), (1, 2)$ or $(2, 1)$, the only edges between $V(P_i)$ and $\{x_j, y_j\}$ are $a_i x_j$ and $b_i y_j$, and the only edges between $V(P_2)$ and $\{x_2, y_2\}$ are $a_i y_j$ and $b_i x_j$,
- for $(i, j) = (1, 1), (1, 2)$ or $(2, 1)$, the only nonedges between $V(Q_j)$ and $\{a_i, b_i\}$ are $a_i y_j$ and $b_i x_j$, and the only nonedges between $V(Q_2)$ and $\{a_2, a_2\}$ are $a_i x_j$ and $b_i y_j$.

In these circumstances we call the quadruple (P_1, P_2, Q_1, Q_2) a *knot* in G. Note that if (P_1, P_2, Q_1, Q_2) is a knot then so is (P_2, P_1, Q_1, Q_2), with a suitable relabelling of the ends of the paths and antipaths.

If $L(H)$ is a degenerate appearance of K_4 in G, it can be viewed as a knot. For, in our usual notation, let $R_{1,3}, R_{1,4}, R_{2,3}, R_{2,4}$ have length 0; let $P_1 = R_{1,2}$, $P_2 = R_{3,4}$, let Q_1 be the antipath $r_{1,3}$-$r_{2,4}$, and Q_2 the antipath $r_{1,4}$-$r_{2,3}$. It is easy to check that this is a knot. In fact, this and its complement are the only knots in Berge graphs, as the next theorem shows.

9.1 *Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G. Then all four of P_1, P_2, Q_1, Q_2 have odd length; and either both P_1, P_2 have length 1, or both Q_1, Q_2 have length 1.*

Proof. Certainly P_1 is odd since x_1-a_1-P_1-b_1-y_2-x_1 is a hole, and similarly the other three are odd. Suppose one of P_1, P_2 has length > 1 and one of Q_1, Q_2 has length > 1. By exchanging P_1, P_2 or Q_1, Q_2 we may therefore assume that P_1, Q_1 both have length > 1. Let Y be the interior of Q_1. Then a_1, b_1, a_2, b_2 are all Y-complete, from the last condition in the definition of a knot, and since a_2 has no neighbours in the interior of P_1 it follows from 2.2 that there is a Y-complete vertex (v say) in the interior of P_1. But x_1, y_1 are not Y-complete, and they are adjacent, so a_1-x_1-y_1-b_1 is an odd path between Y-complete vertices and v has no neighbour in its interior, contrary to 2.2. This proves 9.1. ∎

Nevertheless, it turns out to be advantageous to make only limited use of 9.1; it is better to preserve the symmetry between the paths and the antipaths.

Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G; we define K to be the subgraph of G induced on $V(P_1) \cup V(P_2) \cup V(Q_1) \cup V(Q_2)$. (For brevity we say that the knot *induces* K.) We say a subset $X \subseteq V(K)$ is *local* (with respect to the knot) if X is disjoint from one of $V(P_1), V(P_2)$, and X includes neither of $V(Q_1), V(Q_2)$, and $X \cap (V(P_1) \cup V(P_2))$ is complete to $X \cap (V(Q_1) \cup V(Q_2))$. We say X *resolves* the knot if $V(K) \setminus X$ is local with respect to the knot (Q_1, Q_2, P_1, P_2) in $\overline{G}$; that is, if X includes one of $V(Q_1), V(Q_2)$, and X meets both P_1 and P_2, and X contains at least one end of every edge between $V(P_1) \cup V(P_2)$ and $V(Q_1) \cup V(Q_2)$. Conveniently, these definitions almost agree with what we did for line graphs, because of the following.

9.2 *Let (P_1, P_2, Q_1, Q_2) be a knot in a graph G, inducing K, where Q_1, Q_2 both have length 1, and so $K = L(H)$ is an appearance of K_4. Let $X \subseteq V(K)$. Then:*

- *X is local with respect to the knot if and only if it is local with respect to $L(H)$*
- *X resolves the knot if and only if X saturates $L(H)$ and $X \cap (V(P_1) \cup V(P_2)) \neq \emptyset$.*

The proof is obvious and we omit it. This allows us to unify some portions of 5.9 and 6.2, as follows.

9.3 *Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G, inducing K. Assume that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let F be a connected subset of $V(G) \setminus V(K)$, such that its set of attachments in K is not local. Then either:*

1. *there is a vertex in F such that its neighbour set in K resolves the knot, or*
2. *(up to symmetry) there is a path R in F with ends r_1, r_2 such that r_1, a_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and there are no edges between $R \setminus r_1$ and $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and r_2 has a neighbour in $P_1 \setminus a_1$, and there are no edges between $R \setminus r_2$ and $P_1 \setminus a_1$, or*
3. *(up to symmetry) there is an odd path R in F with ends r_1, r_2 such that r_1, a_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and r_2, b_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and there are no edges between $V(R^*)$ and $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and no edges between R and P_1 except possibly r_1a_1 and r_2b_1, or*
4. *there is a vertex $f \in F$ so that (up to symmetry) f, x_1 have the same neighbours in $V(P_1) \cup V(P_2) \cup V(Q_2)$ and f is not adjacent to y_1.*

Proof By 9.1 there are two cases, depending whether Q_1 and Q_2 have length 1 or P_1, P_2 have length 1.

(1) *If Q_1, Q_2 have length 1 then the theorem holds.*

For assume Q_1, Q_2 have length 1. We claim that we may assume there is no vertex in F whose neighbour set resolves K. For suppose f is such a vertex. If f has a neighbour in $V(P_1) \cup V(P_2)$ then statement 1 of the theorem holds, so we assume it does not. But then f is adjacent to all four of x_1, x_2, y_1, y_2, since it has two neighbours in every triangle of K, and then f-x_1-a_1-P_1-b_1-y_1-f is an odd hole, a contradiction. So we assume there is no such f. Hence we may apply 5.9.If 5.9.1 holds then there is an appearance in G of some K_4-enlargement, a contradiction. So 5.9.2 holds. In the notation of 5.9.2, the edge b_1b_2 of J is of one of two types; either N_{b_1} meets N_{b_2} or it does not. In the first case, there is a path R of G with $V(R) \subseteq F$ and with ends r_1 and r_2, so that r_1 is adjacent to a_1, x_2, and r_2 is adjacent to a_2, y_2, and there are no other edges between $V(P)$ and $K \setminus x_1$. If R has length 0 then statement 4 of the theorem holds, while if R has length > 0 then it is even and there is an overshadowed appearance of K_4 in G, a contradiction. In the second case, one of statements 2 and 3 of the theorem hold. This proves (1).

Henceforth we may therefore assume that one of Q_1, Q_2 has length > 1, and therefore by 9.1, both P_1 and P_2 have length 1. Hence $\overline{K} = L(H)$, where $L(H)$ is a degenerate appearance of K_4 in $\overline{G}$.

(2) *If there exists $f \in F$ such that f is not big with respect to $L(H)$ in $\overline{G}$, then the theorem holds.*

For let $f \in F$ have this property. By hypothesis, the neighbour set of f in K does not resolve the knot P_1, P_2, Q_1, Q_2, and so in $\overline{G}$, the set of neighbours of f in $\overline{K}$ is not local with respect to the knot $(\overline{Q_1}, \overline{Q_2}, \overline{P_1}, \overline{P_2})$. But this set does not saturate $L(H)$; so we can apply 5.9 (or, indeed, 5.8) to $\overline{G}$, and deduce, as before, that either there is a K_4-enlargement that appears in $\overline{G}$ (a contradiction), or (up to symmetry) f, a_1 have the same neighbours in $K \setminus a_1$ (but then statement 2 of the theorem holds), or (up to symmetry) f, x_1 have the same neighbours in $V(P_1) \cup V(P_2) \cup V(Q_2)$ (but then either statement 1 or statement 4 of the theorem holds). This proves (2).

We may therefore assume that every $f \in F$ is big with respect to $L(H)$ in $\overline{G}$. Let X be the set of vertices of K which, in G, have no neighbours in F. By hypothesis, $V(K) \setminus X$ is not local with respect to the knot P_1, P_2, Q_1, Q_2 in G, and hence X does not resolve the knot (Q_1, Q_2, P_1, P_2) in $\overline{G}$.

(3) *If X saturates $L(H)$ in $\overline{G}$ then the theorem holds.*

For assume X saturates $L(H)$; then by 9.2, $X \cap (V(Q_1) \cup V(Q_2)) = \emptyset$, and so $a_1, a_2, b_1, b_2 \in X$. Now one of Q_1, Q_2 has length > 1, say Q_1 without loss of generality. Hence, in $\overline{G}$, the path a_1-y_1-Q_1-x_1-b_1 is odd and has length ≥ 5;its ends are complete to F, and its internal vertices are not. By 2.1, F contains a leap; so there exist nonadjacent $f_1, f_2 \in F$ such that Q_1 is the interior of a path R between them. (All this is in $\overline{G}$ - we will tell the reader when we switch back to G.) Now f_1, f_2 have no common neighbour in Q_2 (because R could be completed to an odd hole through any such common neighbour), so by 2.1, f_1, f_2 is also a leap for the path a_1-y_2-Q_2-x_2-b_1 (this path might have length 3, but still we get a leap by 2.1.3, since $\{f_1, f_2\}$ cannot include the interior of any longer antipath between x_2 and y_2). Hence from the symmetry we may assume that f_1 is adjacent to y_1, y_2, and f_2 to x_1, x_2, and there are no other edges between $\{f_1, f_2\}$ and $V(Q_1) \cup V(Q_2)$. Therefore, back in G, we see that a_1, f_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and so do b_1, f_2, and therefore statement 3 of the theorem holds. This proves (3).

We assume therefore that X does not saturate $L(H)$ in $\overline{G}$, and hence, by (2), we may apply 6.2. We deduce that 6.2.3 holds (since Q_1 has length > 1). But then statement 3 of the theorem holds. This proves 9.3. ∎

9.3 suggests that we should attempt to combine paths into strips, as in the section on "Generalized line graphs", but also we should combine antipaths into "antistrips". Let us make that precise.

Let A, B, C be disjoint subsets of $V(G)$. We call $S = (A, C, B)$ a *strip* if A, B are nonempty, and every vertex of $A \cup B \cup C$ belongs to a path between A and B with interior in C. Such a path is called a *rung* of the strip S, or an S-rung. When $S = (A, C, B)$ is a strip, $V(S)$ means $A \cup B \cup C$. The *reverse* of a strip (A, C, B) is the strip (B, C, A). An *antistrip* is a triple that is a strip in $\overline{G}$, and the corresponding antipaths are called *antirungs*. If P is a rung with ends $a \in A$ and $b \in B$, we speak of the "rung a-P-b" for brevity; the reader can deduce which end is in which set from the names of the ends, because we shall always use a, a', a_1 etc for ends in a set called something like A, and so on.

Let $S = (A, C, B)$ be a strip and $T = (X, Z, Y)$ an antistrip, with $V(S) \cap V(T) = \emptyset$. We say S, T are *parallel* if:

- A is complete to X,

- B is complete to Y,
- A is co-complete to Y,
- B is co-complete to X,
- there are no edges between C and $X \cup Y$, and
- Z is complete to $A \cup B$.

We say S, T are *co-parallel* if S, T' are parallel, where T' is the reverse of T.

Now let S_1, S_2 be strips and T an antistrip, where S_1, S_2, T are pairwise disjoint. We say that S_1, S_2 *agree on* T if either S_1, T are parallel and S_2, T are parallel, or both pairs are co-parallel; and they *disagree* if one pair is parallel and the other pair is co-parallel. If S is a strip and T_1, T_2 are antistrips, pairwise disjoint, we define whether T_1, T_2 agree or disagree on S similarly.

Now let S_1, S_2 be strips, and let T_1, T_2 be antistrips, all pairwise disjoint. We call the quadruple (S_1, S_2, T_1, T_2) a *twist* if S_1, S_2 agree on one of T_1, T_2 and disagree on the other. (Equivalently, T_1, T_2 agree on one of S_1, S_2, and disagree on the other.) Note that if (S_1, S_2, T_1, T_2), then so is (S_1', S_2, T_1, T_2), where S_1' is the reverse of S_1.

A *laceration* in a Berge graph G is a family of strips $S_i = (A_i, C_i, B_i)(1 \leq i \leq m)$ together with a family of antistrips $T_j = (X_j, Z_j, Y_j)(1 \leq j \leq n)$, satisfying the following conditions:

- all the strips and antistrips are pairwise disjoint, and all their rungs and antirungs have odd length
- $m, n \geq 2$
- for $1 \leq i < i' \leq m$ there are no edges between S_i and S_i', and for $1 \leq j < j' \leq n$, T_j is complete to T_j'
- for $1 \leq i \leq m$ and $1 \leq j \leq n$, S_i and T_j are either parallel or co-parallel
- for $1 \leq i < i' \leq m$ there exist distinct j, j' with $1 \leq j, j' \leq n$ such that (S_i, S_i', T_j, T_j') is a twist
- for $1 \leq j < j' \leq n$ there exist distinct i, i' with $1 \leq i, i' \leq m$ such that (S_i, S_i', T_j, T_j') is a twist.

(Note that if we replace some (A_i, C_i, B_i) by its reverse, we obtain another laceration.) We denote the laceration by L, and the union of the vertex sets of all its strips and antistrips by $V(L)$. By analogy with what we did for knots, let us say that a subset $X \subseteq V(L)$ is *local* with respect to L if

- at most one of $X \cap V(S_1), \ldots, X \cap V(S_m)$ is nonempty,
- for $1 \leq j \leq n$, every T_j-antirung has a vertex not in X, and
- $X \cap (V(S_1) \cup \cdots \cup V(S_m))$ is complete to $X \cap (V(T_1) \cup \cdots \cup V(T_n))$.

We say X *resolves* L if $V(L) \setminus X$ is local with respect to the laceration in $\overline{G}$ obtained from L by exchanging the strips and antistrips; that is, if

- there is at most one of $T_1, \ldots, T_n$ that is not a subset of X,
- for $1 \leq i \leq m$, every S_i-rung meets X, and
- X contains at least one end of every edge between $V(S_1) \cup \cdots \cup V(S_m)$ and $V(T_1) \cup \cdots \cup V(T_n)$.

A laceration L in G is *maximal* if there is no laceration L' in G with $V(L) \subset V(L')$.

9.4 *Let G be Berge, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal laceration in G. Let $f \in V(G) \setminus V(L)$, and let X be the set of neighbours of f in $V(L)$. Then either X is local with respect to L, or X resolves L.*

Proof. Let L have strips $S_i = (A_i, C_i, B_i)(1 \le i \le m)$ and antistrips $T_j = (X_j, Z_j, Y_j)(1 \le j \le n)$.

(1) *Let $1 \le i \le m$, and $1 \le j \le n$; let a_i-P_i-b_i be an S_i-rung, and x_j-Q_j-y_j a T_j-antirung. Then either $X \cap V(P_i) \neq \emptyset$, or $V(Q_j) \not\subseteq X$.*

For suppose that X includes $V(Q_1)$ and is disjoint from $V(P_1)$ say. By reversing S_2 we may assume that S_1 and S_2 agree on T_1; and we may assume they disagree on T_2. Let a_2-P_2-b_2 be any S_2-rung, and x_2-Q_2-y_2 any T_2-antirung. Then (P_1, P_2, Q_1, Q_2) is a knot, so by 9.1, we may assume (taking complements if necessary) that Q_1 has length 1. But then f-x_1-a_1-P_1-b_1-y_1-f is an odd hole, a contradiction. This proves (1).

From (1), taking complements if necessary, we may assume that for all $1 \le j \le n$, and for all T_j-rungs Q_j, $V(Q_j) \not\subseteq X$.

(2) *X meets at most one of $V(S_1), \dots, V(S_m)$.*

For suppose that X meets both S_1 and S_2 say. We may assume that S_1, S_2, T_1, T_2 is a twist. For $i = 1, 2$ choose an S_i-rung P_i so that $X \cap V(P_i) \neq \emptyset$, and for $j = 1, 2$ choose any T_j-antirung Q_j. By (1), f has nonneighbours in both Q_1, Q_2. But then (P_1, P_2, Q_1, Q_2) is a knot, and setting $F = \{f\}$ violates 9.3, a contradiction. This proves (2).

We may assume that X is not local with respect to L, and so we may assume that there is an S_1-rung a_1, P_1, b_1 and a T_1-antirung x_1-Q_1-y_1 containing nonadjacent members of X. By reversing each T_j if necessary, we may assume that S_1 is parallel to each T_j. In particular, a_1x_1 is an edge, and so is b_1y_1. Since the interior of Q_1 is complete to $V(P_1)$, we may assume that $x_1 \in X$, and $X \cap (V(P_1) \setminus a_1) \neq \emptyset$. Let $2 \le j \le n$, and let x_j-Q_j-y_j be any T_j-antirung. For definiteness we assume $j = 2$. Now T_1, T_2 agree on S_1, and so there is some S_i on which they disagree, say S_2. Let a_2-P_2-b_2 be any S_2-rung. Then (P_1, P_2, Q_1, Q_2) is a knot, with union K say, and $X \cap V(K)$ is not local with respect to K (since $x_1 \in X$, and $X \cap (V(P_1) \setminus a_1) \neq \emptyset$). By 9.3, it follows that 9.3.2 holds, and hence f,a_1 have the same neighbours in $V(Q_1) \cup V(Q_2)$. In particular, $V(Q_2) \setminus \{y_2\} \subseteq X$. Since $V(Q_2) \not\subseteq X$, it follows that $y_2 \notin X$; since this holds for all Q_2, we deduce that $X \cap V(T_2) = X_2 \cup Z_2$; and since the same holds for all antistrips of L except T_1, we deduce that $X \cap V(T_j) = X_j \cup Z_j$ for $2 \le j \le n$. Since our only assumption about T_1 was that $X \cap X_1 \neq \emptyset$, and since we have shown that the same is true for all T_j, we can replace T_1 by T_2 say, and deduce similarly that $X \cap V(T_1) = X_1 \cup Z_1$. But then we can add f to A_1, contrary to the maximality of the laceration. This proves 9.4. ∎

9.5 *Let G be Berge, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal laceration in G. Let $F \subseteq V(G) \setminus V(L)$ be connected, such that for each $f \in F$, the set of its neighbours in $V(L)$ is local with respect to L. Then the set of attachments of F in $V(L)$ is local with respect to L.*

Proof. Let L have strips $S_i = (A_i, C_i, B_i)(1 \le i \le m)$ and antistrips $T_j = (X_j, Z_j, Y_j)(1 \le j \le n)$. Suppose not, and choose a counterexample F with F minimal. Let X be its set of attachments in $V(L)$.

(1) *$X \not\subseteq V(T_1) \cup V(T_n)$.*

For suppose it is. Since X is not local, we may assume that X includes $V(Q_1)$ for some T_1-antirung x_1-Q_1-y_1. Let $2 \le j \le n$, and let x_j-Q_j-y_j be a T_j-antirung. Then we can choose some S_i, S_i' to make a twist, and if we choose an S_i-ring and S_i'-rung and apply 9.3 to the resultant knot, we deduce (since no vertices of S_i and S_i' are in X) that 9.3.3 holds. This has several consequences. First, it implies that there is an odd path in F with vertices $f_1, \dots, f_k$ say, which is either parallel or co-parallel to Q_1, and either parallel or co-parallel to Q_j; and there are no edges between $\{f_2, \dots, f_{k-1}\}$ and $Q_1 \cup Q_j$. Hence the set of attachments of $\{f_1, \dots, f_k\}$ is not local with respect to L, and so $F = \{f_1, \dots, f_k\}$ from the minimality of F. Second, every vertex of Q_j is in X, and since this holds for all Q_j it follows that $V(T_j) \subseteq X$. By exchanging T_1 and T_j it follows that $V(T_1) \subseteq X$. Moreover, since this holds for all j we deduce that $X = V(T_1) \cup \dots V(T_n)$. This restores the symmetry between T_1 and $T_2, \ldots, T_n$. Third, this shows that there are no edges between $\{f_2, \ldots, f_{k-1}\}$ and $V(T_1) \cup \cdots \cup V(T_n)$. Fourth, for $1 \leq j \leq n$ every vertex in Z_j is adjacent to both f_1, f_k. Since k is even, this proves that either $k = 2$ or $Z_1 \cup \cdots \cup Z_n = \emptyset$. Fifth, every vertex in $X_1 \cup Y_1 \cdots \cup X_n \cup Y_n$ is adjacent to exactly one of f_1, f_n; let U be the set of those adjacent to f_1, and V those adjacent to f_n. For the moment fix j with $1 \leq j \leq n$. Every T_j-antirung has one end in U and the other in V; let M_j be the union of the vertex sets of all T_j-antirungs x_j-Q_j-y_j such that $x_j \in U$, and N_j the union of all those with $x_j \in V$. Since there is no T_j-antirung with both ends in M_j or both ends in Ni_j, it follows that $M_j \cap N_j = \emptyset$, and there are no nonedges betwen M_j and N_j except possibly between $M_j \cap X_j$ and $N_j \cap X_j$, or between $M_j \cap Y_j$ and $N_j \cap Y_j$. Suppose there is such a nonedge; and choose T_j-antirungs x_j-Q_j-y_j,x'_j-Q'_j-y'_j where $x_j \in U$ is nonadjacent to $x'_j \in V$, say. Choose $a_1 \in A_1 \cup B_1$ adjacent to x_j and x'_j; then a-x_j-f_1-$\cdots$-f_k-x'_j-a is an odd hole. This proves that M_j is complete to N_j. Now if M_j is nonempty, then $(M_j \cap X_j, M_j \cap Z_j, M_j \cap Y_j)$ is an antistrip, and similarly if N_J is nonempty it also induces a antistrip. We call these the *offspring* of T_j. (If one of M_j, N_j is empty, then the other equals $V(T_j)$, and so the only offspring of T_j is T_j itself; and otherwise it has two.) Also, there is a new strip $S_0 = (\{f_1\}, \{f_2, \ldots, f_{k-1}\}, \{f_k\})$. Note that

- for all j with $1 \leq j \leq n$, S_0 is parallel or antiparallel with the offspring of T_j
- for all i with $1 \leq i \leq m$, there exists j with $1 \leq j \leq n$ such that S_0, S_i disagree on one of the offspring of T_j, and there exists j so that S_0, S_i agree on one of the offspring of T_j. For if the first were false, say, then each of the T_j's has only one offspring, and we could add f_1 to A_i, $\{f_2, \ldots, f_{k-1}\}$ to C_i, and f_k to B_i, contradicting the maximality of the laceration; whike if the second were false we could do the same with f_1, f_k exchanged.
- if T'_1, T'_2 are each offspring of one of $T_1, \ldots, T_n$, then there exists i with $0 \leq i \leq m$ such that T'_1, T'_2 agree of S_i; and there exists i such that they disagree. For this is clear if they are offspring of different parents, since their parents were in a twist together; while if they are both offspring of the same T_j, then they disagree on S_0 and agree on all of $S_1, \ldots, S_m$.

It follows from these observations that the set of strips $S_0, \ldots, S_m$, together with the set of offspring of $T_1, \ldots, T_n$, forms a new laceration, contrary to the maximality of L. This proves (1).

(2) *X meets exactly one of $S_1, \ldots, S_m$.*

For by (1) it meets at least one of these sets; suppose it meets two, say S_1 and S_2. We may assume that (S_1, S_2, T_1, T_2) is a twist. For $i = 1, 2$ choose an S_i-rung a_i-P_i-b_i so that X meets P_i, and for $j = 1, 2$ let x_j-Q_j-y_j be a Q_j-antirung. Then (P_1, P_2, Q_1, Q_2) is a knot K say, and $X \cap V(K)$ is not local with respect to K. It follows from 9.3 that one of 9.3.1, 9.3.4 holds; and in either case there is a vertex $f \in F$ with neighbours in P_1 and in P_2. Hence the set of neighbours of f in $V(L)$ is not local with respect to L; and so the set of neighbours of f resolves L, by 9.4. But this contradicts a hypothesis of the theorem, and hence proves (2).

(3) *$V(Q_j) \not\subseteq X$, for $1 \leq j \leq n$, and for every T_j-antirung Q_j.*

For suppose that $V(Q_1) \subseteq X$ for some T_1-rung x_1-Q_1-y_1. By (2) we may assume that X meets S_1 and none of $S_2, \ldots, S_m$. Let $2 \leq j \leq n$, and choose i with $2 \leq i \leq m$ so that (S_1, S_i, T_1, T_j) is a twist. Let Q_j be an x_j-T_j-y_j-antirung, let a_1, P_1, b_1 be an S_1-rung so that X meets P_1, and let a_i-P_i-b_i be an S_i-rung. Hence (P_1, P_i, Q_1, Q_j) is a knot. Let us apply 9.3. By (2) it follows that 9.3.3 holds. This has several consequences. First, from the minimality of F, $G|F$ is an odd path f_1-$\cdots$-f_k such that f_1, a_1 have the same neighbours in $V(Q_1 \cup Q_j)$, and so do f_k, b_1, and there are no edges between F and $V(P_1)$ except possibly f_1a_1 and f_kb_1. Since X meets P_1, it follows that at least one of these two edges is present; and therefore they both are, since f_1-$\cdots$-f_k is an odd path and so is P_1 (for otherwise the union of these two path, with one of x_1, y_1, would induce an odd hole). So f_1 is adjacent to a_1 and to no other vertex of P_1, and f_n to b_1 and to no other vertex of P_1. Second, $V(Q_j) \subseteq X$. Since this holds for all Q_j it follows that $V(T_j) \subseteq X$; and by exchanging T_1 and

T_j we deduce that $V(T_1) \cup \cdots \cup V(T_n) \subseteq X$. Moreover $\{f_2, \ldots, f_{k-1}\}$ is complete to $V(T_1) \cup \cdots \cup V(T_n)$. Third, let x'_j-Q'_j-y'_j be some other T_j-antirung. By the same argument applied to the knot (P_1, P_i, Q_1, Q'_j), we deduce that again 9.3.3 holds, and so one of f_1, f_k is adjacent to x'_j and the other to y'_j. Furthermore, the one adjacent to x'_j is also adjacent to a_1; and so in fact f_1 is adjacent to x'_j. Since this holds for all choices of Q_j and of j, it follows that f_1, a_1 have the same neighbours in $V(T_1) \cup \cdots \cup V(T_n)$, and so do f_k, b_1. Hence we can add f_1 to A_1, $\{f_2, \ldots, f_{k-1}\}$ to C_1 and f_k to B_1, contrary to the maximality of the laceration. This proves (3).

Since X is not local with respect to L, we may assume from (2) and (3) that there exist a vertex of $X \cap V(S_1)$ and a vertex of $X \cap V(T_1)$ that are nonadjacent. By reversing $T_1, \ldots, T_n$ we may assume that S_1 is parallel to each T_j. Since every vertex of Z_1 is complete to $V(S_1)$, we may assume that there is an S_1-rung a_1-P_1-b_1 and a T_1-antirung x_1-Q_1-y_1 so that $x_1 \in X$ and $X capV(P_1 \setminus a_1) \neq \emptyset$. Let $2 \leq j \leq n$, and choose i with $2 \leq i \leq m$ so that (S_1, S_i, T_1, T_j) is a twist. Let P_i be an S_i-rung, and let Q_j be a T_j-antirung. So (P_1, P_i, Q_1, Q_j) is a knot K say, and $X \cap V(K)$ is not local with respect to K. By 9.3 and (3) 9.3.2 holds; let R be the path in F satisfying one of 9.3.2. Then the set of attachments of R in $V(L)$ is not local with respect to L, and so $V(R) = F$ from the minimality of F. Hence F is a path with vertices f_1-$\cdots$-f_k say. Since $x_1 \in X$, it follows that one of f_1, f_k is adjacent to x_1, and we may assume that f_1 is adjacent to x_1. By 9.3.2, f_1 is also adjacent to x_j and to all internal vertices of Q_1, Q_j, and to neither of y_1, y_j, and none of $f_2, \ldots, f_{k-1}$ have neighbours in $V(Q_1 \cup Q_j)$, and f_k has a neighbour in $P_1 \setminus a_1$, and f_k has no neighbours in $V(Q_1 \cup Q_j)$. For any other choice of Q_j the same happens, and f_1, f_k cannot become exchanged since f_1 has neighbours in Q_1 and f_k has none. We deduce that f_1 is complete to $X_j \cup Z_j$ and co-complete to Y_j; and $\{f_2, \ldots, f_k\}$ is co-complete to $V(T_j)$. In particular there is a vertex of $X \cap V(S_1)$ and a vertex of $X \cap V(T_j)$ that are nonadjacent, and so by exchanging T_1 and T_j in the above argument, we deduce that f_1 is complete to $X_1 \cup Z_1$ and co-complete to Y_1; and $\{f_2, \ldots, f_k\}$ is co-complete to $V(T_1)$. Since this holds for all j, it follows that a_1, f_1 have the same neighbours in $V(T_1) \cup \cdots \cup V(T_n)$, and there are no edges between $\{f_2, \ldots, f_k\}$ and $V(T_1) \cup \cdots \cup V(T_n)$. But then we can add f_1 to A_1 and $\{f_2, \ldots, f_k\}$ to C_1, contrary to the maximality of the laceration. This proves 9.5. ∎

The main result of this section is the following.

9.6 *Let G be a Berge graph, and let $L(H)$ be an appearance of K_4 in G. Then either G or $\overline{G}$ admits a line graph decomposition, or G is a bicograph.*

Proof. By 5.1 we may assume that $L(H)$ is degenerate, and hence there is a laceration in G; choose a maximal laceration L. Let L have strips $S_i = (A_i, C_i, B_i)(1 \leq i \leq m)$ and antistrips $T_j = (X_j, Z_j, Y_j)(1 \leq j \leq n)$. By 5.1 and 5.2, we may assume that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and by 7.4 that there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L have strips $S_i = (A_i, C_i, B_i)(1 \leq i \leq m)$ and antistrips $T_j = (X_j, Z_j, Y_j)(1 \leq j \leq n)$. By 9.4 we can partition $V(G) \setminus V(L)$ into two sets M, N, where for every vertex in M its set of neighbours in $V(L)$ is local with respect to L, and for every vertex in N, its set of neighbours in $V(L)$ resolves L.

(1) *If there exists $f \in N$ with a nonneighbour in $V(S_1) \cup \cdots \cup V(S_m)$ then the theorem holds.*

For let left f have a nonneighbour in S_1 say. Let N_1 be the co-component of N containing f, and let X be the set of all N_1-complete vertices in $V(G)$. From 9.5 applied in the complement, it follows that X resolves L. Since f has a nonneighbour in $V(S_1)$, there is a vertex u of S_1 not in X. Let U be the component of $V(G) \setminus (X \cup N)$ containing u. We claim that U is disjoint from $V(L) \setminus V(S_1)$, and no vertex in $V(S_2) \cup \cdots \cup V(S_m)$ has a neighbour in U. For suppose not; then there is a path P say in G, from $V(S_1)$ to $V(L) \setminus V(S_1)$, with $(X \cup N) \cap V(P) \subseteq V(S_2) \cup \cdots \cup V(S_m)$; choose such a path minimal. It follows that no internal vertex of P is in $V(L)$ or in $X \cup N$; and since X meets every edge between $V(S_1)$ and $V(L) \setminus V(S_1)$, and there are no edges between $V(S_1)$ and $V(S_2) \cup \cdots \cup V(S_m)$, it follows that Pi^* is nonempty. Now no vertex of P^* is in N, since $N \subseteq N_1 \cup X$; and so there is a component M_1 of M including P^*. From 9.5, the set of attachments of M_1 in $V(L)$ is local with respect to L. Since it has an attachment in $V(S_1)$ it therefore has none in $V(S_2) \cup \cdots \cup V(S_m)$. But the ends of P are attachments of M_1, they are nonadjacent, and one is in $V(S_1)$ and the other is not, a contradiction. This proves that U is disjoint from $V(L) \setminus V(S_1)$. Let X' be the set of vertices in X with neighbours in U, and let $V = V(G) \setminus (U \cup N_1 \cup X')$. Then V is nonempty because $V(S_2) \subseteq V$; and so $U \cup V, N_1 \cup X'$ is a skew partition of G. Since there is a vertex of S_2 in X (because X resolves L), and this vertex is in V, we deduce that the skew partition is loose, and hence by 4.2 G admits an even skew partition. This proves (1).

From (1) we may assume that N is complete to $V(S_1) \cup \cdots \cup V(S_m)$, and by taking complements, that M is co-complete to $V(T_1) \cup \cdots \cup V(T_n)$.

(2) *If M, N are both nonempty then the theorem holds.*

For let M_1 be a component of M, and N_1 a co-component of N. By taking complements we may assume that there is a nonedge between M_1 and N_1. Let X be the set of all N_1-complete vertices in G. Since the set of attachments of M_1 in $V(L)$ is local by 9.5, and since it has no attachments in $V(T_1) \cup \cdots \cup V(T_n)$, we may assume that all its attachments are in $V(S_1)$. Let $V = V(G) \setminus (M_1 \cup N_1 \cup V(S_1))$. Since $V(S_1) \subseteq X$, it follows that $M_1 \cup V, N_1 \cup V(S_1)$ is a skew partition of G, and since there are N_1-complete vertices with no neighbours in M_1 (for instance, any vertex of $V(S_2)$), the skew partition is loose, and by 4.2 G admits an even skew partition. This proves (2).

(3) *If M, N are both empty then the theorem holds.*

For then by 9.1, we may assume that for $1 \leq j \leq n$ all Q_j-antirungs have length 1. If $|V(S_1)| > 2$, then $(V(S_1), V(L) \setminus V(S_1))$ is a 2-join of G; for every vertex in $V(T_1) \cup \cdots \cup V(T_n)$ is adjacent either complete to A_1 and co-complete to $B_1 \cup C_1$, or complete to B_1 and co-complete to $A_1 \cup C_1$ (since all the antirungs have length 1). So we may assume that each S_i has only two vertices. In particular, every S_i-rung has length 1, so by taking complements the same argument shows that we may assume every $V(T_j)$ has only two vertices. But then G is a bicograph and the theorem holds. This proves (3).

From (2) and (3), and taking complements if necessary, we may assume that N is empty and M is nonempty. For $1 \leq i \leq m$ let M_i be the union of the components of M that have an attachment in $V(S_i)$, and let M_0 be the union of the components of M that have no attachments in $V(L)$. Then $M_0, M_1, \ldots, M_n$ are pairwise disjoint and have union M. If M_0 is nonempty then G is not connected, and therefore either it admits an even skew partition, or $|V(G)| \leq 4$ and G is bipartite, so we may assume that M_0 is empty. Since M is nonempty we may assume that M_1 is nonempty. Suppose that $z \in Z_1$. Then z is complete to $V(S_1)$, and hence if we define $V = V(G) \setminus M_1 \cup V(S_1) \cup \{z\}$, then $(M_1 \cup V, V(S_1) \cup \{z\})$ is a skew partition of G, and by 4.1 G admits an even skew partiton. So we may assume that Z_1 is empty, and similarly every Z_j is empty. Then $(M_1 \cup V(S_1), V(G) \setminus (M_1 \cup V(S_1))$ is a 2-join of G. This proves 9.6. ∎

10 The even prism

We proved that if G is Berge and contains a line graph of a non-squared bipartite subdivision of K_4, then G admits a line graph decomposition Now we want to show that the same conclusion holds for Berge graphs that contain an "even" prism. (Incidentally, the results of this section are independent of those in the previous one; these two sections could be in either order. If we were happy to show just that either G or $\overline{G}$ admits a line graph decomposition, then we could use the results of the previous section to slightly simplify this section. But it would make only a very little difference, so we might as well prove the stronger result.) We begin with some results about prisms in general.

For $i = 1, 2, 3$ let R_i be a path in G with distinct ends a_i, b_i, pairwise vertex-disjoint. Suppose that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles in G, and for $1 \leq i < j \leq 3$ the only edges between R_i and R_j are $a_i a_j$ and $b_i b_j$.Then the union of these three paths and the two triangles is a prism in G, and for brevity we speak of "the prism R_1, R_2, R_3", and call the two triangles the "triangles of the prism". Let the prism be K. A subset $X \subseteq V(G)$ *saturates* the prism if at least two vertices of each triangle belong to X; and a vertex is *big* with respect to the prism if its neighbour set saturates it. A subset $X \subseteq V(K)$ is *local* with respect to the prism if either $X \subseteq V(R_i)$ for some i, or X is a subset of one of the triangles.

We start with the obvious

10.1 *Let R_1, R_2, R_3 be a prism in a Berge graph G; then R_1, R_2, R_3 all have the same parity.*

The proof is clear.

A prism is *even* if the three paths $R_, R_2, R_3$ have even length, and *odd* otherwise. Even prisms are easier than odd ones. One reason is that all the prisms contained in a degenerate appearance of K_4 are odd, so if we succeed in growing an even prism to become the line graph of a bipartite subdivision of K_4, this line graph is guaranteed to be nondegenerate.

10.2 *Let R_1, R_2, R_3 be a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Assume no vertex in F is big with respect to the prism. Then there is a path f_1-$\cdots$-f_n in F with $n \geq 1$, such that (up to symmetry) either:*

1. *f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in R_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, and (therefore) G has an induced subgraph which is the line graph of a bipartite subdivision of K_4, or*
2. *$n \geq 2$, f_1 is adjacent to a_1, a_2, a_3, and f_n is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*
3. *$n \geq 2$, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*
4. *f_1 is adjacent to a_1, a_2, and f_n has at least one neighbour in $R_3 \setminus a_3$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K) \setminus a_3$.*

Proof. We may assume that F is minimal such that it is connected and its set of attachments in K is not local. Let X be the set of attachments of F in K. For $1 \leq i \leq 3$, if $X \cap V(R_i) \neq \emptyset$, let c_i and d_i be the vertices of R_i in X closest (in R_i) to a_i and to b_i respectively, and let C_i, D_i be the subpaths of R_i between a_i and c_i, and between d_i and b_i respectively. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$.

We claim that some two-element subset of X is not local. For since $X \not\subseteq B$ we may assume that c_1 exists and $c_1 \neq b_1$. Since $X \not\subseteq V(R_1)$, we may assume d_2 exists. If $d_2 \neq a_2$ then $\{c_1, d_2\}$ is the desired subset; so we may assume $d_2 = a_2$, and similarly $d_3 = a_3$ if d_3 exists. Since $X \not\subseteq A$, it follws that $d_1 \neq a_1$, and then $\{a_2, d_1\}$ is the desired subset. So some two-element subset $\{x_1, x_2\}$ of X is not local. Consequently x_1, x_2 are not adjacent. From the minimality of F, there is a path with vertices $x_1, f_1, \dots, f_n, x_2$ so that $F = \{f_1, \dots, f_n\}$.

(1) *If $n = 1$ then the theorem holds.*

For assume $n = 1$; then $F = \{f_1\}$. Since X is not local it meets at least two of the paths; suppose it only meets R_1 and R_2. Suppose that $c_1 = d_1$. Then we may assume that $c_1 \notin A$ and $c_2 \neq b_2$, by exchanging A and B if necessary; but then c_1 can be linked onto the triangle A, via the paths c_1-C_1-a_1, c_1-f_1-c_2-C_2-a_2, and c_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4, since f has at most one neighbour in A. So c_1 is different from d_1, and similarly c_2 is different from d_2 (and in particular, $c_2 \neq b_2$). Suppose that c_1 is nonadjacent to d_1. Then since f_1 is not big, we may assume it has at most one neighbour in A, by exchanging A and B if necessary; but it can be linked onto A, via f_1-c_1-C_1-a_1, f_1-c_2-C_2-a_2 and f_1-d_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4. So c_1, d_1 are adjacent, and similarly so are c_2, d_2, but then statement 1 of the theorem holds. So we may assume that X meets all three of R_1, R_2, R_3. Since f_1 is not big, we may assume that it has at most one neighbour in A, by exchanging A and B if necessary, and therefore cannot be linked onto A. Since it has neighbours in all three of R_1, R_2, R_3, it follows that for at least two of these paths, the only neighbour of f_1 in this path is in B. We may assume therefore that $c_1 = b_1$ and $c_2 = b_2$. Since X is not local, $c_3 \neq b_3$; but then statement 4 of the theorem holds. This proves (1).

We may therefore assume that $n \geq 2$. Let X_1 be the set of attachments of $F \setminus f_1$, and X_2 the set of attachments of $F \setminus f_n$. From the minimality of F, both X_1 and X_2 are local. Moreover, $X = X_1 \cup X_2$, and for $2 \leq i \leq n-1$, every neighbour of f_i in K belongs to $X_1 \cap X_2$.

(2) *If $X_1 \subseteq A$ and $X_2 \subseteq V(R_1)$ then the theorem holds.*

For then f_1 has at least one neighbour in $R_1 \setminus a_1$, and f_n is adjacent to at least one of a_2, a_3, and there are no other edges between F and $V(K) \setminus a_1$. If f_n is adjacent to both b_2, b_3 then statement 4 of the theorem holds, so we assume it is not adjacent to b_3. But then a_2 can be linked onto the triangle B, via a_2-f_1-$\cdots$-f_n-d_1-D_1-b_1, a_2-R_2-b_2, a_2-a_3-R_3-b_3, contrary to 2.4. This proves (2).

From (2), since both X_1 and X_2 are local, we may assume that either $X_1 \subseteq A$ and $X_2 \subseteq B$, or $X_1 \subseteq V(R_2)$ and $X_2 \subseteq V(R_1)$. In either case $X_1 \cap X_2 = \emptyset$, so none of $f_2, \ldots, f_{n-1}$ has any neighbours in $V(K)$. So X_1 is the set of neighbours of f_n in $V(K)$, and x_2 is the set of neighbours of f_1 in $V(K)$.

(3) *If $X_1 \subseteq A$ and $X_2 \subseteq B$ then the theorem holds.*

For then we may assume that f_n is adjacent to a_1 and f_1 to b_2. Suppose first that n has the same parity as the length of R_1. Since a_2-R_2-b_2-f_1-$\cdots$-f_n-a_2 is not an odd hole, it follows that f_n is not adjacent to a_2, and similarly f_1 is not adjacent to b_1. Since a_3-R_3-b_3-b_2-f_1-$\cdots$-f_n-a_1-a_3 is not an odd hole, either f_n is adjacent to a_3 or f_1 to b_3, and not both, as we saw before. But then statement 4 of the theorem holds. Now suppose that n has different parity from the length of R_1. Since a_1-a_2-R_2-b_2-f_1-$\cdots$-f_n-a_1 is not an odd hole, f_n is adjacent to a_2, and similarly f_1 to b_1. Since f_1 is not big it follows that $n \geq 2$. If there are no more edges between F and $V(K)$ then statement 3 of the theorem holds, so we may assume that f_n is adjacent to a_3. By the same argument as before it follows that f_1 is adjacent to b_3, and then statement 2 of the theorem holds. This proves (3).

From (2) and (3) we may assume that $X_1 \subseteq V(R_2)$ and $X_2 \subseteq V(R_1)$. So f_1 is adjacent to the vertices of R_1 that are in X, and f_n to those of R_2 in X. If $c_1 = d_1$, then from the symmetry we may assume that $c_1 \neq a_1$, and $c_2 \neq b_2$; but then c_1 can be linked onto A, via c_1-C_1-a_1, c_1-f_1-$\cdots$-f_n-c_2-C_2-a_2, c_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4. So $c_1 \neq d_1$ and similarly $c_2 \neq d_2$; and in particular $c_2 \neq b_2$. If c_1, d_1 are nonadjacent, then we may assume f_1 has at most one neighbour in A, by exchanging A and B if necessary, since f_1 is not big; but f_1 can be linked onto A via f_1-c_1-C_1-a_1, f_1-$\cdots$-f_n-c_2-C_2-a_2, f_1-d_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4. So c_1, d_1 are adjacent, and similarly so are c_2, d_2; but then statement 1 of the theorem holds. This proves 10.2. ▮

10.3 *Let R_1, R_2, R_3, K, F be as in 10.2, and suppose that 10.2.1 holds. Then either R_1 and R_2 both have length 1, or G admits a line graph decomposition.*

Proof. For let f_1-$\cdots$-f_n be a path in F so that f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in P_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$. Then $G|(V(K) \cup \{f_1, \ldots, f_n\}$ is a line graph of a bipartite subdivision of K_4. By 5.1 we may assume it is degenerate. Hence the prism is odd, for all prisms contained in a degenerate appearance of K_4 are odd. So R_3 s odd, and therefore so is the path f_1-$\cdots$-f_n, and the other four "rungs" of this line graph have length 0. In particular, R_1 and R_2 both have length 1. This proves 10.3. ▮

There is also a tighter version of 10.2, the following.

10.4 *Let R_1, R_2, R_3 be a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that no vertex in F is big with respect to K.*

Let x_1 be an attachment of F in the interior of R_1, and assume that there is another attachment x_2 of F not in R_1. Assume G does not admit a line graph decomposition. Then there is a path f_1-$\cdots$-f_n in F such that x_1 is adjacent to f_1 and to none of $f_2, \ldots, f_n$, and (up to symmetry) f_n is adjacent to a_2, a_3 and to no other vertices of $R_2 \cup R_3$, and none of $f_1, \ldots, f_{n-1}$ has any neighbours in $R_2 \cup R_3$.

Proof. We may assume F is minimal such that it is connected, x_1 is one of its attachments, and it has some other attachment x_2 in $R_2 \cup R_3$. Hence there is a path x_1-v_1-$\cdots$-v_m-x_2 where $F = \{v_1, \ldots, v_m\}$. By 10.2, there is a subpath f_1-$\cdots$-f_n of v_1-$\cdots$-v_m such that one of 10.2.1-4 holds. From the minimality of F, v_m is the only vertex of F with a neighbour in $V(R_2) \cup V(R_3)$, and in particular, at most one vertex of f_1-$\cdots$-f_n has a neighbour in $V(R_2) \cup V(R_3)$. We deduce that f_1-$\cdots$-f_n does not satisfy 10.2.2 or 10.2.3. Suppose it satisfies 10.2.1. By 10.3 we may assume that the path f_1-$\cdots$-f_n is odd, and it joins two of R_1, R_2, R_3 that are both of length 1. Since R_2 has length ≥ 2 (because x_1 is in its interior) it follows that f_1, f_n are distinct vertices of F both with neighbours in $V(R_2) \cup V(R_3)$, a contradiction. So we may assume that f_1-$\cdots$-f_n satisfies 10.2.4. Since f_1 has two neighbours in one of A,B it follows that $f_1 = v_m$, and so $f_n = v_{m+1-n}$. We may assume by exchanging A and B if necessary that v_m has at least two neighbours in A, and so f_n has neighbours in a unique $V(R_i \setminus a_i)$ with $1 \leq i \leq 3$. Suppose first that f_n has neighbours in $V(R_3 \setminus a_3)$, and hence f_1 is adjacent to a_1, a_2 and to no other vertex of $R_1 \cup R_2$. From the minimality of F, n = 1; but then f_1 can be linked onto the triangle B, via the path between f_1 and a_1 with interior in $(F \setminus f_1) \cup (V(R_1) \setminus a_1)$, the path f_1-a_2-R_2-b_2, and the path between f_1 and b_3 with interior in $R_3 \setminus a_3$, a contradiction. So f_n has no neighbours in $V(R_3 \setminus a_3)$, and similarly no neighbours in $V(R_2 \setminus a_2)$, and so it has neighbours in $V(R_1 \setminus a_1)$, and f_1 is adjacent to a_2 and a_3 and to no other vertices of $R_2 \cup R_3$. But then the theorem holds. This proves 10.4. ∎

Another useful corollary of 10.2 is the following.

10.5 *Let R_1, R_2, R_3 be a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that if the prism is even then no vertex in F is big with respect to K. Assume that the set of attachments of F in K is not local, but none are in $V(R_3)$. Assume also that G does not admit a line graph decomposition. Then $|F| \geq 2$, and the set of attachments of F in K is precisely $\{a_1, b_1, a_2, b_2\}$.*

Proof. If there is a big vertex $v \in F$, then since it has no neighbours in R_3, it is adjacent to a_1 and b_2, and since v-a_1-a_3-R_3-b_3-b_2-v is a hole, it follows that the prism is even, contrary to the hypothesis. So there is no big vertex in F. By 10.4 no internal vertex of R_1 or R_2 is an attachment of F. By 10.2, there is a path in F satisfying one of 10.2.1-4; and since it has no attachments in R_3, it must satisfy 10.2.1 or 10.2.3, and in either case a_1, b_1, a_2, b_2 are all attachments of F. Since no vertex in F is big it follows that $|F| \geq 2$. This proves 10.5. ∎

The next result is very closely related to 7.4 - indeed, they are in some sense the same theorem, or can be viewed as instances of the same theorem. Unifying them would perhaps make this paper a little shorter, but more difficult to follow, and we decided not to do it. In any case the proof of the next result is fairly short.

10.6 *Let R_1, R_2, R_3 be an even prism in a Berge graph G, and let y be big with respect to the prism. Then G admits a line graph decomposition.*

Proof. Let the prism be K. Let Y be a maximal co-connected set of vertices, each of which is big with respect to K. Let X be the set of all Y-complete vertices in G. By 7.2, X saturates K. Consequently there is one of R_1, R_2, R_3 with both ends in X, say R_1. Let

$$\begin{aligned} X_1 &= \{a_1, b_1\} \\ X_2 &= X \cap V(K) \setminus X_1 \\ X_0 &= X \setminus V(K). \end{aligned}$$

(1) *If $F \subseteq V(G)$ is connected and some vertex of $V(R_1^*)$ has a neighbour in F, and so does some vertex of $V(R_2) \cup V(R_3)$, then $F \cap (X_0 \cup X_1 \cup Y)$ is nonempty.*

Suppose for a contradiction that some F exists not satisfying (1), and choose it minimal. Hence $G|F$ is a path, disjoint from K. Consequently $F \cap X = \emptyset$. Suppose some vertex in $v \in F$ is big with respect to K. Then since $v \notin X$ it follows that v has a nonneighbour in Y, and so $Y \cup v$ is co-connected; the maximality of Y therefore implies that $v \in Y$, and hence $F \cap Y \neq \emptyset$ and the claim holds. So we may assume that no vertex in F is big. Let x_1 be an attachment of F in R_1^*. By 10.4, we may assume there is a path $f_1\text{-}\cdots\text{-}f_n$ in F such that x_1 is adjacent to f_1 and to none of $f_2, \ldots, f_n$, and f_n is adjacent to a_2, a_3 and to no other vertices of $R_2 \cup R_3$, and none of $f_1, \ldots, f_{n-1}$ has any neighbours in $R_2 \cup R_3$. Now there is a path P from f_n to b_1 with interior in $\{f_1, \ldots, f_{n-1}\} \cup V(R_1 \setminus a_1)$. Hence P, R_2, R_3 form a prism, and so by 7.3, $f_n \in X$, a contradiction. This proves (1).

It follows from (1) that there is a partition of $V(G) \setminus (X_0 \cup X_1 \cup Y)$ into two sets L and M say, where there is no edge between L and M, and $V(R_1^*) \subseteq L$ and $V(R_2) \cup V(R_3) \subseteq M$. So $(L \cup M, X_0 \cup X_1 \cup Y)$ is a skew partition of G. Since at least two vertices of A are in X and only one is in X_1, there is a vertex of X in M, and so the skew partition is loose. By 4.2 the result follows. This proves 10.6. ∎

The main result of this section is the following.

10.7 *Let G be a Berge graph, containing an even prism. Then G admits a line graph decomposition.*

Proof. Since G contains an even prism, we can choose in G a collection of nine sets

$$\begin{matrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{matrix}$$

with the following properties:

- all these sets are nonempty and pairwise disjoint
- for $1 \leq i < j \leq 3$ A_i is complete to A_j and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$
- for $1 \leq i \leq 3$, every vertex of $A_i \cup B_i \cup C_i$ belongs to a path between A_i and B_i with interior in C_i
- some path between A_1 and B_1 with interior in C_1 is even.

We call this collection of nine sets a *hyperprism*. Let H be the subgraph of G induced on the union of the nine sets. Choose the hyperprism with $V(H)$ maximal. For $1 \leq i \leq 3$, a path from A_i to B_i with interior in C_i is called an i-rung. Let us write $S_i = A_i \cup B_i \cup C_i$ for $1 \leq i \leq 3$, and $A = A_1 \cup A_2 \cup A_3$, and $B = B_1 \cup B_2 \cup B_3$.

(1) *For $1 \leq i \leq 3$, all i-rungs have even length.*

For we are given that some 1-rung R_1 say has even length. Let R_2 be an 2-rung; then the union of R_1 and R_2 induces a hole, and so R_2 is even. Hence every 2- or 3-rung is even, and hence so is every 1-rung. This proves (1).

A subset $X \subseteq V(H)$ is *local* (with respect to the hyperprism) if X is a subset of one of S_1, S_2, S_3, A or B.

(2) *For every connected subset F of $V(G) \setminus V(H)$, its set of attachments in H is local.*

For suppose not. Choose F minimal, and let X be the set of attachments of F in H. Suppose first that therie exists $x_1 \in X \cap C_1$. Since X is not local, we may assume that there exists $x_2 \in X \cap S_2$. For $i = 1, 2, 3$ choose an i-rung R_i with ends $a_i \in A_i$ and $b_i \in B_i$, so that for $i = 1, 2$, $x_i \in V(R_i)$.Then R_1, R_2, R_3 is an even prism K say. By 10.6 we may assume no vertex in F is big with respect to K; so by 10.4, we may assume that there is a path $f_1\text{-}\cdots\text{-}f_n$ in F such that f_1 is adjacent to a_2, a_3, and f_n has at least one neighbour in $R_1 \setminus a_1$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K) \setminus a_1$. Since this holds for all choices of R_3 it follows that f_1 is complete to A_3 and there are no edges between $\{f_1, \dots, f_n\}$ and $B_3 \cup C_3$. Since $a_3 \in X$ the same conclusion follows for all choices of R_2, and so f_1 is complete to A_2 and there are no edges between $\{f_1, \dots, f_n\}$ and $B_2 \cup C_2$. But then we can add f_1 to A_1 and $\{f_2, \dots, f_n\}$ to C_1, contradicting the maximality of the hyperprism. It follows that $X \cap C_1 = \emptyset$, and similarly $X \cap C_2, X \cap C_3 = \emptyset$. We claim there is a 2-element subset of X which is also not local. For we may assume $X \cap A_1 \neq \emptyset$; and hence if X meets B_2 or B_3 our claim holds. If not, then it meets B_1 (since it is not a subset of A) and meets $A_2 \cup A_3$ (since it is not a subset of S_1), and again the claim holds. So there is a subset $\{x_1, x_2\}$ of X which is not local. We may assume that $x_1 \in A_1$ and $x_2 \in B_2$. From the minimality of F, there is a path $x_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}x_2$ with $F = \{f_1, \dots, f_n\}$.

Suppose first that n is even. For any 3-rung R_3 with ends $a_3 \in A_3$ and $b_3 \in B_3$, $x_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}x_2\text{-}b_3\text{-}R_3\text{-}a_3\text{-}x_1$ is not an odd hole, and so some vertex of R_3 is in X. Since $X \cap C_3 = \emptyset$, and a_3 has no neighbour in $\{f_2, \dots, f_n\}$ from the minimality of F, and similarly b_3 has no neighbour in $\{f_1, \dots, f_{n-1}\}$, it follows that either f_1 is adjacent to a_3, or f_n to b_3 (and not both, since otherwise $f_1\text{-}\cdots\text{-}f_n\text{-}b_3\text{-}R_3\text{-}a_3$ is an odd hole). From the symmetry we may assume that f_n is adjacent to b_3. By exchanging S_2 and S_3 it follows that for every 2-rung with ends $a_2 \in A_2$ and $b_2 \in B_2$, either f_1 is adjacent to a_2 or f_n to b_2, and not both. Suppose that f_n is complete to $B_2 \cup B_3$; then f_1 has no neighbours in $S_2 \cup S_3$, and we can add f_1 to A_1, f_n to B_1 and $f_2, \dots, f_{n-1}$ to C_1, contrary to the maximality of the hyperprism. So f_n is not complete to $B_2 \cup B_3$, and hence f_1 has a neighbour in one of A_2, A_3, say A_3; and by exchanging S_1 and S_2 it follows that for every 1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$, either f_1 is adjacent to a_1 or f_n to b_1 and not both. In particular, f_1 has no neighbours in B and f_n has none in A. For $i = 1, 2, 3$ let A'_i be the set of neighbours of f_1 in A_i, and let $A''_i = A_i \setminus A'_i$; let B''_i be the set of neighbours of f_n in B_i, and let $B'_i = B_i \setminus B''_i$. We have shown so far that every i-rung is either between A'_i and B'_i or between A''_i and B''_i. Let C'_i be the union of the interiors of the i-rungs between A'_i and B'_i, and C''_i the union of the interiors of the i-rungs between A''_i and B''_i. We observe that $C_i = C'_i \cup C''_i$. Moreover, $C'_i \cap C''_i = \emptyset$, for otherwise there would be an i-rung between A'_i and B''_i. For the same reason there are no edges between $A'_i \cup C'_i$ and $C''_i \cup B''_i$, and no edges between $A''_i \cup C''_i$ and $C'_i \cup B'_i$. We claim that A'_i is complete to A''_i. For if not, let R'' be an i-rung with ends $a'' \in A''_i$ and $b'' \in B''_i$, and let $a' \in A'_i$ be nonadjacent to a. Since we have seen that f_1 has neighbours in at least two of A_1, A_2, A_3, we may choose $a \in A'_j$ for some $j \neq i$. Then $a\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b''\text{-}R''\text{-}a''\text{-}a$ is an odd hole, a contradiction. So A'_i is complete to A''_i for each i, and similarly B'_i is complete to B''_i for each i. We showed already that we may assume that A'_1, A''_2, A'_3, A''_3 are all nonempty. But then the nine sets

$$\begin{array}{ccc} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C'_2 \cup C'_3 & B'_2 \cup B'_3 \\ A''_1 \cup A''_2 \cup A''_3 \cup \{f_1\} & C''_1 \cup C''_2 \cup C''_3 \cup \{f_2, \dots, f_n\} & B''_1 \cup B''_2 \cup B''_3 \end{array}$$

form a hyperprism, contrary to the maximality of $V(H)$. This completes the argument when n is even.

Now assume n is odd. f_1 has a neighbour a_1 say in A_1; let R_1 be a 1-rung with ends a_1 and b_1 say. Similarly let R_2 be a 2-rung with ends a_2 and b_2, where $b_2 \in B_2$ is adjacent to f_n. Since $a_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b_2\text{-}b_1\text{-}R_1\text{-}a_1$ is not an odd hole, it follows that $b_1 \in X$, and similarly $a_2 \in X$. From the minimality of F, one of b_1, a_2 is adjacent to f_1 and the other to f_n, and neither has any more neighbours in F. Suppose that f_n is not adjacent to b_1; so f_1 is adjacent to b_1, and $n \geq 2$, and f_n is adjacent to a_2. But then $b_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b_2\text{-}b_1$ is an odd hole, a contradiction. This proves that f_n is adjacent to b_1 and f_1 to a_2. Hence for all $1 \leq i \leq 3$, and for every i-rung with ends $a \in A$ and $b \in B$, $a \in X$ if and only if $b \in X$, and if so then f_1 is adjacent to a and f_n to b. Consequently, for every vertex in $X \cap A$, f_1 is its unique neighbour in F, and for every vertex in $X \cap B$, f_n is its unique neighbour in F. For $1 \leq i \leq 3$, let

$$A_i' = A_i \cap X$$
$$B_i' = B_i \cap X$$
$$A_i'' = A_i \setminus X$$
$$B_i'' = B_i \setminus X.$$

Let C_i' be the union of the interior of the i-rungs between A_i' and B_i', and C_i'' the union of the interior of the i-rungs between A_i'' and B_i''. We have seen that every i-rung is of one of these two types, and so $C_i = C_i' \cup C_i''$. Moreover, since there is no rung between A_i' and B_i'', it follows that $C_i' \cap C_i'' = \emptyset$, and there are no edges between $A_i' \cup C_i'$ and $C_i'' \cup B_i''$, and similarly no edges between $A_i'' \cup C_i''$ and $C_i' \cup B_i'$. We have seen that f_1 has neighbours in at least two of A_1, A_2, A_3, and f_n has neighbours in at least two of B_1, B_2, B_3. We claim that also F_1 has nonneighbours in at least two of A_1, A_2, A_3, and the same for f_n. For suppose not, and f_1 is complete to $A_1 \cup A_2$ say. Then f_n is complete to $B_1 \cup B_2$; by 10.6 it follows that $n > 1$, and so we can add f_1 to A_3, f_n to B_3 and $f_2, \ldots, f_{n-1} to C_3$, contrary to the maximality of $V(H)$. This proves that F_1 has nonneighbours in at least two of A_1, A_2, A_3, and similarly f_n has nonneighbours in at least two of B_1, B_2, B_3. Let $1 \leq i \leq 3$; we claim that A_i' is complete to A_i''. For we may assume that $i = 1$; suppose that $a' \in A_1'$ and $a'' \in A_1''$ are nonadjacent, and let R'' be a 1-rung with ends a'', b''. Choose $a \in A_2'' \cup A_3''$ and $b \in B_2' \cup B_3'$; then a, b are not adjacent since all rungs have even length, and so a-a'-f_1-$\cdots$-f_n-b-b''-R''-a''-a is an odd hole, a contradiction. This proves that A_i' is complete to A_i'' for $i = 1, 2, 3$, and similarly B_i' is complete to B_i''. We have seen that we may assume that A_1', A_2' are nonempty. But then

$$\begin{array}{ccc} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2' \cup C_3' & B_2' \cup B_3' \\ A_1'' \cup A_2'' \cup A_3'' \cup \{f_1\} & C_1'' \cup C_2'' \cup C_3'' \cup \{f_2, \ldots, f_{n-1}\} & B_1'' \cup B_2'' \cup B_3'' \cup \{f_n\} \end{array}$$

is a hyperprism, contrary to the maximality of $V(H)$. This proves (2).

Suppose F is a component of $V(G) \setminus V(H)$, and all its attachments are in A. Then $(V(G) \setminus A, A)$ is a skew partition of G. We must show that G admits an even skew partition. Choose $b_2 \in B_2$ and $a_3 \in A_3$. Then $B_1 \cup C_1 \cup \{b_2\}$ is connected, and all vertices in A_1 have neighbours in it. By 2.6, $(A_1, B_1 \cup C_1 \cup \{b_2\})$ is balanced, and so by 2.7.1, so is (A_1, F). By 4.5, G admits an even skew partition. So we may assume there is no such F, and the same for B.

From (2) it follows that for every component of $V(G) \setminus V(H)$, all its attachments in H are a subset of one of S_1, S_2, S_3. Let X be the union of S_1 and all components of $V(G) \setminus V(H)$ whose attachment set is a subset of S_1, and let $Y = V(G) \setminus X$. Then $|Y| \geq 4$, and so either (X, Y) is a 2-join in G, or $|X| = 3$ and both A_1, B_1 have one element and X is the vertex set of a path between these two vertices. We may assume the latter, and the same for S_2 and S_3; and so $|V(G)| = 9$, and G is a line graph. This proves 10.7. ∎

11 Step-connected strips

Our next target is the statement analogous to 10.7 for long odd prisms, but we need to creep up on it in stages. (A warning: we shall not prove the exact analogue, and we don't know if it is true. We need to permit more types of decomposition, namely 2-joins in $\overline{G}$, and M-joins.) The key idea is to start with a prism of three paths, R_0, R_1, R_2, where R_0 has length ≥ 3, and to grow the union of the other two paths into a kind of strip (*one* strip, not two) with a richer internal structure than we have seen hitherto, what we call being "step-connected". If we expand the union of these two paths into a maximal step-connected strip, then the remainder of the graph attaches to this structure in ways that we can exploit. In this section we introduce step-connected strips, and prove some preliminary lemmas about them.

Let (A, C, B) be a strip in G. A *step* is a pair a_1-R_1-b_1, a_2-R_2-b_2 of rungs so that

- $V(R_1) \cap V(R_2) = \emptyset$
- a_1 is adjacent to a_2, and b_1 to b_2, and there are no other edges between $V(R_1)$ and $V(R_2)$.

The edges a_1a_2 and b_1b_2 such that there exists a step as above are called *stepped* edges. We say that the strip is *step-connected* if every vertex of $A \cup B \cup C$ is in a step, and for every partition (X, Y) of A or of B into two nonempty sets, there is a step R_1, R_2 so that R_1 has an end in X and R_2 has an end in Y. (This second condition is equivalent to requiring that the subgraph of G with vertex set A and edges the stepped edges within A be connected, and the same for B.)

Let (A, C, B) be a step-connected strip in a Berge graph G. A vertex $v \in V(G) \setminus A \cup B \cup C$ is a *left-star* for the strip if it is complete to A and co-complete to $B \cup C$, and it is a *right-star* if it is complete to B and co-complete to $A \cup C$. A *stride* (with respect to S) is a path a-R-b such that a is a left-star, b is a right-star, and there are no edges between the interior of R and $V(S)$. Here we distinguish between a-R-b and b-R-a; we follows the convention that when describing a stride relative to a strip, the end which is the left-star is listed first.

11.1 *Let G be a Berge graph, admitting no line graph decomposition, and let $S = (A, C, B)$ be a step-connected strip in G. Let a_0-R_0-b_0 be a stride. Suppose that $v \in V(G) \setminus V(S)$ has a neighbour in $A \cup C$, and has no neighbour in B; and that P is a path in $G \setminus (V(S) \cup \{a_0\})$ from v to b_0, and there are no edges between P^* and $V(S)$. Then v is a left-star.*

Proof. Let F be a connected subset of $V(P)$, containing v and disjoint from $V(R_0)$, and with an attachment in $R_0 \setminus a_0$.

(1) *For every step a_1-R_1-b_1, a_2-R_2-b_2 , if v has a neighbour in $R_1 \cup R_2$ then v is adjacent to a_1, a_2 and to no other vertices of $R_1 \cup R_2$.*

For assume v has a neighbour in R_1 say, and hence in $R_1 \setminus b_1$. Now R_0, R_1, R_2 is a prism K say, and no vertex in F is big with respect K since no vertex in F is adjacent to b_1 or b_2. Yet F has an attachment in $R_0 \setminus a_0$ and one in $R_1 \setminus b_1$, so its set of attachments is not local. Since b_1 is not an attachment of F, it follows from 10.5 that F has an attachment in R_2; and therefore v has a neighbour in $R_2 \setminus b_2$. If v has any neighbours in $R_1 \cup R_2$ different from a_1, a_2, say a neighbour in the interior of R_1, then v can be linked onto the triangle b_0, b_1, b_2, via the paths v-P-b_0, from v to b_1 with interior in $R_1 \setminus a_1$, and from v to b_2 with interior in R_2; but this contradicts 2.4. This proves (1).

From (1) it follows that v has no neighbour in C (since every vertex is in a step), and therefore v has at least one neighbour in A; and from (1) again, v has no nonneighbour in A (for otherwise we could choose the step in (1) with v adjacent to a_1 and not to a_2, since the strip is step-connected.) This proves 11.1. ∎

11.2 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph; let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a stride. Let $v \in V(G)$ $V(S)$ have a neighbour in $V(S)$, and be nonadjacent to b_0. Let P be a path in $G \setminus (V(S) \cup \{a_0\})$ from v to b_0, and let Q be a path in $G \setminus (V(S) \cup \{b_0\})$ from v to a_0, such there are no edges from $P^* \cup Q^*$ to $V(S)$. Then either v is B-complete, or v is a left-star.*

Proof. If v has no neighbours in B, then by 11.1 either G admits a line graph decomposition or v is a left-star, so we may assume v has a neighbour in B. Since we may assume it is not B-complete, there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that v is adjacent to b_1 and not to b_2. Let $F \subseteq V(Q)$ be connected, containing v and disjoint from $V(R_0)$, with an attachment in $R_0 \setminus b_0$. Now R_0, R_1, R_2 form a prism K say, and no vertex of F is big with respect to K since none of them has two neighbours in $\{b_0, b_1, b_2\}$. But there is an attachment of F in $R_0 \setminus b_0$, and b_1 is also an attachment of F, so its set of attachments is not local with respect to the prism. By 10.2, one of 10.2.1-4 holds. By 9.6, 10.2.1 does not hold. Also 10.2.2, 10.2.3 do not hold, since v is the only vertex in F with neighbours in $A \cup B$. So 10.2.4 holds, and therefore F has an attachment in R_2, and so v has a neighbour in R_2. But then v can be linked onto the triangle $\{b_0, b_1, b_2\}$, via v-Q-b_0, v-b_1, and the path from v to b_2 with interior in R_2, contrary to 2.4. This proves 11.2. ∎

We remark:

11.3 *Let G be a Berge graph, not admitting a line graph decomposition, and let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a stride. Then every rung of the strip has odd length.*

Proof. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step. Then these three paths form a prism, and by 10.7 they all have odd length. In particular R_0 has odd length. For any rung a-R-b, the hole a_0-P_0-b_0-b-R-a-a_0 has even length, and so R is odd. This proves 11.3. ∎

11.4 *Let G be a Berge graph, not admitting a line graph decomposition, and let $S = (A, C, B)$ be a step-connected strip in G. Let $Z \subseteq V(G) \setminus (A \cup B \cup C)$ be connected, such that there are no edges between Z and $A \cup B \cup C$. There is no co-connected set $Q \subseteq V(G) \setminus (A \cup B \cup C \cup Z)$ such that:*

- *some right-star has a neighbour in Z and a nonneighbour in Q,*
- *some vertex in B has a nonneighbour in Q,*
- *some left-star with a neighbour in Z is Q-complete,*
- *every vertex in Q has a neighbour in Z,*
- *every vertex in Q has a neighbour in $A \cup B \cup C$, and*
- *no vertex in Q is a left-star.*

Proof. Suppose that such a set Q exists. Let a_0 be a left-star with a neighbour in Z complete to Q, and let b_0 be a right-star with a neighbour in Z and a nonneighbour in Q. Let R_0 be a path between a_0 and b_0 with interior in Z. Hence a_0-R_0-b_0 is a stride. By 11.3 R_0 and every rung has odd length. Since some vertex in B has a nonneighbour in Z, there is an antipath q_1-$\cdots$-q_n in Q such that q_1 is not adjacent to b_0 and q_n is not adjacent to some vertex in B. Choose such an antipath with n minimum. Let B_1 be the set of neighbours of q_n in B, and $B_2 = B \setminus B_1$. So $B_2 \neq \emptyset$, and by 11.1 it follows that $B_1 \neq \emptyset$. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 with $b_1 \in B_1$ and $b_2 \in B_2$.

(1) *$n \geq 2$.*

For suppose $n = 1$. Then q_1 is adjacent to a_0 and to b_1, and not to b_0, so by 10.5 q_1 has a neighbour in $R_2 \setminus b_2$. Since q_1 also has a neighbour in Z, it can be linked onto the triangle $\{b_0, b_1, b_2\}$, via a path from q_1 to b_0 with interior in Z, the path q_1-b_1, and the path from q_1 to b_2 with interior in R_2, contrary to 2.4. This proves (1).

(2) *$(A \cup B \cup C, \{b_0, q_1, \ldots, q_n\})$ is balanced.*

For there exists $b_1 \in B_1$, which is complete to $\{q_1, \ldots, q_n\}$ from the minimality of n. But b_1 has no neighbour in Z, so by 2.6, $(Z, \{b_0, q_1, \ldots, q_n\})$ is balanced. Since Z is connected and every vertex in $\{b_0, q_1, \ldots, q_n\}$ has a neighbour in Z, the claim follows from 2.7.1. This proves (2).

Now the path a_0-a_2-R_2-b_2-b_1 is odd, and its ends are complete to $\{q_1, \ldots, q_n\}$; so by (2) and 2.1, there are two adjacent vertices u, v in this path, both complete to $\{q_1, \ldots, q_n\}$. Suppose that the hole a_0-R_0-b_0-b_2-R_2-a_2-a_0 has length ≥ 6. Then one of u, v is nonadjacent to both b_0, b_2, say v, and hence n is odd, since v-b_0-q_1-$\cdots$-q_n-b_2-v is an antihole; but b_1 is adjacent to b_0 and b_2, and has no other neighbours in this hole, and is complete to $\{q_1, \ldots, q_n\}$, contrary to 3.3. So the hole has length 4, and in particular a_2 is adjacent to b_2 and is complete to $\{q_1, \ldots, q_n\}$, and a_0 is adjacent to b_0. Hence n is odd, because b_1-v-b_0-q_1-$\cdots$-q_n-b_2-a_0-b_1 is an antihole, and so a_2-b_0-q_1-$\cdots$-q_n-b_2 is an odd antipath, contrary to (1). This proves 11.4. ∎

11.5 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph; and let $S = (A, C, B)$ be a step-connected strip in G. Let $Z \subseteq V(G) \setminus V(S)$ be connected, such that there are no edges between Z and $V(S)$. Assume that there is a left- and right-star, both with neighbours in Z. Then there is no co-connected set $Q \subseteq V(G) \setminus (V(S) \cup Z)$ such that:*

- *some vertex in A has a nonneighbour in Q,*
- *some vertex in B has a nonneighbour in Q,*
- *some left-star with a neighbour in Z is Q-complete,*
- *every vertex in Q has a neighbour in Z,*
- *every vertex in Q has a neighbour in $A \cup B \cup C$, and*
- *no vertex in Q is a left-star.*

Proof. Suppose that some such Q exists, and for fixed G, choose Z and Q with $|Z| + |Q|$ maximum so that all the hypotheses of the theorem remain satisfied (possibly exchanging "left" and "right"). Let N be the set of vertices of G not in Z but with a neighbour in Z. Hence $Q \subseteq N$, and every left- or right-star with a neighbour in Z is in N.

(1) *Every vertex in N has a neighbour in $A \cup B \cup C$.*

For suppose $v \in V(G) \setminus Z$ has a neighbour in Z and has none in $A \cup B \cup C$. Let $Z' = Z \cup \{v\}$. Certainly Z' is connected and disjoint from $A \cup B \cup C$, and there are no edges between Z' and $A \cup B \cup C$; and Z' is disjoint from Q since every vertex in Q has a neighbour in $A \cup B \cup C$. Every left-star with a neighbour in Z also has a neighbour in Z', and vice versa for vertices in N. In particular, no vertex in Q is a left-star with a neighbour in Z'. It follows that the hypotheses of the theorem remain true, contrary to the maximality of $|Z| + |Q|$. This proves (1).

(2) *There is no left- or right-star in Q, and every left- and right-star with a neighbour in Z is Q-complete.*

For we are given that there is no left-star in Q. Suppose there is a right-star with a neighbour in Z, either in Q or with a nonneighbour in Q. Then there is an antipath with interior in Q, between B and some right-star with a neighbour in Z; but the set of vertices in such an antipath contradicts 11.4. So there is no right-star in Q, and every right-star with a neighbour in Z is Q-complete. We are given that there is a right-star with a neighbour in Z, and so all hypotheses of the theorem are true with "left" and "right" exchanged. It follows by the same argument, therefore, that every left-star with a neighbour in Z is Q-complete. This proves (2).

Since $Q \subseteq N$ is co-connected, it is contained in some co-connected component of N, say N_1.

(3) *There is a left- or right-star in N_1.*

For let N_2 be the union of all the co-components of N different from N_1. Assume that no left- and right-star is in N_1. Let $Y = V(G) \setminus Z \cup N$; then there are no edges between Z and Y, from definition of N. Also, $A \cup B \cup C \subseteq Y$, so in particular $Y \neq \emptyset$, and hence $(Z \cup Y, N)$ is a skew partition of G. By (1), every vertex in N has a neighbour in $A \cup B \cup C$ and in Z, and so every vertex in N_1 has a neighbour in B (since otherwise it would be a left-star by 11.1 and therefore belong to N_2). Now $(B \cup C, N_1)$ is balanced, by 2.6, since any left-star is complete to N_1 and has no neighbour in $B \cup C$. Since $B \cup C$ is connected (because every vertex of $B \cup C$ is in a step and the strip is step-connected), it follows from 2.7.1 that (Z, N_1) is balanced. From 4.5, G admits an even skew partition, a contradiction. This proves (3).

From (3), $N_1 \neq Q$; and hence there is a vertex $v \in N \setminus Q$ with a nonneighbour in Q. From the maximality of $|Z| + |Q|$, replacing Q by $Q \cup \{v\}$ violates one of the hypotheses of the theorem. But $v \notin Z$ since it has a neighbour in $A \cup B \cup C$ by (1); and v is not a left-star since all left-stars in N are Q-complete; and so no left-star in N is $Q \cup \{v\}$-complete. Since they are all Q-complete, it follows that v is nonadjacent to every left-star in N . Similarly v is nonadjacent to every right-star in N.

(4) *v is complete to $A \cup B$.*

For suppose not; then from the symmetry we may assume that v has a nonneighbour in B. By 11.2, v is a left-star, a contradiction. This proves (4).

Choose an antipath q_1-$\cdots$-q_k in Q, such that $q_1 = v$ and q_k has a nonneighbour in $A \cup B$, with k minimum. From (4), $k \geq 2$. From the minimality of K, $\{q_1, \ldots, q_{k-1}\}$ is complete to $A \cup B$. Let A_1 be the set of neighbours of q_k in A, and $A_2 = A \setminus A_1$, and define $B_1, B_2 \subseteq B$ similarly. So $A_2 \cup B_2$ is nonempty.

(5) *k is even.*

For $A_2 \cup B_2$ is nonempty. If there exists $a_2 \in A_2$, let $b_0 \in N$ be a right-star; then b_0-q_1-$\cdots$-q_k-a_2-b_0 is an antihole, so it follows that k is even. The result follows similarly if B_2 is nonempty.

(6) *A_1 is complete to B_2, and A_2 is complete to B_1.*

For suppose that $a_1 \in A_1$ and $b_2 \in B_2$ are nonadjacent. Let $b_0 \in N$ be a right-star; then by (5), b_0-q_1-$\cdots$-q_k-b_2-a_1-b_0 is an odd antihole, a contradiction. So A_1 is complete to B_2 and similarly A_2 is complete to B_1. This proves (6).

(7) *A_1, B_1, A_2, B_2 are all nonempty.*

For we may assume that A_2 is nonempty. Since the strip is step-connected, every vertex in A has a nonneighbour in B, and so by (6), $B_1 \neq B$. Hence B_2 is also nonempty. Since q_k has a neighbour in $A \cup B \cup C$ it follows that it has a neighbour in B, by 11.1, and similarly it has a neighbour in A. This proves (7).

Now the strip is step-connected, and so there is a step a_1-R-b_2, a_2-R'-b_1 with $a_1 \in A_1$ and $a_2 \in A_2$. Since a_1 is not adjacent to b_1 it follows that $b_1 \in B_1$ by (6), and similarly $b_2 \in B_2$. Also by (6), R and R' both have length 1. Let $a_0 \in N$ be a left-star and $b_0 \in N$ a right-star. Since q_1-a_1-a_0-b_0-b_2-q_1 is not an odd hole, it follows that a_0 is not adjacent to b_0.

For every vertex $v \in V(G) \setminus Z$, let Z_v be the set of vertices in Z adjacent to v.

(8) *$Z_{a_0} \cap Z_{b_0} = \emptyset$, and every path in Z between Z_{a_0} and Z_{b_0} meets both Z_{q_1} and Z_{q_k}.*

For if $z \in Z_{a_0} \cap Z_{b_0}$, then z-a_0-a_1-b_2-b_0-z is an odd hole, so $Z_{a_0} \cap Z_{b_0} = \emptyset$. Let p_1-P-p_2 be a path in Z between Z_{a_0} and Z_{b_0}, with $V(P)$ minimal, where $p_1 \in Z_{a_0}$ and $p_2 \in Z_{b_0}$. Hence a_0-p_1-P-p_2-b_0-b_1-a_2-a_0 is a hole, and so P is odd. If P does not meet Z_{q_1} then q_1-a_1-a_0-p_1-P-p_2-b_0-b_1-q_1 is an odd hole, whike if P does not meet Z_{q_k} then q_k-a_0-p_1-P-p_2-b_0-q_k is an odd hole, in both cases a contradiction. This proves (8).

(9) *Every path in Z between Z_{q_1} and Z_{q_k} meets both Z_{a_0} and Z_{b_0}.*

For suppose not; then since Z is connected and $Z_{a_0} \cap Z_{b_0} = \emptyset$, there is a connected subset Z' of Z meeting both Z_{q_1} ,Z_{q_k} and meeting exactly one of Z_{a_0}, Z_{b_0}. From the symmetry we may assume Z' meets Z_{a_0} and not Z_{b_0}. Define $q_{k+1} = a_2$; then q_{k+1} has no neighbour in Z', so we may choose i with $1 \leq i \leq k+1$ minimum so that q_i has no neighbour in Z'. So $i \geq 2$, since q_1 has a neighbour in Z' (becuase Z' meets Z_{q_1}). If i is odd, then b_0-q_1-$\cdots$-q_i is an odd antipath; its internal vertices have neighbours in Z', and its ends do not, and

a_1 is complete to its interior and has no neighbours in Z', contrary to 2.2 in the complement. If i is even, then b_1-a_0-q_1-$\cdots$-q_i is an odd antipath, and its internal vertices have neighbours in Z' and its ends do not, and again a_1 is complete to its interior and has no neighbours in Z', contrary to 2.2 in the complement. This proves (9).

Let z_1-z_2-$\cdots$-z_n be a minimal path in Z between Z_{a_0} and Z_{b_0}, where $z_1 \in Z_{a_0}$ and $z_n \in Z_{b_0}$. Then $n \geq 2$ by (8), and by (8) and (9) it follows that z_1-z_2-$\cdots$-z_n is also a minimal path between Z_{q_1} and Z_{q_k}, so we may assume that $z_1 \in Z_{q_1}$, $z_n \in Z_{q_k}$, and no other vertex of the path is in either set. Then z_1-z_2-$\cdots$-z_n-q_k-a_0-z_1 and z_1-z_2-$\cdots$-z_n-b_0-b_1-q_1-z_1 are both holes, of different parity, a contradiction. This proves 11.5. ∎

12 Attachments in a staircase

For the next step of our approach towards the long odd prism, let us fix a little more than just the strip. Let $S = (A, C, B)$ be a step-connected strip in a Berge graph G, and let a_0-R_0-b_0 be a stride of length ≥ 3. We call the pair $K = (S, R_0)$ a *staircase*, and define $V(K) = V(R_0) \cup V(S)$. The staircase is *maximal* if there is no staircase $(S' = (A', C', B'), a_0'$-R_0'-$b_0')$ such that $A \subseteq A', B \subseteq B', C \subseteq C', V(S) \subset V(S')$.

Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a staircase in G. Some definitions (all with respect to K):

- A subset $X \subseteq V(K)$ is *local* if X is a subset of one of $V(S), V(R_0), A \cup \{a_0\}, B \cup \{b_0\}$
- $v \in V(G) \setminus V(K)$ is *minor* if its set of neighbours in $V(K)$ is local
- $v \in V(G) \setminus V(K)$ is *major* if it has neighbours in all of A,B and $V(R_0)$
- $v \in V(G) \setminus V(K)$ is *left-diagonal* if v is $(A \cup \{b_0\})$-complete, and *right-diagonal* if it is $(B \cup \{a_0\})$-complete
- $v \in V(G) \setminus V(K)$ is *central* if it is $(A \cup B)$-complete, and is nonadjacent to both a_0 and b_0

First let us examine the possible types of vertices outside the staircase.

12.1 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a maximal staircase in a Berge graph G, and let $v \in V(G) \setminus V(K)$. Then exactly one of the following holds:*

1. *v is minor, or*
2. *v is major; and in that case, it is either left- or right-diagonal or central, or*
3. *v is a left-star with a neighbour in $P_0 \setminus a_0$, or a right-star with a neighbour in $P_0 \setminus b_0$.*

Proof.

(1) *If v is left- or right-diagonal then the theorem holds.*

For assume v is right-diagonal say. If it has no neighbours in $A \cup C$ then statement 3 of the theorem holds, so we assume there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that v has a neighbour in $R_1 \setminus b_1$. Hence it can be linked onto the triangle $\{a_0, a_1, a_2\}$, via v-a_0, the path from v to b_1 with interior in $R_1 \setminus b_1$, and the path from v to b_2 with interior in R_2, and so by 2.4, v has neighbours in B. So it is major, and therefore statement 2 holds. This proves (1).

(2) *If v is adjacent to both a_0, b_0 then the theorem holds.*

For then it has a neighbour in R_0^*, since R_0 is odd and has length ≥ 3 and v is adjacent to both its ends; and we may assume that v has a neighbour in $V(S)$. If it has no neighbour in B then it is a left-star by 11.1, and statement 3 of the theorem holds, so we may assume it has neighbours in B and similarly in A. Hence it is major. By 11.5 (taking $Q = \{v\}$ and $Z = V(R_0^*)$) it does not have nonneighbours in both A and B, so it is either left- or right-diagonal and the claim follows from (1). This proves (2).

(3) *If v is adjacent to a_0 and not to b_0 then the theorem holds.*

For we may assume v has a neighbour in $V(S)$. If v has a neighbour in R_0^*, then by 11.2 it is either B-complete (when it is right-diagonal and the claim follows from (1)) or a left-star (when statement 3 holds). So we may assume it has no neighbour in R_0^*. We may assume it has a neighbour in $B \cup C$, for otherwise it is minor; let a_1-R_1-b_1, a_2-R_2-b_2 be a step so that v has a neighbour in $R_1 \setminus a_1$, and in addition so that v is not adjacent to b_2 if possible. By 10.5, v has a neighbour in R_3. If a_3 is its only neighbour in R_3, then the strip $S' = (A \cup \{v\}, C, B)$ is step-connected, since v-R-b_1, a_2-R_2-b_2 is an S'-step where R is the path from v to b_1 with interior in $R_1 \setminus a_1$; and this is contrary to the maximality of the staircase. So v has a neighbour in $R_2 \setminus a_2$; and hence v can be linked onto the triangle $\{b_0, b_1, b_2\}$ via v-a_0-R_0-b_0, and for $i = 1, 2$, the path from v to b_i with interior in $R_i \setminus a_i$. By 2.4 it follows that v is adjacent to both b_1, b_2; and hence from our choice of the step R_1, R_2, and since the strip is step-connected, it follows that v is right-diagonal, and the claim follows from (1). This proves (3).

(4) *If v is nonadjacent to both a_0, b_0 then the theorem holds.*

For then we may assume that v has a neighbour in R_0^* and in $V(S)$, since otherwise it is minor. If it is a left-star then statement 3 holds, so we assume not; and then by 11.2, it is B-complete. Similarly it is A-complete and therefore central, and statement 2 holds. This proves (4).

But (2)-(4) cover all the possibilities, up to symmetry, and this completes the proof of 12.1. ∎

Now let us do the same thing for connected sets.

12.2 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a maximal staircase in a Berge graph G, and let $F \subseteq V(G) \setminus V(K)$ be connected, so that its set of attachments in $V(K)$ is not local with respect to K. Then F contains either:*

- *a major vertex, or*
- *a stride u-R-v, so that there are no edges between $V(R)$ and $V(R_0)$, or*
- *(up to symmetry) a path u-R-v, where u is a left-star, v has a neighbour in $R_0 \setminus a_0$, and there are no edges between $V(R \setminus u)$ and $V(S)$.*

Proof. Let X be the set of attachments of F in $V(K)$. We may assume that F is minimal (connected) so that X is not local. Now a subset of $V(K)$ is local if and only if it does not meet both $A \cup C$ and $V(R_0 \setminus a_0)$ and does not meet both $B \cup C$ and $V(R_0 \setminus b_0)$; so we may assume that X meets both $A \cup C$ and $V(R_0 \setminus a_0)$, and therefore from the minimality of F, there is a path f_1-$\cdots$-f_k where $F = \{f_1, \ldots, f_k\}$ and f_1 is the unique vertex of F with a neighbour in $A \cup C$, and f_k is the unique vertex of F with a neighbour in $V(R_0 \setminus a_0)$.

(1) *If f_1 is A-complete then the theorem holds.*

For assume f_1 is A-complete. If there is no edge between F and $B \cup C$, then statement 3 of the theorem holds, so we assume that there is such an edge. Choose i with $1 \le i \le k$ minimum so that f_i has a neighbour in $B \cup C$. Suppose first that there is no edge between $\{f_1, \ldots, f_i\}$ and $V(R_0)$. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step so that f_i has a neighbour in $R_1 \setminus a_1$, and in addition so that f_i is nonadjacent to b_2 if possible. With respect to the prism formed by R_0, R_1, R_2, the set of attachments of $\{f_1, \ldots, f_i\}$ is not local, and so by 10.5, $i \ge 2$ and its attachments in the prism are a_1, a_2, b_1, b_2. Hence the only edges between $\{f_1, \ldots, f_i\}$ and $V(R_1 \cup R_2)$

are $f_1a_1, f_1a_2, f_ib_1, f_ib_2$. From our choice of the step it follows that f_i is B-complete. Consequently any step satisfies the condition we imposed on R_1, R_2, and so the same conclusion follows for every step; that is, statement 2 of the theorem holds. Now assume that there is an edge between $\{f_1, \ldots, f_i\}$ and $V(R_0)$. Suppose that $i < k$; then there is no edge between $\{f_1, \ldots, f_i\}$ and $R_0 \setminus a_0$, from the minimality of F, and so a_0 is an attachment of $\{f_1, \ldots, f_i\}$. But this set also has an attachment in $B \cup C$, so its set of attachments is not local, contrary to the minimality of F. This proves that $i = k$. If $k = 1$ then by 12.1, f_1 is major and the theorem holds. So assume $k \geq 2$. Hence from the minimality of F, there are no edges between $\{f_2, \ldots, f_k\}$ and $V(P_0 \setminus b_0)$; and so b_0 is the unique neighbour of f_k in R_0. Also, there are no edges between $\{f_1, \ldots, f_{k-1}\}$ and $B \cup C$, from the minimality of i. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 so that f_i is adjacent to b_1, and in addition so that f_i is nonadjacent to b_2 if possible. Since R_1 is odd and a_1-f_1-$\cdots$-f_k-b_1-R_1-a_1 is a hole, it follows that k is even. Since a_2-f_1-$\ldots$-f_k-b_0-b_2-R_2-a_2 is not an odd hole, f_i is adjacent to b_2, and therefore to all B from our choice of the step. Since a_1-f_1-$\cdots$-f_k-b_0-R_0-a_0-a_1 is not an odd hole and R_0 is odd, it follows that f_1 is adjacent to a_0. But then we can add f_1 to A, f_k to B, and $\{f_2, \ldots, f_{k-1}\}$ to C, contrary to the maximality of the staircase. This proves (1).

By (1), we may assume there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that f_1 has a neighbour in $R_1 \setminus a_1$, and a_2 is not adjacent to f_1. Then R_0, R_1, R_2 form a prism K' say, and the set of attachments of F in $V(K')$ is not local with respect to K'. Suppose that some vertex v in F is big with respect to K'. Then we claim v is major with respect to K. For it has a neighbour in A and in B, and if it has none in R_0 then it is adjacent to all of a_1, a_2, b_1, b_2, in which case v-a_1-a_0-R_0-b_0-b_2-v is an odd hole. So v is major, and hence the theorem holds. Hence we may assume that no vertex in F is big with respect to K', and so we may apply 10.2. By 9.6, 10.2.1 does not hold. Since no vertex of F is adjacent to a_2, 10.2.2 does not hold.

Suppose that 10.2.3 holds. Since f_1 is not adjacent to a_2, it follows that f_1 is adjacent to a_0, a_1 , and there exists i with $2 \leq i \leq k$ such that f_i is adjacent to b_0, b_1, and there are no other edges between $\{f_1, \ldots, f_i\}$ and $V(K')$. Then we can add f_1 to A, f_i to B and $\{f_2, \ldots, f_{i-1}\}$ to C, contrary to the maximality of the staircase. So 10.2.3 does not hold.

Hence 10.2.4 holds, that is, there is a path p_1-P-p_2 in F, such that for some j with $0 \leq j \leq 2$, either:

- p_1 is adjacent to the two vertices in $\{a_0, a_1, a_2\} \setminus \{a_j\}$, and p_2 has neighbours in $R_j \setminus a_j$, and there are no other edges between $V(P)$ and $V(K') \setminus a_j$, or
- p_1 is adjacent to the two vertices in $\{b_0, b_1, b_2\} \setminus \{b_j\}$, and p_2 has neighbours in $R_j \setminus b_j$, and there are no other edges between $V(P)$ and $V(K') \setminus b_j$

From the minimality of F, $F = V(P)$. If $j > 0$ then in the first case we can add p_1 to A and $V(P \setminus p_1)$ to C, contrary to the maximality of the staircase; and in the second case we do the same with A and B exchanged. So $j = 0$. The first case is impossible since no vertex in F is adjacent to a_2; and the second case is impossible since $f_1 \in F = V(P)$ and f_1 has a neighbour in $R_1 \setminus b_1$. This proves 12.2. ∎

Next we need a lemma about attachments sets that are local with respect to the staircase, and not local with respect to some prism, as follows.

12.3 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a maximal staircase in a Berge graph G, and let $F \subseteq V(G) \setminus V(S)$ be connected, containing a left-star and with an attachment in $B \cup C$. Then F contains either a major vertex or a stride.*

Proof. We may assume $|F|$ is minimum (possibly exchanging A and B); so F is the vertex set of a path f_1-$\cdots$-f_k, where f_1 is the unique left-star in F, and f_k is the only vertex in F with a neighbour in $B \cup C$. We may assume there is no major vertex in F. If there is a right-star in F, then it must be f_k; and then from the minimality of $|F|$ (exchanging A and B), no vertex of F different from f_1 has a neighbour in $A \cup C$, and so f_1-$\cdots$-f_k is a stride. So we may assume that there is no right-star in F. In particular, $b_0 \notin F$. Also, f_k is neither major nor a right-star, so by 12.1 it is not B-complete. Now $F \cup V(R_0 \setminus b_0)$ contains no major vertex, and no stride (since it contains no right-star). Hence by 12.2, F is disjoint from $V(R_0)$, and there are no edges between F and $V(R_0 \setminus b_0)$. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step such that f_k has a neighbour in $R_1 \setminus a_1$, and f_k is nonadjacent to b_2. (This exists since f_k is not B-complete.) If f_k has a neighbour in R_2, then its neighbour set in the prism formed by R_0, R_1, R_2 is not local with respect to that prism, and therefore by 10.5 f_k has a neighbour in R_0; and then by 12.1 it is major, a contradiction. So f_k has no neighbours in R_2. From the minimality of $|F|$, there are no edges between F and $R_2 \setminus a_2$. Hence b_2 is not an attachment of F; and so by 10.5 applied to the prism formed by R_0, R_1, R_2, it follows that b_0 has neighbours in F. Suppose that f_k is the unique neighbour of b_0 in F. Then since f_k is not major, its unique neighbour in R_1 is b_1. But then we can link a_1 onto the triangle $\{b_0, b_1, f_k\}$, via a_1-a_0-R_0-b_0, a_1-R_1-b_1, and the path from a_1 to f_k with interior in F, contrary to 2.4. So b_0 has a neighbour in $F \setminus f_k$. Let f_1-$\cdots$-f_i-b_0 be a path, where $1 \le i < k$. We claim that this is a stride. For there are no edges between $F \setminus f_k$ and $B \cup C$, from the minimality of $|F|$; and if there are edges between $\{f_2, \dots, f_i\}$ and A, then we can replace F by the shorter path b_0-f_i-$\cdots$-f_2, contrary to the minimality of $|F|$. Hence f_1-$\cdots$-f_i-b_0 is a stride R say. Let $F' = \{f_{i+1}, \dots, f_k\}$. Then F' is connected and disjoint from $V(R)$, and F' has attachments in $R_1 \setminus a_1$, and in $R \setminus b_0$. Suppose it has none in R_2. Then by 10.5 and 10.2 applied to the prism formed by R, R_1, R_2, F' contains a path with one end adjacent to a_1, f_1, the other end adjacent to b_0, b_1, and with no more edges between this path and $V(R) \cup V(R_1)$. Since the only vertex of F' adjacent to f_1 is f_2, and that only if $i = 1$, and the only vertex in F adjacent to b_1 is f_k, it follows that $i = 1$, and the only edges between $\{f_2, \dots, f_k\}$ and $V(R) \cup V(R_1)$ are $f_k b_1, f_k b_0, f_2 a_1, f_2 f_1$. But then a_1 can be linked onto the triangle $\{b_0, b_1, f_k\}$, via a_1-a_0-R_0-b_0, a_1-R_1-b_1, a_1-f_2-$\cdots$-f_k, and so by 2.4 it has two neigbours in that triangle, a contradiction. It follows that F' has attachments in R_2. Since there are no edges between $\{f_2, \dots, f_k\}$ and $R_2 \setminus a_2$, it follows that a_2 has a neighbour in f_j-$\cdots$-f_k. Now a_2 is not adjacent to f_k, and b_0 has a neighbour in F different from f_k, so there is a path between a_2 and b_0 with interior in $F \setminus f_k$. From the minimality of F, all vertices of $F \setminus f_k$ belong to this path. Hence b_0 has no neighbour in $\{f_2, \dots, f_{k-1}\}$. Therefore b_0 is adjacent to f_1 (since f_k is not its only neighbour) and to f_k (since it has a neighbour in F'). Since a_2 has a neighbour in $\{f_2, \dots, f_k\}$, the minimality of $|F|$ implies that a_2 is adjacent to f_2 and to no other vertex of $F \setminus f_1$. But then the path f_2-$\cdots$-f_k-b_0 can be completed to a hole via b_0-f_1-f_2 and via b_0-R_0-a_0-a_2-f_2, and one of these is odd, a contradiction. This proves 12.3. ∎

Now we turn to co-connected sets of major vertices. We have already defined what it is for a staircase to be maximal in G. We say a staircase $K = (S = (A, C, B), a_0$-R_0-$b_0)$ is *strongly maximal* if it is maximal, and in addition, either $C \neq \emptyset$, or there is no staircase (R', S') in $\overline{G}$ with $V(S) \subset V(S')$. The next result shows that there are no central vertices.

12.4 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a strongly maximal staircase in a Berge graph G. There is no co-connected set $Q \subseteq V(G) \setminus V(K)$ satisfying:*

- *some vertex of A is Q-complete, and some vertex of B is Q-complete*
- *a_0, b_0 are not Q-complete, and*
- *some vertex of R_0 is Q-complete.*

In particular, there are no central vertices.

Proof. Assume such a set Q exists, and choose it maximal. Let a_0-S-s and b_0-T-t be the subpaths of R_0 such that s is the unique Q-complete vertex of S, and t is the unique Q-complete vertex of T.

(1) *S, T both have odd length, and therefore s, t are different.*

For choose $a \in A \cap X$ and $b \in B \cap X$; then a-a_0-S-s has length > 1, and its ends are Q-complete and its internal vertices are not, and b is also Q-complete and has no neighbours in the interior of a-a_0-S-s. By 2.2, this path is even, and so S is odd, and similarly T is odd. Since R_0 is odd it follows that s, t are different. This proves (1).

(2) Every vertex in $A \cup B$ is Q-complete.

For suppose some vertex in A say is not Q-complete. Choose a step a_1-R_1-b_1, a_2-R_2, b_2 so that a_1 is Q-complete and a_2 is not. Since s, t are different it follows that t is nonadjacent to both a_0, a_2; and so by 2.8, Q cannot be linked onto the triangle $\{a_0, a_1, a_2\}$. Hence there is no Q-complete vertex in R_2. Assume s, t are nonadjacent; then the subpath of R_0 between them is odd, and a_1 has no neighbour in its interior, so by 2.2 it contains another Q-complete ertex u say; and then s-S-a_0-a_2-R_2-b_2-b_0-T-t is an odd path, its ebds are Q-complete and its internal vertices are not, and u has no neighbour in its interior, contrary to 2.2. So s, t are adjacent. Hence the hole a_0-R_0-b_0-b_2-R_2-a_2-a_0 has length ≥ 6, and the only Q-complete vertices in it are the adjacent vertices s, t. By 2.11 Q contains a hat or a leap; and in either case there is a vertex $q \in Q$ with no neighbours in R_2. But q is adjacent to s and a_1, contrary to 10.5 applied to the prism formed by R_0, R_1, R_2. This proves (2).

(3) *Every major vertex is in either in Q or complete to Q.*

For let v be a major vertex, and suppose $v \notin Q$, and Q' is co-connected, where $Q' = Q \cup \{v\}$. From 12.1, v is either left- or right-diagonal, or central; and in either case it has neighbours $a_1 \in A$ and $b_1 \in B$ that are nonadjacent. It follows that a_1-a_0-R_0-b_0-b_1 is an odd path of length ≥ 5, and its ends are Q'-complete. From the maximality of Q, none of its internal vertices are Q'-complete, and so by 2.1, Q' contains a leap q_1, q_2 say. So neither of q_1, q_2 has neighbours in the interior of R_0; but this is impossible since one of them is in Q and is therefore adjacent to s. This proves (3).

(4) *There is no edge uv of $G \setminus V(K)$ such that u is a left-star, v is a right-star, and u, v are not Q-complete.*

For suppose uv is such an edge. Since u, v have nonneighbours in Q and Q is co-connected, there is an antipath u-q_1-$\cdots$-q_k-v with $q_1, \ldots, q_k \in Q$. Choose a step a_1-R_1-b_1, a_2-R_2, b_2. Then a_1-b_2-u-q_1-$\cdots$-q_k-v-a_1 is an antihole, so k is even. Hence every Q-complete vertex w say is adjacent to one of u, v, for otherwise w-u-q_1-$\cdots$-q_k-v-w would be an odd antihole. In particular, there are no Q-complete vertices in C; and therefore a_1-R_1-b_1 is an odd path with both ends Q-complete and no internal vertex Q-complete. Since a_2 is Q-complete and has no neighbour in the interior of R_1, it follows from 2.2 that R_1 has length 1, and similarly R_2 has length 1. Since this step was arbitrary, and every vertex is in a step, it follows that $C = \emptyset$. Hence no Q-complete vertex W in R_0 is adjacent to both u and v; for otherwise w-b_1-u-q_1-$\cdots$-q_k-v-a_1-w would be an odd antihole. Hence every Q-complete vertex in R_0 is adjacent to exactly one of u, v. Suppose they are all adjacent to v. Then s-S-a_0-a_1-R_1-b_1 is an odd path, its ends are $(Q \cup \{v\})$-complete, its internal vertices are not, and t is also $(Q \cup \{v\})$-complete and has no neighbour in its interior, contrary to 2.2. So they are not all adjacent to v, and similarly not all to u. By 2.3 applied to the path a_1-a_0-R_0-b_0-b_2, there are an odd number of edges of this path (and therefore of R_0) with both ends Q-complete. Now we claim that in the hole a_0-R_0-b_0-b_1-R_1-a_1-a_0 there are an even number of edges with both ends $(Q \cup \{u\})$-complete; for certainly a_1 and some vertex in the interior of R_0 are $(Q \cup \{u\})$-complete, so either there are no more $(Q \cup \{u\})$-complete vertices in the hole (when our claim is true) or we can apply 2.3 to deduce the claim. This proves that there are an even number in the hole. But they all belong to R_0; so there are an even number of edges of R_0 with both ends $(Q \cup \{u\})$-complete. Similarly there are an even number with both ends $(Q \cup \{v\})$-complete. Since no vertex of R_0 is both $(Q \cup \{u\})$-complete and $(Q \cup \{v\})$-complete, and there are an odd number of edges in R_0 with both ends Q-complete, it follows that there is an edge ab of Q with both ends Q-complete, and not both $(Q \cup \{u\})$-complete, and not both $(Q \cup \{v\})$-complete. Since a, b are both adjacent to either u or v, we may assume that a is adjacent to u and not v, and b is adjacent to v and not u. Define $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$; then $(A', \emptyset, B')$ is a strip in $\overline{G}$. Now a-b_1, a_1-b is a step of this strip (in $\overline{G}$) for any edge $a_1 \in A$ and $b_1 \in B$ that are adjacent in G, and every vertex of $A \cup B$ is in such an edge, and so the strip is step-connected. Hence $((A', \emptyset, B'), v$-q_k-$\cdots$-q_1-$u)$ is a staircase in $\overline{G}$, contrary to the fact that K is strongly maximal. This proves (4).

(5) *Every path in G from an A-complete vertex to a vertex with a neighbour in $B \cup C$ contains either a vertex in Q or a Q-complete vertex.*

For suppose not, and choose a path p_1-$\cdots$-p_k say, with k minimum such that p_1 is A-complete and p_k has a neighbour in $B \cup C$, and none of $p_1, \dots, p_k$ is in Q or Q-complete. Since $A \cup B$ is complete to Q it follows that none of $p_1, \dots, p_k$ is in $A \cup B$. Now p_1 is not in C since no vertex in C is A-complete (because they are all in steps), and if some $p_i \in C$ for $i > 1$, then p_1-$\cdots$-p_{i-1} is a shorter path with the same properties, contrary to the minimality of k. So none of $p_1, \dots, p_k$ is in $V(S)$. (Some may be in R_0, however.) Since none of $p_1, \dots, p_k$ is major, it follows from 12.3 and the minimality of k that p_1-$\cdots$-p_k is a stride. From (4), since none of $p_1, \dots, p_k$ is Q-complete, it follows that $k > 2$. Let a_1-R_1-b_1, a_2, R_2, b_2 be a step. From the hole a_1-p_1-$\cdots$-p_k-$b1$-R_1-a_1 it follows that k is even; and so a_1-p_1-$\cdots$-p_k-$b2$ is an odd path of length ≥ 5; its ends are Q-compete, and its internal vertices are not. By 2.1, Q contains a leap a, b; so a-p_1-$\cdots$-p_k-b is a path. But then $(A \cup \{a\}, C, B \cup \{b\})$ is a step-connected strip S' say (since for every nonadjacent $a' \in A$ and $b' \in B$, the two paths a-b', a'-b make a step in this strip), and so $(S', p_1$-$\cdots$-$p_k)$ is a staircase, contrary to the maximality of (S, R_0). This proves (5).

Let X be the set of all Q-complete vertices in G; let M be the component of $G \setminus (Q \cup X)$ that contains a_0, and N the union of all the other components. By (5), $b_0 \in N$, so N is nonempty, and hence $(M \cup N, Q \cup X)$ is a skew partition of G. Choose $b \in B$; then $b \in X$, and it has no neighbour in M by (5). Hence the skew partition is loose, and so G admits an even skew partition, by 4.2. This proves the main statement of the theorem.

Now note that if q is a central vertex, then setting $Q = \{q\}$ violates what we just proved; so there are no central vertices. This proves 12.4. ∎

12.5 *Let G be a Berge graph, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a strongly maximal staircase in a Berge graph G. Let q_1-$\cdots$-q_k be an antipath such that $q_2, \dots, q_{k-1}$ are both left- and right-diagonal, and q_1 is left- and not right-diagonal, and q_k is right- and not left-diagonal. Then q_1 is a left-star and q_k is a right-star.*

Proof. First, obviously $k \geq 2$. Let $Q = \{q_1, \dots, q_k\}$.

(1) *If q_1 is adjacent to a_0 and q_k to b_0 then the theorem holds.*

For then both a_0, b_0 are Q-complete, and q_1 has a nonneighbour in B (for otherwise it would be a right diagonal), and q_k has a nonneighbour in A. Since R_0 has odd length ≥ 3, it follows that each of $q_1, \dots, q_k$ has a neighbour in R_0^*. By 11.5 (taking Z to be the interior of R_0) it follows that Q contains a left-star, which must be q_1; and similarly q_k is a right-star. Then the theorem holds. This proves (1).

(2) *If q_1 is adjacent to a_0 and q_k is nonadjacent to b_0 then the theorem holds.*

For in this case, q_1 has a nonneighbour in B, say b_3. From the antihole a_0-b_3-q_1-$\cdots$-q_k-b_0-a_0 we deduce that k is odd. Now R_0 is odd, of length ≥ 3, and its ends are complete to $Q \setminus q_k$, and so is b_3, and b_3 has no neighbour in the interior of R_0, so by 2.2, there is a $(Q \setminus q_k)$-complete vertex in the interior of R_0, say t. Let T be the subpath of t to b_0, and let us choose t with T of minimum length, that is, so that t is the unique $(Q \setminus q_k)$-complete vertex of T . If t is nonadjacent to q_k then t-b_3-q_1-$\cdots$-q_k-t is an odd antihole (since $k \geq 2$) , a contradiction. Hence t is Q-complete, and in particular, all of $q_1, \dots, q_k$ have neighbours in the interior of R_0. By 11.4 it follows that Q contains a left-star, which must be q_1. We may assume that q_k is not a right-star, for otherwise the theorem holds. Since q_k is right-diagonal, from 12.1 it follows that q_k is major and therefore has a neighbour in A. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 so that q_k is adjacent to a_1, and if possible nonadjacent to a_2. Then t-T-b_0-b_1-R_1-a_1 is a path, and both its ends are Q-complete, and none of its internal vertices are Q-complete (since q_1 is a left-star). By 3.2 applied to t-T-b_0-b_1-R_1-a_1 and

Q, it follows that t-T-b_0-b_1-R_1-a_1 has length 4, and so R_1 has length 1 and T has length 2; let its middle vertex be u say. Also from 3.2, u is complete to $Q \setminus q_1$-complete, and nonadjacent to q_1. Suppose that q_k is nonadjacent to a_2. Then there is no Q-complete vertex in R_2. If t is nonadjacent to a_0 then a_0-a_2-R_2-b_2-b_0-u-t is an odd path of length ≥ 5; its ends are Q-complete and its internal vertices are not, so by 2.1, Q contains a leap, which is impossible since every vertex in Q is adjacent to one of b_0, b_2. If t is adjacent to a_0, then a_0-a_2-R_2-b_2-b_0-R_0-a_0 is a hole of length ≥ 6, and the only Q-complete vertices in it are a_0, t, and these are adjacent; so by 2.11 there is a hat or a leap in Q; and agin this is impossible since every vertex in Q is adjacent to one of b_0, b_2. This proves that $q_k a_2$ is an edge. From our choice of the step, it follows that q_k is A-complete. But therefore any step satisfies the condition we imposed on the step R_1, R_2; and therefore every path in every step has length 1, that is $C = \emptyset$. Then $S = (A \cup \{t\}, \emptyset, B \cup \{u\})$ is a step-connected strip in $\overline{G}$, and $(S', b_0\text{-}q_k\text{-}\cdots\text{-}q_1)$ is a staircase in $\overline{G}$, contradicting that (S, R_0) is strongly maximal. This proves (2).

(3) *If q_1 is nonadjacent to a_0 and q_k is nonadjacent to b_0 then the theorem holds.*

For then a_0-q_1-$\cdots$-q_k-b_0-a_0 is an antihole, so k is even. Let A_1 be the set of vertices in A adjacent to q_k, and $A_2 = A \setminus A_1$; and let B_1 be the set of vertices in B adjacent to q_1, and $B_2 = B \setminus B_1$. If $a_1 \in A_1$ and $b_2 \in B_2$, then a_1-b_2-q_1-$\cdots$-q_k-b_0-b_2 is not an odd antihole, and so a_1 is adjacent to b_2; and hence A_1 is complete to B_2, and similarly A_2 is complete to B_1. If A_1, B_1 are both empty then by 12.1, the theorem holds; so we may assume that A_1 is nonempty. Choose $a_1 \in A_1$. Since a_1 is in a step, it has a nonneighbour in B, say b_1. Since a_1 is B_2-complete it follows that $b_1 \in B_1$. Then a_1, b_1 are both Q-complete, so by 12.4, no internal vertex of R_0 is Q-complete. So a_1-a_0-R_0-b_0-b_1 is an odd path of length ≥ 5, and its ends are Q-complete, and its internal vertices are not. By 2.1, Q contains a leap. Since every vertex of Q except q_1, q_k has ≥ 2 neighbours in R_0, it follows that $k = 2$ and q_1, q_2 both have no neighbours in the interior of R_0. Then $S' = (A \cup \{q_2\}, C, B \cup \{q_1\})$ is a step-connected strip (since a_1-q_1, q_k-b_1 is a step of it), and (S', R_0) is a staircase, contrary to the maximality of (S, R_0). This proves (3).

From (1),(2),(3), the theorem follows. This proves 12.5. ∎

13 The long odd prism

In this section we apply the results of the previous section to prove that a Berge graph containing a long odd prism has a decomposition unless it is a line graph.

Let $K = ((A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a staircase in a Berge graph G. From 12.1 there are three possible kinds of B-complete vertices; right-stars, vertices complete to both A and B, and B-complete vertices adjacent to some but not all of A. The most difficult step in handling the long odd prism is when there is a vertex of the third kind. In that case, we shall construct a subset of B-complete vertices, including all these "mixed" vertices and some of the others, such that they and their common neighbours form a cutset of the graph, and thereby give us a skew partition. We define the set recursively as follows: initially let X be the set of all B-complete vertices adjacent to some but not all of A. Then enlarge X by repeatedly applying the following two rules, in any order:

1. if there is a $A \cup B$-complete vertex v that is not in X and not X-complete, add v to X
2. if there is a stride a-R-b such that a is not X-complete and b is not in X, add b to X.

The process eventually stops with some set X. We shall prove that X and its common neighbours (say Y) separate A (or at least the part of A that is not X-complete) from b_0, and this will provide an even skew partition. To prove that $X \cup Y$ separates G as described, we have to show that every path from A to b_0 meets $X \cup Y$, and it turns out that there are only two kinds of paths to worry about; strides, and 1-vertex paths consisting of a major vertex. Any stride a-R-b is automatically hit, because of the rule above; if $a \notin Y$ then $b \in X$. The 1-vertex paths are trickier. Let v be a major vertex. If it is B-complete, then by the rule above, either it is Y or X, so assume it is not B-complete. By 12.1, it is left- and not right-diagonal, and now we have to show it belong to Y. If only we knew that every vertex in X was adjacent to a_0, then it follows easily that $v \in Y$, because of 12.5. So that is what we need to do - to prove that every vertex in X is adjacent to a_0.

Let us start again, more formally. Let $K = ((A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a staircase in a Berge graph G. We define a *right-sequence* to be a sequence $x_1, \dots, x_t$, with the following properties (which we refer to the "right-sequence axioms"):

1. $x_1, \dots, x_t$, are all distinct and are all B-complete
2. for $1 \leq i \leq t$, if x_i is A-complete then there exists h with $1 \leq h < i$ such that x_h is nonadjacent to x_i
3. for $1 \leq i \leq t$, if x_i is A-co-complete then there is a stride $r\text{-}R\text{-}x_i$ such that r has a nonneighbour in $\{x_1, \dots, x_{i-1}\}$.

Any initial subsequence of a right-sequence is therefore another right-sequence. We say x_i is *earlier* than x_j if $i < j$. Let $X = \{x_1, \dots, x_t\}$. For each $x_i \in X$ that has an earlier nonneighbour, we define its *predecessor* to be x_h, where h is minimum such that $1 \leq h < i$ and x_h is nonadjacent to x_i. From the third axiom, every x_i either has a nonneighbour in A or a predecessor, so we can follow the sequence of predecessors until we get to some vertex that is not A-complete. For each x_i we therefore define the *trajectory* of x_i to be the sequence $w_1\text{-}\cdots\text{-}w_n$ with the following properties:

- $n \geq 1$, and $w_1 = x_i$
- w_n has a nonneighbour in A
- for $1 \leq j < n$, w_j is A-complete, and w_{j+1} is the predecessor of w_j.

Clearly the trajectory is unique, and is an antipath. If $v \in V(G)$ is A-complete, not in X and not X-complete, we define the *trajectory* of v to be the antipath $v\text{-}w_1\text{-}\cdots\text{-}w_n$, where w_1 is the earliest nonneighbour of v in X, and $w_1\text{-}\cdots\text{-}w_n$ is the trajectory of w_2.

Let a be a left-star. If it is not X-complete, we define the *birth* of a to be the earliest nonneighbour of a in X. Now let b be a right-star. A stride $a\text{-}R\text{-}b$ is said to be *b-optimal* if a is not X-complete, and there is no stride $a'\text{-}R'\text{-}b$ such that a' is not X-complete and the birth of a' is earlier than the birth of a.

13.1 *Let G be Berge, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a strongly maximal staircase in G, and let $x_1, \dots, x_t$ be a right-sequence. Let b be a right-star, and let $a\text{-}R\text{-}b$ be a b-optimal stride. Let $a\text{-}w_1\text{-}\cdots\text{-}w_n$ be the trajectory of a. Then n is odd, and either:*

- *b is the unique vertex of R which is $\{w_1, \dots, w_n\}$-complete, or*
- *R has length 1, and there exists some even m with $1 \leq m < n$ such that $a\text{-}w_1\text{-}\cdots\text{-}w_m\text{-}b$ is an antipath.*

Proof. We proceed by induction of t, and assume the result holds for all smaller values of t. Hence we may assume that $w_1 = x_t$, for otherwise the result follows by induction. Let $W = \{w_1, \dots, w_n\}$; then every vertex in B is W-complete.

(1) *n is odd.*

For choose $a_2 \in A$ nonadjacent to w_n, and $b_1 \in B$ nonadjacent to a_2; then $b_1\text{-}a\text{-}w_1\text{-}\cdots\text{-}w_n\text{-}a_2\text{-}b_1$ is an antihole, so n is odd. This proves (1).

(2) *If w_n has a neighbour in A then the theorem holds.*

For choose a step $a_1\text{-}R_1\text{-}b_1, a_2\text{-}R_2\text{-}b_2$ such that w_n is adjacent to a_1 and not to a_2. Then a_1,b_2 are W-complete. Suppose first that there are no W-complete vertices in R. Then $a_1\text{-}a\text{-}R\text{-}b\text{-}b_2$ is an odd path between

W-complete vertices. If R has length 1 then there is an antipath Q joining a, b with interior in W, and since it can be completed to an antihole via $b - a_1$-b_2-a, it has odd length and the theorem holds. So we may assume R has length > 1, and hence by 2.1 W contains a leap. Since all vertices of W except w_1 are adjacent to a, the leap is $w1, w2$; and hence the only edges between w_1, w_2 and R are $w_1 b$ and $w_2 a$. Since n is odd it follows that $n > 2$ and so w_1, w_2 are both $A \cup B$-complete. But then $S' = (A \cup \{w_2\}, C, B \cup \{w_1\}$ is a step-connected strip, and $(S', a\text{-}R\text{-}b)$ is a staircase, contrary to the maximality of (S, R_0). So we may assume there are W-complete vertices in R. If b is the only one then the theorem holds, so assume there is another. But then W can be linked onto the triangle $\{a, a_1, a_2\}$, via a subpath of $R \setminus b$, the 1-vertex path a_1, and a subpath of R_2. Since b_1 is W-complete and nonadjacent to both a, a_2, this contradicts 2.8. This proves (2).

From (2) we may assume that w_n has no neighbour in A. Let $w_n = x_s$ say. From the third axiom, there is a stride r'-R'-w_n, such that r' has a nonneighbour in $\{x_1, \ldots, x_{s-1}\}$, and therefore we may choose it to be w_n-optimal.

(3) *R' is disjoint from R, and there are no edges between $V(R) \setminus a$ and $V(R') \setminus w_n$.*

Suppose that $(R \setminus a) \cup (R' \setminus w_n)$ is connected. Then it contains a path between r' and b, with interior in the union of the interior of R and R', and therefore this path is a stride. But R is b-optimal, and the birth of r' is earlier than the birth of a, a contradiction. So $R \setminus a$ is disjoint from $R' \setminus w_n$, and there are no edges between them. Since $a \neq r'$ (because their births are different), and $b \neq w_n$ (because R is optimal for b) it follows that R is disjoint from R'. This proves (3).

Let v_1-$\cdots$-v_m be the trajectory of r', and let $V = \{v_1, \ldots, v_m\}$. Let $W' = \{a, w_1, \ldots, w_{n-1}\}$. Since each of $v_1, \ldots, v_m$ is earlier than w_n, it follows from the definition of trajectory that $v_1, \ldots, v_m$ are all W'-complete. By induction on t, it follows that either w_n is the unique V-complete vertex in R', or R' has length 1 and there is an odd antipath between r' and w_n with interior in V.

(4) *If $n = 1$ then the theorem holds.*

For let $n = 1$, and choose a step a_1-R_1-b_1, a_2-R_2-b_2. Suppose first that a has no neighbour in R'. Now a is V-complete, and either w_n is the unique V-complete vertex in R', or R' has length 1 and there is an odd antipath Q between r' and w_n with interior in V. In the first case, a-a_1-r'-R'-w_n is an odd path, its ends are V-complete, its internal vertices are not, and the V-complete vertex has no neighbour in its interior, contrary to 2.2. In the second case, a-r'-Q-w_n-a is an odd antihole. This proves that a has a neighbour in R'. Now suppose it has a neighbour different from r'; then R' has length > 1, and so w_n is the unique V-complete vertex in R'; and there is a path P' say from a to w_n with interior in $R' \setminus r'$. Since the ends of this path are V-complete and its internal vertices are not, and the V-complete vertex b_1 has no neighbour in its interior, it is even by 2.2. But it can be completed to an odd hole via w-n-b_1-R_1-a_1-a, a contradiction. This proves that r' is the unique neighbour of a in R'. Since a-r'-R'-w_n-b_1-b-R-a is not an odd hole, it follows from (3) that w_n has neighbours in R. If b is its unique neighbour then the theorem holds, so we assume not. Then there is a path P say from w_n to a with interior in $R \setminus b$. Since the union of P, P' is a hole it follows that P is even; but P can be completed via a-a_1-R_1-b_1-w_n, a contradiction. This proves (4).

We may therefore assume that $n \geq 3$ (since it is odd.)

(5) $C = \emptyset$.

For suppose not, and choose a step a_1-R_1-b_1, a_2-R_2-b_2 where R_1 has length > 1. Since R_1 is odd, and its ends are $(W \setminus w_n)$-complete, and the $(W \setminus w_n)$-complete vertex b_2 has no neighbour in its interior, there is a $(W \setminus w_n)$-complete vertex v in the interior of R_1, by 2.2. But then v is nonadjacent to both a and w_n, since they are left- and right-stars respectively, and so v-a-w_1-$\cdots$-w_n-v is an odd antihole, a contradiction. This proves (5).

(6) *If b is not $(W \setminus w_n)$-complete and no two adjacent vertices of R are both $(W \setminus w_n)$-complete then the theorem holds.*

For choose a step a_1-R_1-b_1, a_2-R_2-b_2. Then a_1-a-R-b-b_2 is an odd path, its ends are $(W \setminus w_n)$-complete, and no two adjacent vertices in this path are $(W \setminus w_n)$-complete. Suppose first that R has length ≥ 3. Then by 2.1 there is a leap in $W \setminus w_n$; and so there are nonadjacent vertices $x, y \in W \setminus w_n$ such that x-a-R-n-y is a path. But then $((A \cup \{x\}, C, B \cup \{y\}), a\text{-}R\text{-}b)$ is a staircase, contrary to the maximality of (S, R_0). So R has length 1, and there exists i with $1 \leq i < n$ such that a-w_1-$\cdots$-w_i-b is an odd antipath. But then the theorem holds. This proves (6).

(7) *If no vertex in R is W-complete then the theorem holds.*

For by (6) we may assume that there is a vertex v of R which is $(W \setminus w_n)$-complete. Hence v is nonadjacent to w_n. Since $n \geq 3$ and is odd, it follows that a-w_1-$\cdots$-w_n-v-a is not an odd antihole, and so v is adjacent to a. Consequently v is the unique $(W \setminus w_n)$-complete vertex in R. From (6) we may assume that $v = b$, and R has length 1. Choose a step a_1-R_1-b_1, a_2-R_2-b_2. Then b_1-a-w_1-$\cdots$-w_n-b is an odd antipath, of length ≥ 5. All its internal vertices have neighbours in the connected set $(R' \setminus w_g) \cup \{a_2\}$, and its ends do not. By 2.1 is the complement; there are adjacent vertices x, y in $(R' \setminus w_g) \cup \{a_2\}$, such that x-a-w_1-$\cdots$-w_n-y is an odd antipath. Since x is adjacent to w_n, it follows that x is the neighbour of w_n in R', and therefore either y is the second neighbour of x in R', or R' has length 1 and $y = a_1$. Assume first that R' has length > 1, and so both x, y belong to the interior of R'. Hence x, y are both co-complete to $A \cup B$, and so $((B \cup \{x\}, \emptyset, A \cup \{y\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, contradicting that (S, R_0) is strongly maximal. Now assume that R' has length 1. Then $x = r'$ and $y = a_1$, and $((B \cup \{r'\}, \emptyset, A \cup \{b\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, a contradiction as before. This proves (7).

We may therefore assume that some vertex of $R \setminus b$ is W-complete, for otherwise the theorem holds. Let a-P-v be the subpath of $R \setminus b$ such that v is the unique W-complete vertex of P. Let us apply 3.2 to the path v-P-a-a_1-R_1-b_1, and the even antipath a-w_1-$\cdots$-w_n-a_1. Both ends of the path are complete to the interior of the antipath, so by 3.2 it follows that P has length 2, and if q denotes its middle vertex then q is nonadjacent to w_n and adjacent to $w1, \ldots, w_{n-1}$. But then $((B \cup \{p\}, \emptyset, A \cup \{q\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, a contradiction. This completes the proof of 13.1. ∎

13.2 *Let G be Berge, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a strongly maximal staircase in G, and let $x_1, \ldots, x_t$ be a right-sequence. Then $x_1, \ldots, x_t$ are all adjacent to a_0.*

Proof. Suppose the theorem is false, and choose t is small as possible so that the statement of the theorem does not hold. So $t \geq 1$, and $x_1, \ldots, x_{t-1}$ are all adjacent to a_0, and x_t is not.

(1) *a_0-R_0-b_0 is not an optimal stride for b_0.*

For suppose it is, and let a_0-w_1-$\cdots$-w_n be the trajectory of a_0. Since R_0 has length > 1 it follows from 13.1 that b is the unique W-complete vertex of R_0, where $W = \{w_1, \ldots, w_n\}$. Suppose first that $n = 1$. Then w_1 is nonadjacent to a_0 and has a nonneighbour in A, and so by 12.1 it is a right-star. By axiom 3 there is a stride r-R-w_1 such that r has a nonneighbour in X earlier than w_1. Since a_0-R_0-b_0 is optimal for b_0, it follows as in the proof of 13.1 that R is disjoint from R_0, and there are no edges between $R_0 \setminus a_0$ and $R \setminus w_1$. The only edge from w_1 to R_0 is w_1b_0, by 13.1. Choose an S-rung a_1-R_1-b_1. Since a_1-a_0-R_0-b_0-w_1-R-r-a_1 is not an odd hole it follows that a_0 has neighbours in R. If it has a neighbour different from r, then the path from a_0 to w_1 with interior in $R \setminus r$ can be completed via r-b_0-R_0-a_0 and via r-b_1-R_1-a_1-a_0, and one of these is odd, a contradiction. So the unique neighbour of a_0 in R is r. But then we can add r to A, w_1 to B and the interior of R to C, contradicting the maximality of (S, R_0). So $n \geq 2$. Now all of $w_1, \ldots, w_{n-1}$ are left-diagonals, and all of $w_2, \dots, w_n$ are right-diagonals. But w_1 is not a right-diagonal, and w_n is not a left-diagonal, and w_1 is not a right-star, contrary to 12.5. This proves (1).

Now since a_0 has a nonneighbour in $\{x_1, \dots, x_t\}$, it follows that there is an optimal stride r-R-b_0 for b_0. From (1), r has a nonneighbour in $\{x_1, \dots, x_{t-1}\}$. From the minimality of t (replacing R_0 by R) it follows that R has length 1, and so rb_0 is an edge. Let r-w_1-$\cdots$-w_n be the trajectory of r; so w_1 is earlier than x_t. Let $W = \{w_1, \dots, w_n\}$; hence a_0 is W-complete. By 13.1, n is odd.

(2) *b_0 is W-complete.*

For suppose not. Then by 13.1, there exists i with $1 \le i < n$ so that r-w_1-$\cdots$-w_i-b_0 is an odd antipath. Now $r, w_1, \dots, w_{i-1}$ are all left-diagonals; $w_1, \dots, w_i$ are all right-diagonals; r is not a right-diagonal (since it is a left-star); and w_i is not a left-diagonal (since it is nonadjacent to b_0) and not a right- or left-star (since it is $A \cup B$-complete, because $i < n$). This contradicts 12.5, and so proves (2).

(3) *a_0 is adjacent to r, and w_n is a right-star.*

Let a_1-R_1-b_1 be an S-rung with w_n nonadjacent to a_1. Since a_0-r-w_1-$\cdots$-w_n-a_1-b_0-a_0 is not an odd antihole it follows that a_0 is adjacent to r. So each of $r, w_1, \dots, w_{n-1}$ is left-diagonal, each of $w_1, \dots, w_n$ is right-diagonal, r is not right-diagonal, w_n is not left-diagonal, and the claim follows from 12.5. This proves (3).

(4) *There is no $(W \cup \{r\})$-complete in the interior of R_0.*

For suppose there is, v say. Let a_1-R_1-b_1 be an S-rung. Then a_0-a_1-R_1-b_1-b_0 is an odd path; both its ends are $(W \cup \{r\})$-complete; and the $(W \cup \{r\})$-complete vertex v has no neighbour in its interior, so by 2.2 there is a $(W \cup \{r\})$-complete vertex in R_1. But by (3), w_n is a right-star and r is a left-star, so they have no common neighbour in R_1, a contradiction. This proves (4).

(5) $n = 1$.

For assume $n > 1$. Now R_0 is odd, and both its ends are $(W \cup \{r\})$-complete. Suppose first that R_0 has length ≥ 5. By 2.1 there is a leap; two nonadjacent vertices $x, y \in W \cup \{r\}$ joined by an odd path P whose interior is the interior of R_0. Choose $b_1 \in B$; then b_1-x-P-y-b_1 is not an odd hole, and so one of x, y is nonadjacent to b_1. Since b_1 is W-complete, we may assume $y = r$; and hence $x = w_1$ since that is the only vertex in W nonadjacent to r. Choose $a_1 \in A$; then since a_1-r-P-w_1-a is not an odd hole it follows that a_1 is not adjacent to w_1 and so $n = 1$. Now assume that R_0 has length 3, and let its internal vertices be x, y (in some order). By 2.1 there exists an odd antipath Q joining x, y with interior in $W \cup \{r\}$. If $r \notin V(Q)$ then b_1-x-Q-y-b_1 is an odd antihole, where $b_1 \in B$; and if $w_n \notin V(Q)$ then a_1-x-Q-y-a_1 is an odd antihole, where $a_1 \in A$. Hence x-r-w_1-$\cdots$-w_n-y is an antipath. We claim that $C = \emptyset$. For suppose there is an S-rung a_1-R_1-b_1 say of length > 1. Then a_1-R_1-b_1-b_0-r-a_1 is a hole of length ≥ 6; and r-w_1-$\cdots$-w_n-a_1 is an even antipath of length ≥ 4; and a_0 is complete to the antipath, and has no other neighbours on the hole; and at least two vertices of the hole are complete to the interior of the antipath, namely b_0 and b_1. This contradicts 3.3. So $C = \emptyset$. Hence $((B \cup \{x\}, \emptyset, A \cup \{y\}), r\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, a contradiction. This proves (3).

From (4) and (5) there is an odd path P joining r and w_1 with interior equal to the interior of R_0. From (3), w_1 is a right-star, and from axiom 3 there is a stride r'-R-w_1 (and we may choose it optimal for w_1) such that r' has a nonneighbour in $\{x_1, \dots, x_t\}$ earlier than w_1. Now R' is disjoint from R_0, and there are no edges between $R_0 \setminus a_0$ and $R' \setminus w_1$; for otherwise there would be a stride from r' to b_0, contradicting that r-b_0 is optimal for b_0. Suppose that r has a neighbour in R'; then the path between r and w_1 with interior in R' can be completed to holes via w_1-b_0-r and via w_1-P-r, a contradiction since one of these holes is odd. So r has no neighbour in R'. Let r'-v_1-$\cdots$-v_m be the trajectory of r'. Since $v_1, \dots, v_m$ are earlier than w_1, and w_1 is the earliest nonneighbour of r, it follows that r is adjacent to all of $v_1, \dots, v_m$. Now by 13.1, either

- w_1 is the unique $\{v_1, \dots, v_m\}$-complete vertex in R'; but then w_1-R'-r'-a_1-r (where $a_1 \in A$ is nonadjacent to v_m) is an odd path; its ends are $\{v_1, \dots, v_m\}$-complete and its internal vertices are not; and the $\{v_1, \dots, v_m\}$-complete vertex b_1 (for any $b_1 \in B$ nonadjacent to a_1) has no neighbour in its interior, contrary to 2.2.
- R' has length 1, and there is an odd antipath Q between r' and w_1 with interior in $\{v_1, \dots, v_m\}$; but then r-r'-Q-w_1-r is an odd antihole, a contradiction.

This completes the proof of 13.2. ∎

Now we are ready to apply 13.2 to produce a skew partition, in the proof of the following.

13.3 *Let G be Berge, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a strongly maximal staircase in G. Then every B-complete vertex is either A-complete or A-co-complete.*

Proof. Suppose there is a B-complete vertex which is neither A-complete not A-co-complete, say x_1. Then the 1-vertex sequence x_1 is a right-sequence; so there exists a right-sequence $x_1, \dots, x_t$ of maximum length, with $t \geq 1$. Let $X = \{x_1, \dots, x_t\}$, and let Y be the set of all $A \cup X$-complete vertices in $V(G)$. So $a_0 \in Y$ by 13.2.

(1) *$X \cup Y \cup B$ meets the interior of every path in G from $A \cup C$ to b_0.*

For suppose P is a path from $A \cup C$ to b_0, disjoint from $X \cup Y \cup B$. We may assume P is minimal, and therefore no internal vertex of P is in $V(S)$. Let P be from $p \in A \cup C$ to b_0. By 12.3, $P \setminus p$ contains either a major vertex or a stride. Suppose first that it contains a stride a-R-b say. Since a is a left-star it follows that $a \neq b_0$, and so a belongs to the interior of P, and therefore is not in $X \cup Y \cup B$. Also $b \notin X$, since $b_0 \notin X$ by 13.2, and no vertex of the interior of P is in X. But then we can set $x_{t+1} = b$, contradicting the maximality of the right-sequence. So $P \setminus p$ contains no stride. Now assume it contains a major vertex v say. Since b_0 is not major it follows that v is in the interior of P, and so is not in $X \cup Y \cup B$. Hence v is not $X \cup A$-complete. Suppose v is B-complete. Since it is major it has a neighbour in A. If it is not A-complete we can set $x_{t+1} = v$ and obtain a longer right-sequence, a contradiction; and if v is A-complete then since it is not $X \cup A$-complete, it is not X-complete and so again we can set $x_{t+1} = v$ and obtain a longer right-sequence, a contradiction. So v is not B-complete. By 12.1 and 12.4, v is left-diagonal, and not right-diagonal. Let v-w_1-$\cdots$-w_n be the trajectory of v. Then each of $w_1, \dots, w_n$ is right-diagonal, since they are all $A \cup \{a_0\}$-complete. Since w_n has a nonneighbour in A, it is not left-diagonal; and so there is a minimum i with $1 \leq i \leq n$ such that w_i is not left-diagonal. By 12.5 applied to the sequence $v, w_1, \dots, w_i$, we deduce that v is a left-star and w_i is a right-star. But v is not a left-star since it is major, a contradiction. This proves (1).

Now since S is step-connected, it follows that $A \cup C$ is connected; and therefore belongs to a component A_1 of $G \setminus (X \cup Y \cup B)$. Let A_2 be the union of all the other components. So by (1), $b_0 \in A_2$, and $(A_1 \cup A_2, X \cup Y \cup B)$ is a skew partition of G (since $Y \cup B$ is complete to X, and X is nonempty). We need to find an even skew partition. By 4.2 we may assume this skew partition is not loose; so every X-complete vertex in G either belongs to B or is also A-complete. So every vertex in $Y \cup B$ has a neighbour in $A \cup C$, so $A \cup C$ is a kernel for this skew partition, in $\overline{G}$. By 4.7 it suffices to show that any two nonadjacent vertices in $Y \cup B$ are joined by an even path with interior in $A \cup C$, and any two adjacent vertices of $A \cup C$ are joined by an even antipath with interior in $Y \cup B$. Now let $u, v \in Y \cup B$ be nonadjacent. If they are both adjacent to b_0, then any path joining them with interior in $A \cup C$ (and there is one) is even, since it can be completed to a hole via v-b_0-u. So we may assume that u is nonadjacent to b_0, and hence $u \notin B$, so $u \in Y$. If they are both in Y, then they are joined by an even path u-a_1-v for any $a_1 \in A$. So we may assume that $v \in B$. Since u is nonadjacent to b_0 and to v, it is neither left- nor right-diagonal, and so from 12.1 and 12.4, it is a left-star. Let a_1-R_1-v be an S-rung; then u-a_1-R_1-v is the desired even path between u and v. Now for antipaths, let uv be an edge with $u, v \in A \cup C$. They both therefore have nonneighbours in B, and since $B \cup \{a_0\}$ is co-connected, they are joined by an antipath Q with interior in $B \cup \{a_0\}$. It suffices to show that Q is even, since $Q^* \subseteq Y \cup B$.

If $a_0 \notin Q^*$, then Q is even since b_0-u-Q-v-b_0 us an antihole. So a_0 is in Q^*. But there are no edges between a_0 and B, and so a_0 is nonadjacent to every other vertex in the interior of Q; and since Q is an antipath, it therefore has at most 3 internal vertices, so its length is ≤ 4. If it is odd, then it has length 3, that is, there are nonadjacent vertices $u' \in Y$ and $v' \in B$, joined by an odd path with interior in $A \cup C$. But we have already shown that they are joined by an even path, and the result follows from 4.3. This proves 13.3. ∎

Now we come to the main result of this section. (M-joins were defined in section 1.)

13.4 *Let G be Berge, such that neither G nor $\overline{G}$ admits a line graph decomposition, and G is not a bicograph. Suppose that G contains a long odd prism as an induced subgraph. Then G admits an M-join.*

Proof. Since G contains a long odd prism, it contains a staircase; and therefore (possibly by replacing G by its complement) there is a strongly maximal staircase $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ say in G. Let A_0 be the set of all left-stars, B_0 the set of all right-stars, and N the set of all vertices that are $A \cup B$-complete. So by 13.3, every A-complete vertex is in $A_0 \cup N$, and every B-complete vertex is in $B_0 \cup N$. Let $H = G \setminus (V(K) \cup A_0 \cup B_0 \cup N)$.

(1) *Let F be a component of H, and let X be the set of attachments of F in $V(K)$. Then either $X \cap V(S) = \emptyset$, or $X \subseteq V(S)$ and X meets both $A \cup C$ and $B \cup C$.*

We may assume that X meets $V(S)$, and therefore from the symmetry we may assume that X meets $A \cup C$. Since no vertex in F is A- or B-complete, and therefore no vertex in F is major or a left- or right-star, it follows from 12.2 that X is disjoint from $R_0 \setminus a_0$. If X meets $B \cup C$ then similarly X is disjoint from $R_0 \setminus b_0$, and so $X \subseteq V(S)$ and the claim holds. We assume therefore that $X \subseteq A \cup \{a_0\}$. Suppose there is a vertex v of G, not in $A \cup A_0 \cup N \cup F$ but with a neighbour in F. Then $v \notin V(H) \cup X$, and so $v \in B_0$. By 12.3 applied to $F \cup \{v\}$, it follows that $F \cup \{v\}$ contains a major vertex or a left-star, a contradiction. So there is no such v. Hence $V(G) \setminus (A \cup A_0 \cup N), A \cup A_0 \cup N)$ is a skew partition of G, since F is a component of $V(G) \setminus (A \cup A_0 \cup N)$ and b_0 is in a different component, and $A, A_0 \cup N$ are both nonempty and complete to each other. Now by 2.6, $(B \cup C, A)$ is balanced, since a_0 is complete to A and co-complete to $B \cup C$; and therefore from 2.7, (F, A) is balanced (since $B \cup C$ is connected and all vertices in A have neighbours in it). Hence from 4.5, G admits an even skew partition, a contradiction. This proves (1).

Let M be the union of all components of H with no attachment in $V(S)$. Then M is nonempty since it contains the interior of R_0. Let D be the union of all the other components of H. Hence $V(G)$ is partitioned into $A, B, C, D, A_0, B_0, N, M$, where possibly C, D or N may be empty.

(2) $N \neq \emptyset$.

For assume that $N = \emptyset$. Then the only edges between $V(S) \cup D$ and $A_0 \cup B_0 \cup M$ are the edges from A to A_0 and those from B to B_0; and since $A_0 \cup B_0 \cup M$ contains at least 4 vertices (the vertices of R_0) and both A and B contain at least two, this is a 2-join in G, a contradiction. This proves (2).

(3) $C \cup D = \emptyset$.

For assume that $C \cup D$ is nonempty. We claim there are no edges between edges between $C \cup D$ and $A_0 \cup B_0 \cup M$. For certainly there are none between C and $A_0 \cup B_0 \cup M$; suppose that some $u \in D$ has a neighbour in $A_0 \cup B_0 \cup M$. This neighbour is not in M since M and D are unions of disjoint sets of components of H, so we may assume that u has a neighbour $v \in A_0$. Now the component F of H containing u has an attachment in $B \cup C$, by (1), and so by 12.3 applied to $F \cup \{v\}$, it follows that $F \cup \{v\}$ contains a stride or major vertex. But v is a left-star, and F contains no right-star or major vertex, a contradiction. This proves that there are no edges between $C \cup D$ and $A_0 \cup B_0 \cup M$. Since N is complete to $A \cup B$, it follows that $C \cup D \cup A_0 \cup B_0 \cup M, N \cup A \cup B$ is a skew partition of G. By 4.2, it is not loose, and so there is no N'-complete vertex in R_0, where N' is a co-component of N. Let $a_1\text{-}R_1\text{-}b_1, a_2\text{-}R_2\text{-}b_2$ be a step; then $a_1\text{-}a_0\text{-}R_0\text{-}b_0\text{-}b_2$ is an odd path of length ≥ 5; its ends are N'-complete, and its internal vertices are not. By

2.1, there is a leap in N', and so there exist nonadjacent x, y in N so that x-a_0-R_0-b_0-y is a path. But then $((A \cup \{x\}, C, B \cup \{y\}), a_0\text{-}R_0\text{-}b_0)$ is a staircase, contradicting the maximality of (S, R_0). This proves (3).

But then the six sets A, B, A_0, B_0, M, N form an M-join in G. This proves 13.4. ∎

This is the only place in the entire paper where we use M-joins. It is natural to ask whether M-joins are really necessary, or whether 1.2 remains true if we omit them. It appears that the second holds; one of us (Chudnovsky) has what seems to be a proof, which if correct will appear in her PhD thesis. But in this paper we tolerate M-joins.

Let us say a graph G is *nonconforming* if:

- G is Berge
- G and $\overline{G}$ are not line graphs, and G is not a bicograph
- G and $\overline{G}$ do not admit 2-joins, and
- G does not admits an M-join or even skew partition.

The remainder of the paper is a proof of the following.

13.5 *If G is nonconforming then either G or $\overline{G}$ is bipartite.*

Clearly any counterexample to 1.2 is nonconforming, so 13.5 will imply 1.2.

Henceforth in this paper we shall only be concerned with nonconforming graphs, and 13.4 shows that they do not contain long prisms, so henceforth we can exclude long prisms as well. It turns out that for such graphs, there is a useful strengthening of 2.1 - the second alternative of that theorem can no longer hold.

13.6 *Let G be nonconforming, and let P be a path in G with odd length. Let $X \subseteq V(G)$ be co-connected, so that both ends of P are X-complete. Then either:*

1. *there is an edge of P so that both its ends are X-complete, or*
2. *P has length 3 and there is an odd antipath joining the internal vertices of P with interior in X.*

Proof. Let P be P is p_1-$\cdots$-p_n. By 2.1, we may assume that P has length ≥ 5 and X contains a leap u, v say - so u-p_2-$\cdots$-p_{n-1}-v is a path. But then the three paths p_1-v, u-p_n, p_2-$\cdots$-p_{n-1} form a long prism, contrary to 13.4. This proves 13.6. ∎

14 The double diamond

We are now finished with prisms - we cannot dispose of the prism where all three paths have length 1 (yet), and we have disposed of all others. Now we turn to a different type of subgraph, the double diamond.

Let G be Berge. If A, B are disjoint subsets of $V(G)$, we say a *square* in (A, B) is a hole a_1-b_1-b_2-a_2-a_1 of length 4, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. The pair (A, B) is *square-connected* if:

- $|A|, |B| \geq 2$,
- for every partition (X, Y) of A with X, Y nonempty, there is a square a_1-b_1-b_2-a_2-a_1 with $a_1 \in X$ and $a_2 \in Y$
- for every partition (X, Y) of B with X, Y nonempty, there is a square a_1-b_1-b_2-a_2-a_1 with $b_1 \in X$ and $b_2 \in Y$.

It follows that if (A, B) is square-connected then every vertex of $A \cup B$ is in a square. An *antisquare* is a square in $\overline{G}$; that is, an antihole a_1-b_1-b_2-a_2-a_1 with $a_1, a_2 \in A$ and $b_1, b_2 \in B$; and (A, B) is antisquare-connected if it is square-connected in $\overline{G}$. For strips in which every rung has length 1 (and from now on, those are the only kind of strips we shall need), being square-connected is equivalent to being step-connected. We have renamed the concepts because we wanted to improve our notation for a step.

We say a quadruple (A, B, C, D) of subsets of $V(G)$ is a *cube* in G if it satisfies the following conditions:

- A, B, C, D are pairwise disjoint and nonempty
- A is complete to C and B to D, and A is co-complete to D and B to C
- (A, B) is square-connected, and (C, D) is antisquare-connected.

If G contains a double diamond, then it contains a cube in which A, B, C, D all have two elements, and that turns out to be the right approach to the double diamond - grow the cube until it is maximal, and analyze how the remainder of G attaches to it. That is our goal in this section. A cube (A, B, C, D) is *maximal* if there is no cube (A', B', C', D') with $A \subseteq A', B \subseteq B', C \subseteq C', D \subseteq D'$ such that $(A, B, C, D) \neq (A', B', C', D')$. The subgraph $G|(A \cup B \cup C \cup D)$ is called the graph *formed* by the cube. Note that if (A, B, C, D) is a cube in G, then (C, D, B, A) is a cube in $\overline{G}$. (This is very convenient, because it reduces our work by half - we are going to have the usual minor vertices and major vertices, and whatever we can prove about minor vertices is also true in the complement for major ones.)

14.1 *Let G be nonconforming. Let (A, B, C, D) be a maximal cube in G, forming K, let $v \in V(G) \setminus V(K)$, and let X be the set of neighbours of v in $V(K)$. Then either*

- *X is a subset of one of $A \cup B, C \cup D, A \cup C, B \cup D$, and $X \cap (A \cup C)$ is complete to $X \cap (B \cup D)$, or*
- *X includes one of $A \cup B, C \cup D, A \cup D, B \cup C$, and $(A \cup D) \setminus X$ is co-complete to $(B \cup C) \setminus X$.*

Proof. Note that under taking complements the two outcomes become exchanged. If $X \subseteq A \cup B$, and there exists $a \in X \cap A$ and $b \in X \cap B$, nonadjacent, then choose $c \in C$ and $d \in D$, adjacent, and v-a-c-d-b-v is an odd hole. So if $X \subseteq A \cup B$ then the theorem holds. Similarly it holds if $X \subseteq C \cup D$; and trivially it holds if X is a subset of one of $A \cup C, B \cup D$. So we may assume that X meets both A and D. By taking complements, we may also assume that none of $A \cup B, C \cup D, A \cup D, B \cup C$ is a subset of X, that is, either X includes neither of A,C or it includes neither of B,D. These two cases are exchanged when we pass to the complement; so we may assume by taking complements that X includes neither of B, D. Let $A_1 = A \cap X$, and $A_2 = A \setminus A_1$; and define B_1, B_2 etc similarly. We have shown so far that A_1, B_2, D_1, D_2 are nonempty. Choose an antisquare c-d_1-d_2-c'-c such that $d_1 \in D_1$ and $d_2 \in D_2$, and choose $b_2 \in B_2$. Since v-c-d_2-b_2-d_1-v is not an odd hole, it follows that $c \in C_2$. Hence A_1 is complete to B_1; for if $a_1 \in A_1$ and $b_1 \in B_1$ are nonadjacent then v-a_1-c-d_2-b_1-v is an odd hole. If $A_1 = A$, then since A_1 is complete to B_1 it follows that B_1 is empty; but then we can add v to C (because v-d_2-d_1-c'-v becomes a new antisquare), contrary to the maximality of the cube. So A_2 is nonempty. Hence there is a square a_1-b_1-b_2-a_2-a_1 with $a_1 \in A_1$ and $a_2 \in A_2$. Since a_1 is nonadjacent to b_2 and complete to B_1, it follows that $b_2 \in B_2$; but then v-a_1-a_2-b_2-d_1-v is an odd hole, a contradiction. This proves 14.1. ∎

Say a vertex $v \in V(G) \setminus V(K)$ is *minor* if the first case of 14.1 applies to it, and *major* if the second case applies. Then every such vertex is either minor or major and not both; and by taking complements, the minor and major vertices are exchanged.

14.2 *Let G be nonconforming. Let (A, B, C, D) be a maximal cube in G, forming K, let $F \subseteq V(G) \setminus V(K)$ be a connected set of minor vertices, and let X be the set of attachments of F in $V(K)$. Then X is a subset of one of $A \cup B, C \cup D, A \cup C, B \cup D$. Moreover, $X \cap (A \cup C)$ is complete to $X \cap (B \cup D)$.*

Proof. Suppose the first assertion is false, and choose F minimal with this property. We may assume that X meets both of A, D. Since all vertices in F are minor, it follows that F is a path f_1-f_2-$\cdots$-f_k of length ≥ 1. We may assume f_1 is the unique vertex of F with a neighbour in A, and f_k is the unique vertex of F with a neighbour in D. Let X_1,X_2 be the sets of attachments in $V(K)$ of $F \setminus f_k$, $F \setminus f_1$ respectively. From the minimality of F it follows that X_1 is a subset of one of $A \cup B$, $A \cup C$, and X_2 is a subset of one of $B \cup D$, $C \cup D$.

(1) *Not both $X_1 \subseteq A \cup B$ and $X_2 \subseteq B \cup D$.*

For suppose that both these hold. Choose $a \in A$ is adjacent to f_1, and $d \in D$ is adjacent to f_k, and $c \in C$ is adjacent to d; then a-f_1-$\cdots$-f_k-d-c-a is a hole, and so k is odd. Suppose first that f_1 is complete to A. Since it is minor, it has no neighbours in B (for no vertex in B is A-complete). Let a_1-b_1-b_2-a_2-a_1 be a square. Since a_1-b_1, a_2-b_2, f_1-$\cdots$-f_k-d is not a long prism (where $d \in D$ is adjacent to f_k) it follows that there are edges between B and F. Choose i with $1 \leq i \leq k$ minimum so that f_i has a neighbour in B. If f_i is not complete to B, choose a square a_1-b_1-b_2-a_2-a_1 so that f_i is adjacent to b_1 and not to b_2. Then b_2 can be linked onto the triangle $\{f_1, a_1, a_2\}$, via b_2-f_i-$\cdots$-f_1, b_2-b_1-a_1, b_2-a_2, contrary to 2.4. So f_i is complete to B. Let a_1-b_1-b_2-a_2-a_1; then since a_1-b_1, a_2-b_2-f_1-$\cdots$-f_i is not a long prism by 13.4, it follows that $i = 2$. But $k > 2$ since k is odd; so we can add f_1 to C and f_2 to D, contrary to the maximality of the cube. This proves (1) if f_1 is A-complete. Now assume f_1 is not A-complete, and choose a square a_1-b_1-b_2-a_2-a_1 so that f_1 is adjacent to a_1 and not to a_2. Since a_1-f_1-$\cdots$-f_k-d-b_2-a_2-a_1 is not an odd hole (where $d \in D$ is adjacent to f_k), it follows that b_2 has a neighbour in F. Choose i minimum so that b_2 is adjacent to f_i. Let $c \in C$ and $d \in D$ be any adjacent pair of vertices. Then the three paths a_1-b_1, a_2-b_2, c-d form a prism, and since the set of attachments of $\{f_1, \ldots, f_i\}$ in this prism is not local, and does not include a_2, it has an attachment in the third path c-d, by 10.5; and hence $i = k$, and f_k is D-complete. Again, let $c \in C$ and $d \in D$ be adjacent. Then the prism formed by a_1-f_1-$\cdots$-f_k,a_2-b_2,c-d is long, contrary to 13.4. This proves (1).

(2) *Not both $X_1 \subseteq A \cup C$ and $X_2 \subseteq C \cup D$.*

For assume these both hold. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1, and choose $d \in D$ adjacent to f_k. If a_2 is adjacent to f_1 then a_1-b_1, a_2-b_2, f_1-$\cdots$-d form a long odd prism, a contradiction. If a_2 is not adjacent to f_1 then a_1 can be linked onto the triangle $\{b_1, b_2, d\}$, via a_1-b_1,a_1-a_2-b_2, a_1-f_1-$\cdots$-f_k-d, a contradiction. This proves (2).

(3) *Not both $X_1 \subseteq A \cup B$ and $X_2 \subseteq C \cup D$.*

For assume these both hold. Then $X_1 \cap X_2 = \emptyset$, and so f_1 is the unique neighbour in F of the vertices in X_1, and f_k is the unique neighbour of those in X_2. From (1), $X_2 \not\subseteq B \cup D$ and so $X_2 \cap C \neq \emptyset$; and similarly from (2), $X_1 \cap B \neq \emptyset$. Also we are given that $X_1 \cap A, X_2 \cap D \neq \emptyset$. Since a_1-f_1-$\cdots$-f_k-c_1-a_1 is a hole (where $a_1 \in A \cap X_1$ and $c_1 \in C \cap X_2$ it follows that k is even. Since f_1 is minor, $X_1 \cap A$ is complete to $X_1 \cap B$, and so A, B are not subsets of X_1; and similarly C, D are not subsets of X_2. So all the eight sets $A \cap X_1, A \setminus X_1$ etc are nonempty. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2; and choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_k is adjacent to d_1 and not to d_2. It follows that f_1 is nonadjacent to b_2, since $X_1 \cap A$ is complete to $X_1 \cap B$. If $d_1 \in X_2$ then a_1-f_1-$\cdots$-f_k-d_1-c_2-b_2-a_2-a_1 is an odd hole; and otherwise a_1-f_1-$\cdots$-f_k-d_1-b_2-d_2-c_1-a_1 is an odd hole, a contradiction. This proves (3).

(4) *Not both $X_1 \subseteq A \cup C$ and $X_2 \subseteq B \cup D$.*

For assume both these hold. Then again, the only edges between $V(K)$ and F are between X_1 and f_1 and between X_2 and f_k. By (1) and (2), again all four of the sets $A \cap X_1, B \cap X_2, C \cap X_1, D \cap X_2$ are nonempty. There are two cases, depending on the parity of k. First assume k is odd. Then $A \cap X_1$ is co-complete to $B \cap X_2$ (for if ab where an edge there, then a-f_1-$\cdots$-f_k-b-a would be an odd hole), and so $A \setminus X_1, B \setminus X_2$

are nonempty; and similarly $C \setminus X_1, D \setminus X_2$ are nonempty. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2; and choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_k is adjacent to d_1 and not to d_2. Since a_1-f_1-$\cdots$-f_k-d_1-b_2-a_2-a_1 is not an odd hole it follows that $b_2 \in X_2$. But then the the three paths a_1-b_1, c_2-d_1, f_1-$\cdots$-f_k-b_2 form a long prism, contrary to 13.4. Now assume k is even. Then $A \cap X_1$ is co-complete to $B \setminus X_2$ (for if $a \in A \cap X_1$ is adjacent to $b \in B \setminus X_2$ then a-f_1-$\cdots$-f_k-d-b-a is an odd hole, where $d \in X_2 \cap D$) . Similarly $A \setminus X_1$ is co-complete to $B \cap X_2$, $C \cap X_1$ is co-complete to $D \setminus X_2$, and $C \setminus X_1$ is co-complete to $D \cap X_2$. Choose $a \in A \cap X_1$ and a neighbour b of a in B; then $b \in X_2$. Similarly choose $c \in C \cap X_1$ and $d \in D \cap X_2$, adjacent. Then the three paths a-b, c-d, f_1-$\cdots$-f_k form a prism, and so $k = 2$ by 13.4. If f_1 is C-complete then since $C \cap X_1 = C$ is co-complete to $D \setminus X_2$, it follows that f_2 is D-complete; and then we can add f_1 to A and f_2 to B, contrary to the maximality of the cube. So $C \not\subseteq X_1$. Choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_1 is adjacent to c_1 and not to c_2. It follows that f_2 is adjacent to d_2 and not to d_1. If f_1 is A-complete, then as before f_2 is B-complete, and we can add f_1 to C and f_2 to D (because f_1-d_2-f_2-c_1-f_1 is a new antisquare), a contradiction. So f_1 has a nonneighbour in A, and we can choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2. It follows that f_2 is adjacent to b_1 and not to b_2. But then a_1-f_1-f_2-d_2-b_2-d_1-c_2-a_1 is an odd hole, a contradiction. This proves (4).

From (1)-(4), the first assertion of the theorem follows. Now let us prove the second assertion. We may assume X meets both $A \cup C$ and $B \cup D$, and so from what we just proved, either $X \subseteq C \cup D$ or $X \subseteq A \cup B$. Suppose first that $X \subseteq C \cup D$. Choose $c \in C \cap X$ and $d \in D \cap X$, nonadjacent, and choose a path P joining them with interior in F_1. Let a_1-b_1-b_2-a_1-a_1 be a square; then the three paths a_1-b_1, a_2-b_2, c-P-d form a long prism, a contradiction.

So $X \subseteq A \cup B$. Assume $X \cap A$ is not complete to $X \cap B$, and choose a path a-f_1-$\cdots$-f_k-b, where $a \in A, b \in B$ are nonadjacent and $f_1, \dots, f_k \in F$, with k minimum. Since f_1 is minor, its neighbours in A are complete to its neighbours in B, and so $k \geq 2$. Let A' be the set of all vertices $a \in A$ such that a is adjacent to f_1 and there is a nonneighbour b of a in B adjacent to f_k. By assumption $A' \neq \emptyset$. Define B' similarly in B. If $A' = A$ and $B' = B$, then f_1 is A-complete, and so there are no edges between $\{f_1, \dots, f_{k-1}\}$ and B, from the minimality of k; and similarly f_k is B-complete and there are no edges between $\{f_2, \dots, f_k\}$ and A. Choose a square a_1-b_1-b_2-a_2-a_1; then a_1-b_1, a_2-b_2-f_1-$\cdots$-f_k is a prism, so $k = 2$, and we can add f_1 to C and f_2 to D, contrary to the maximality of the cube. So we may assume that $A' \neq A$. Choose a square a_1-b_1-b_2-a_2-a_1 so that $a_1 \in A'$ and $a_2 \notin A'$. Choose $c \in C$ and $d \in D$, adjacent. Choose $b \in B'$ nonadjacent to a_1 (this exists from the definition of A'). From the minimality of k, a_1-f_1-$\cdots$-f_k-b is a path. From the hole a_1-f_1-$\cdots$-f_k-b-d-c-a_1 we deduce that k is even. Since b is not adjacent to a_1, b is different from b_1. Suppose that f_k is adjacent to b_2. Then the set of attachments of $\{f_1, \dots, f_k\}$ with respect to the prism formed by a_1-b_1, a_2-b_2, c-d is not local, and yet it has no attachment in c-d, so by 10.5, both a_2 and b_1 are attachments. Since a_2, b_1 are nonadjacent, it follows from the minimality of k and 10.2 that a_2 is adjacent to f_1 and b_1 to f_k, contradicting that $a_2 \notin A'$.

So f_k is not adjacent to b_2. Then b is different from b_2. Since c has no neighbour in the connected set $F' = \{f_1, ldots, f_k, b\}$, and the set of attachments of F' is not local with respect to the prism formed by $a1$-b_1, a_2-b_2, c-d, it follows from 10.5 that F' has an attachment in a_2-b_2. If a_2 is not an attachment then b_2 is, and from the minimality of k it follows that b is the unique neighbour of b_2 in F'; but then a_1-f_1-$\cdots$-f_k-b, a_2-b_2,c-d form a long prism, a contradiction. So a_2 is an attachment of F'. Since a_2-a_1-f_1-$\cdots$-f_k-b-a_2 is not an odd hole, a_2 has a neighbour in $\{f_1, ldots, f_k\}$. If b_1 also has a neighbour in $\{f_1, ldots, f_k\}$, then (since a_2, b_1 are nonadjacent) from the minimality of k and 10.2 it follows that a_2 is adjacent to f_1 and b_1 to f_k, and hence $a_2 \in A'$, a contradiction. So b_1 has no neighbour in $\{f_1, ldots, f_k\}$. Since a_1-f_1-$\cdots$-f_k-b-b_1-a_1 is not an odd hole it follows that b_1 is not adjacent to b, and therefore has no neighbours in F'. Let P be the path between a_2 and b with interior in F'. From 10.5, a_1 has a neighbour in $P \setminus a_2$. But the only neighbour of a_1 in F' is f_1, so f_1 is in $P \setminus a_2$, and therefore f_1 is adjacent to a_2, and there are no other edges between a_2 and F'. But then a_1-b_1, a_2-b_2, f_1-$\cdots$-f_k-b-d is a long prism, a contradiction. This proves 14.2. ∎

The main result of this section is the following:

14.3 *Let G be nonconforming. Then G does not contain a double diamond as an induced subgraph.*

Proof. Suppose that G contains a double diamond; then it contains a cube, and so there is a maximal cube

(A, B, C, D) in G, forming K. Let F be the set of all minor vertices in $V(G) \setminus V(K)$, and Y the set of all major ones.

(1) *Every co-component Y_1 of Y is complete to one of $A \cup B, C \cup D, A \cup D, B \cup C$, and every edge from $A \cup C$ to $B \cup D$ has a Y_1-complete end.*

This is immediate from 14.2 by taking complements.

(2) *There is no co-component of Y that is complete to $A \cup D$ or $B \cup C$.*

For suppose such a component exists, say Y_1. From the symmetry we may assume it is complete to $A \cup D$. Define L to be the union of C and all components of F with an attachment in C, and M to be the union of B and all other components of F; and define X to be the set of all Y_1-complete vertices of G not in $L \cup M$. So all major vertices belong to $Y_1 \cup X$, and the four sets $L, M, X \cup A \cup D, Y_1$ are nonempty and partition $V(G)$; and since Y_1 is complete to $X \cup A \cup D$, and there are no edges between L, M by 14.2, it follows that $(L \cup M, X \cup A \cup D \cup Y_1)$ is a skew partition of G. By 4.2 it is not loose. We claim it is even. For by 2.6, (L, D) is balanced, since any vertex in B is D-complete and L-co-complete. Let $u, v \in L$ be adjacent, and suppose they are joined by an odd antipath Q_1 with interior in Y_1. If they both have nonneighbours in D, then since D is co-connected they are also joined by an antipath Q_2 with interior in D, which is also odd since its union with Q_1 is an antihole, contradicting that (L, D) is balanced. So we may assume that u is D-complete. Hence $u \notin C$, and so u belongs to some component F_1 of F with an attachment in C. Since u is minor, all its neighbours in C are adjacent to all its neighbours in D, and hence it has no neighbours in C; so $v \in F_1$. Since F_1 has an attachment in C and in D (because u has neighbours in D) it follows that F has no attachments in A, and so u, v have no neighbours in A. But then a-u-Q_1-v-a is an odd antihole (where $a \in A$), a contradiction. Next suppose there exist nonadjacent $u, v \in Y_1$, joined by an odd path P_1 with interior in L. By what we just proved about odd antipaths, it follows that P has length ≥ 5. Now $C \cup D$ is co-connected, and there is no $C \cup D$-complete vertex in L, since every vertex in L is minor or belongs to C. Hence the ends of P are $C \cup D$-complete and its internal vertices are not. But this contradicts 13.6. By 4.5, G admits an even skew partition, a contradiction. This proves (2).

By taking complements, we deduce:

(3) *There is no component of F such that its set of attachments in K is a subset of one of $A \cup C, B \cup D$.*

(4) *There do not exist both a component F_1 of F with set of attachments contained in $A \cup B$ and a co-component Y_1 of Y complete to $A \cup B$.*

For assume that such F_1, Y_1 exist. Define $M = C \cup D \cup (F \setminus F_1)$, and X to be the set of all Y_1-complete vertices in $V(G) \setminus (M \cup F_1)$. So $A \cup B \subseteq X$, and the four sets F_1, M, Y_1, X are all nonempty and form a partition of $V(G)$. Since Y_1 is complete to X and there are no edges between F_1 and M, it follows that $(F_1 \cup M, Y_1 \cup X)$ is a skew partition of G. Suppose that $u, v \in F_1$ are adjacent and are joined by an odd antipath Q_1 with interior in Y_1. Since a-u-Q-v-a is not an odd antihole (where $a \in A$), it follows that one of u, v has a neighbour in A, say u. Since there is an attachment of F_1 in B, and no vertex in B is A-complete, it follows from 14.2 that not every vertex in A is an attachment of F_1, and so u is not A-complete. Choose a square a_1-b_1-b_2-a_2-a_1 such that u is adjacent to a_1 and not to a_2. Since a_2-u-Q-v-a_2 is not an antihole, and u is not adjacent to a_2,it follows that va_2 is an edge. Since u is minor, it is not adjacent to b_2; and since b_2-u-Q-v-b_2 is not an antihole, v is adjacent to b_2. Similarly b_1 is adjacent to u and not v. But then $G|\{a_1, a_2, b_1, b_2, u, v, c, d\}$ (where $c \in C$ and $d \in D$ are adjacent) is $L(K_{3,3} \setminus e)$, a contradiction. This proves (4).

(5) *There do not exist both a component F_1 of F with set of attachments contained in $C \cup D$ and a co-component Y_1 of Y complete to $C \cup D$.*

For assume such F_1, Y_1 exist. Define $M = A \cup B \cup (F \setminus F_1)$, and X to be the set of all Y_1-complete vertices in $V(G) \setminus (M \cup F_1)$. So $C \cup D \subseteq X$, and the four sets F_1, M, Y_1, X are all nonempty and form a partition of $V(G)$. Since Y_1 is complete to X and there are no edges between F_1 and M, it follows that $(F_1 \cup M, Y_1 \cup X)$ is a skew partition of G. Suppose that $u, v \in F_1$ are adjacent and joined by an odd antipath Q with interior in Y_1. Choose $c \in C$ and $d \in D$, nonadjacent. Since c-u-Q-v-c is not an odd antihole, c is adjacent to one of u, v, and similarly so is d. So u, v are both attachments of F_1, contrary to 14.2. Hence there are no such u, v. It follows by taking complements that there are no two nonadjacent $u, v \in Y_1$ joined by an odd path with interior in F_1; and so by 4.5, G admits an even skew partition, a contradiction. This proves (5).

Now if $Y = \emptyset$, then by (3) it follows that G admits a 2-join, a contradiction. So Y is nonempty, and by taking complements, F is nonempty. By (4), passing to the complement if necessary, we may assume that there is no co-component of Y that is complete to $A \cup B$. Hence Y is complete to $C \cup D$, by (2). Since Y is nonempty, it follows from (5) that there is no component F_1 of F with set of attachments contained in $C \cup D$; so by (3), all attachments of F belong to $A \cup B$. Choose a co-component Y_1 of Y. By (3), Y_1 is not A-complete or B-complete. Let X be the set of Y_1-complete vertices in $A \cup B \cup C \cup D$. Let L be the union of $A \setminus X$ and all components of F that have an attachment in $A \setminus X$; and let M be the union of $B \setminus X$ and all other components of F. Since by 14.2 X meets every edge from A to B, and for every component F_1 of F, X contains either all attachments of F_1 in A, or all attachments of F_1 in B, it follows that there is no edge between L and M. Since $L, M, X \cup (Y \setminus Y_1), Y_1$ is a partition of $V(G)$, and Y_1 is complete to $X \cup (Y \setminus Y_1)$, it follows that $(L \cup M, X \cup (Y \setminus Y_1) \cup Y_1$ is a skew partition of G. We may assume it is not loose. Suppose that $u, v \in L$ are adjacent and joined by an odd antipath Q with interior in Y_1. Then one of A, B contains neither of u, v, say A; and then c-u-Q-v-c is an odd antihole, where $c \in C$, a contradiction. So there is no such u, v. Suppose now that $u, v \in Y_1$ are nonadjacent and joined by an odd path P with interior in L. If no vertex of P is in A, then again c-u-P-v-c is an odd hole, where $c \in C$; so P meets A and similarly B. Hence there is a minimal subpath P' of P from A to B, and none of its internal vertices are in $A \cup B$. Since no vertex of P' is in X, and every edge from A to B has an end in X, it follows that P' has length > 1, and so has nonempty interior, which therefore lies in some component F_1 of F. But then the ends of P' are attachments of F_1, and therefore are adjacent by 14.2, a contradiction. By 4.5, G admits an even skew partition, a contradiction. Hence there is no such graph G. This proves 14.3.

We can now prove a form of Chvatal's skew partition conjecture [?], that no minimum imperfect graph G admits a skew partition,because it is a consequence of the following.

14.4 *Let G be nonconforming. Then G does not admit a skew partition.*

Proof. Suppose G admits a skew partition. It does not admit an even skew partition; so by 4.10, one of $G, \overline{G}$ contains as an induced subgraph either a long prism, or a double diamond, or $L(K_{3,3} \setminus e)$. By taking complements we may assume that G contains this subgraph. But this contradicts one of 13.4,14.3 and 9.6. This proves 14.4. ∎

Consequently we have the following:

14.5 *Let G be nonconforming, and let $X, Y \subseteq V(G)$ be nonempty, disjoint, and complete to each other. Then $V(G) \setminus (X \cup Y)$ is nonempty and connected, and if $|X| > 1$ then every vertex in X has a neighbour in $V(G) \setminus (X \cup Y)$.*

Proof. If $X \cup Y = V(G)$ then $\overline{G}$ is not connected, so either admits a skew partition or is a line graph of a bipartite graph, in either case a contradiction (by 14.4). So $G \setminus (X \cup Y)$ is nonnull. Suppose that $V(G) \setminus (X \cup Y)$ is not connected; then $(V(G) \setminus (X \cup Y), X \cup Y)$ is a skew partition, contrary to 14.4, since X, Y are nonempty. So Now suppose some $X \in X$ has no neighbour in $V(G) \setminus (X \cup Y)$. Hence $V(G) \setminus ((X \setminus x) \cup Y)$ is not connected, and since G admits no skew partition it follows that $X = \{x\}$. This proves 14.5. ∎

15 Consequences

Disposal of the long prism and double diamond has a number of consequences that we show in this section. Here is one:

If Y is a co-connected subset of $V(G)$ and uv is an edge of $G \setminus Y$, we say that uv is *Y-complete* if u,v are both Y-complete.

15.1 *Let G be nonconforming. Let C be a cycle in G of length ≥ 6, with vertices $p_1, \dots, p_n$ in order, and let $1 < h < i$ and $i + 1 < j < n$. Let C be induced except possibly for an edge $p_h p_j$. Let $Y \subseteq V(G) \setminus V(C)$ be co-connected, such that the only Y-complete vertices in C are p_n, p_1, p_i, p_{i+1} for some i with $3 \leq i \leq n-3$. Suppose there is a path F from p_h to p_j (possibly of length 1), such that there are no edges between its interior and $V(C) \setminus \{p_h, p_j\}$. Then some vertex of F is Y-complete.*

Proof. Assume no vertex of F is Y-complete. Since the hole p_1-p_2-$\cdots$-p_h-F-p_j-$\cdots$-p_n-p_1 is even, and the path p_1-$\cdots$-p_h-$\cdots$-p_i is even (by 2.2), it follows that the path p_i-p_{i-1}-$\cdots$-p_h-F-p_j-$\cdots$-p_n is odd, and therefore has length 3 by 13.6. So F has length 1, and $i = h + 1$ and $n = j + 1$. Similarly $h = 2$ and $j = i + 2$, and so $n = 6$. Then p_1, p_4 are adjacent, so there is an antipath Q joining them with interior in Y. But then in $\overline{G}$, the three paths p_1-Q-p_4, p_3-p_6, p_5-p_2 form a long prism, a contradiction. This proves 15.1. ∎

There is a considerable strengthening of 3.2, the following.

15.2 *Let G be nonconforming. Let p_1-$\cdots$-p_m be a path in a Berge graph G. Let $2 \leq s \leq m-2$, and let p_s-q_1-$\cdots$-q_n-p_{s+1} be an antipath, where where $n \geq 2$. Assume that p_1, p_m are both adjacent to all of $q_1, \dots, q_n$. Then n is even and $m = 4$.*

Proof. From 3.2 and the symmetry, we may assume that n is odd, and the only nonedges between $\{p_{s-2}, p_{s-1}, p_s, p_{s+1}, p_{s+2}\}$ and $\{q_1, \dots, q_n\}$ are $p_{s-1}q_n, p_s q_1, p_{s+1} q_n$. But then $\overline{G}|\{\{p_{s-2}, p_{s-1}, p_s, p_{s+1}, p_{s+2}, q_1, \dots, q_n\}$ is a long prism, contrary to 13.4. This proves 15.2. ∎

Second, there is a strengthening of 10.2.

15.3 *Let G be nonconforming. Let R_1, R_2, R_3 be a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. (So each R_i has length 1.) Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Assume no vertex in F is big with respect to the prism. Then there is a path f_1-f_2 in F such that (up to symmetry) either:*

1. *f_1 is adjacent to a_1, a_2, a_3, and f_2 is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, f_2\}$ and $V(K)$, or*
2. *f_1 is adjacent to a_1, a_2, and f_2 is adjacent to b_3, and there are no other edges between $\{f_1, f_2\}$ and $V(K) \setminus a_3$.*

Let us apply 10.2. By 9.6, 10.2.1 does not hold, and by 13.4 and 14.3, 10.2.3 does not hold. So one of 10.2.2, 10.2.4 holds, and the corresponding path must have length 1 by 13.4. This proves 15.3. ∎

As a consequence, 10.5 can be strengthened.

15.4 *Let G be nonconforming. Let R_1, R_2, R_3 be a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected. Assume that the set of attachments of F in K is not local; then F has attachments in each of R_1, R_2, R_3.*

Proof. This is immediate from 15.3. ∎

That in turn has another useful consequence:

15.5 *Let G be nonconforming, let C be a hole in G, and let $Q \subseteq V(G) \setminus V(C)$ be co-connected. Let P be a path in C of length > 1 so that its ends are Q-complete and its internal vertices are not. Then P has even length.*

Proof. The claim is trivial if C has length 4, so we assume it has length ≥ 6. Let the vertices of C be $p_1, \dots, p_n$ in order, and let P be $p_1\text{-}\cdots\text{-}p_k$ say, where $3 \leq k < n$. Assume k is even. Then by 13.6 applied to P we deduce that P has length 3, so $k = 4$. By 2.2 every Q-complete vertex is adjacent to one of p_2, p_3, so there are none in the interior of the odd path $p_k\text{-}p_{k+1}\text{-}\cdots\text{-}p_n\text{-}p_1$. By 13.6 this path also has length 3, so $n = 6$. But then in $\overline{G}$, the three paths $p_1\text{-}p_4, p_5\text{-}p_2, p_3\text{-}p_6$ form a prism; and the attachments of the connected set Q in this prism are precisely p_1, p_3, p_4, p_6, contrary to 15.4. This proves 15.5. ∎

There is also a strengthening of 3.3; we no longer need the vertex z.

15.6 *Let G be nonconforming, let C be a hole in G of length ≥ 6, with vertices $p_1, \dots, p_m$ in order, and let Q be an antipath with vertices $p_1, q_1, \dots, q_n, p_2$, with length ≥ 4 and even. There is at most one vertex in $\{p_3, \dots, p_m\}$ complete to either $\{q_1, \dots, q_{n-1}\}$ or $\{q_2, \dots, q_n\}$, and any such vertex is one of p_3, p_m.*

Proof. Suppose first that one of $q_1, \dots, q_n$ belongs to the hole. Since it is adjacent to at least one of p_1, p_2 (since Q is an antipath), we may assume that it is p_m; and since it is nonadjacent to p_2, it follows that $p_m = q_n$. So $p_3 \neq q_1$ (since q_1 is adjacent to q_n), and therefore no more of $q_1, \dots, q_n$ belong to C. Suppose that there exists i with $3 \leq i < m$ complete to either $\{q_1, \dots, q_{n-1}\}$ or $\{q_2, \dots, q_n\}$. If $i < m-1$ then p_i is not adjacent to $p_m = q_n$, so p_i is complete to $\{q_1, \dots, q_{n-1}\}$; but then $p_i\text{-}p_1\text{-}q_1\text{-}\cdots\text{-}q_n\text{-}p_i$ is an odd antihole. So $i = m-1$. By 15.5 applied to the path $p_{m-1}\text{-}p_m\text{-}-1\text{-}p2$ it follows that p_{m-1} is not complete to $\{q_1, \dots, q_{n-1}\}$, and therefore it is complete to $\{q_2, \dots, q_n\}$ and nonadjacent to q_1. But then $p_2\text{-}p_{n-1}\text{-}q_1\text{-}\cdots\text{-}q_n\text{-}p_2$ is an odd antihole, a contradiction. So there is no such i, and therefore the theorem holds in this case.

So we may assume that none of $q_1, \dots, q_n$ belong to C. Let $X = \{q_1, \dots, q_n\}$, and let Y_1, Y_2 be the sets of vertices in $\{p_3, \dots, p_m\}$ complete to $X \setminus q_n$,$X \setminus q_1$ respectively. As in the proof of 3.3, we have

(1) $Y_1 \subseteq Y_2 \cup \{p_m\}$, *and* $Y_2 \subseteq Y_1 \cup \{p_3\}$.

(2) *If* $Y_1 \not\subseteq \{p_m\}$ *then* $p_3 \in Y_1 \cap Y_2$, *and if* $Y_2 \not\subseteq \{p_3\}$ *then* $p_m \in Y_1 \cap Y_2$.

For assume $Y_1 \not\subseteq \{p_m\}$, and choose i with $3 \leq i \leq m-1$ minimum so that $p_i \in Y_1$. By (1), $p_i \in Y_2$, so we may assume $i > 3$, for otherwise the claim holds. By 15.5 applied to the co-connected set $X \setminus q_n$, i is even. The path $p_1\text{-}\cdots\text{-}p_i$ is odd, and between $X \setminus q_1$-complete vertices, so by 15.5 it contains another in its interior, say p_h. From the minimality of i, $p_h \notin Y_1$, so by (1) $h = 3$, and 15.5 applied to the path $p_3\text{-}\cdots\text{-}p_i$ implies that $i = 4$. Choose j with $4 \leq j \leq m$ maximum so that $p_j in Y_2$. By (1), p_j is X-complete. By 15.2 applied to $p_j\text{-}\cdots p_m\text{-}p_1\text{-}\cdots\text{-}p_4$ we deduce that $j \leq 5$, and so $j \neq m$. By 15.5 applied to the path $p_j\text{-}\cdots\text{-}p_m\text{-}p_1$ and co-connected set $X \setminus q_1$, it follows that j is odd, and so $j = 5$. From 15.5 applied to the path $p_j\text{-}\cdots\text{-}p_m\text{-}p_1\text{-}p_2$ and co-connected set $X \setminus q_n$, we deduce that there exists k with $6 \leq k \leq m$ such that $p_k \in Y_1$. Since it is not in Y_2, it follows from (1) that $k = m$, and so $p_m \in Y_1 \setminus Y_2$. But then $p_3\text{-}q_1\text{-}\cdots\text{-}q_n\text{-}p_m\text{-}p_3$ is an odd antihole, a contradiction. This proves (2).

Now not both p_3, p_m are in $Y_1 \cap Y_2$, for otherwise Q could be completed to an odd antihole via $p_2\text{-}p_m\text{-}p_3\text{-}p_1$. Hence we may assume $p_3 \notin Y_1 \cap Y_2$, and so from (2), $Y_1 \subseteq \{p_m\}$. By (1), $Y_2 \subseteq \{p_3\} \cup Y_1$, and so $Y_1 \cup Y_2 \subseteq \{p_3, p_m\}$. We may therefore assume that $Y_1 \cup Y_2 = \{p_3, p_m\}$, for otherwise the theorem holds. In particular, $p_3 \in Y_2$. If also $p_m \in Y_2$, then $p_3\text{-}p_4\text{-}\cdots\text{-}p_m$ is an odd path between $X \setminus q_1$-complete vertices, and none of its internal vertices are $X \setminus q_1$-complete, contrary to 15.5. So $p_m \notin Y_2$, and so $p_m \in Y_1$; but then $p_3\text{-}q_1\text{-}q_2\text{-}\cdots\text{-}q_n\text{-}p_m\text{-}p_3$ is an odd antihole, a contradiction. This proves 15.6. ∎

This implies a strengthening of 3.1:

15.7 *Let G be nonconforming. Let C be a hole of length > 4 and D an antihole of length > 4. Then $|C \cap D| \leq 2$.*

Proof. Assume that $|C \cap D| \geq 3$; then by taking complements if necessary, we may assume that there are three vertices in $C \cap D$ such that exactly one pair of them is adjacent. Hence we can number the vertices of C as $p_1, \ldots, p_m$ in order, and the vertices of Q as $p_1, q_1, \ldots, q_n, p_2, p_k$ for some k with $3 \leq k \leq m-1$. (Possibly the hole and antihole also share some fourth vertex.) Hence the antipath p_1-q_1-$\cdots$-q_n-p_2 has length ≥ 4 and even. The vertex p_k is complete to $\{q_1, \ldots, q_n\}$, and different from p_3, p_m, contrary to 15.6. This proves 15.7. ∎

16 Odd wheels

16.1 *Let C be a hole in a nonconforming graph G, and let $Y \subseteq V(G) \setminus V(C)$ be co-connected. If there are at least two Y-complete vertices in C, then either there are exactly two and they are adjacent, or there are an even number of Y-complete edges in C.*

Proof. This is immediate from 15.5. ∎

A *wheel* in a graph G is a pair (C, Y), satisfying:

- C is a hole of length ≥ 6
- Y is a non-null co-connected set disjoint from C
- there are two disjoint edges of C, both Y-complete.

We call C the *rim* and Y the *hub* of the wheel.

Let us say that distinct vertices u, v of the rim of a wheel (C, Y) have the same *wheel-parity* if there is a path of the rim joining them containing an even number of Y-complete edges; (and hence by 16.1, so does the second path, if u, v are nonadjacent); and opposite wheel-parity otherwise.

16.2 *Let (C, Y) be a wheel in a nonconforming graph G. Let $v \in V(G) \setminus (V(C) \cup Y)$, such that v is not Y-complete, and let X be the set of its neighbours in C. Suppose that there exist vertices in X with opposite wheel-parity. Then in every path of C between them there is a $Y \cup \{v\}$-complete edge. Moreover, either:*

- *v has only two neighbours in C, and they are adjacent and both Y-complete, or*
- *there is a 3-vertex path p_1-p_2-p_3 in C, so that p_1, p_2, p_3 are all $Y \cup \{v\}$-complete, and every other neighbour of v is C has the same wheel-parity as p_1, or*
- *$(C, Y \cup \{v\})$ is a wheel.*

Proof.

(1) *Let P be a path in C of length ≥ 1, such that its ends are adjacent to v and have opposite wheel-parity. Then either some internal vertex of P is a neighbour of v, or P has length 1.*

Let C have vertices $p_1, \ldots, p_n$ in order, and let P be the path p_1-$\cdots$-p_j say, where $j < n$. We assume no internal vertex of P is a neighbour of v, and that $j \geq 3$. Choose $Y' \subseteq Y$ minimal such that Y' is co-connected and there are an odd number of Y'-complete edges in P. From the hole v-p_1-$\cdots$-p_j-v it follows that j is odd. Since p_1, p_j have different wheel-parity (with respect to the wheel (C, Y')), this hole contains an odd number of Y'-complete edges; and so from 15.5, it contains just one Y'-complete edge and only two Y'-complete vertices. Hence there exists i with $1 \leq i < j$ so that x_i, x_{i+1} are the only Y'-complete vertices in

P. Since j is odd, it follows that exactly one of $i-1, j-i$ is even; so (by replacing P by its reverse if necessary) we may assume that i is odd. So p_j is different from p_{i+1}, and hence p_j is not Y'-complete. There are two disjoint Y'-complete edges in C, so one of them does not use p_i; and therefore it does not use p_1 either (for p_1 is not Y'-complete unless $i = 1$). Hence both its ends are in $\{p_{j+1}, \dots, p_n\}$. Consequently $n \geq j+2$, and since n is even and j is odd it follows that $n \geq j+3$. Therefore there is a Y'-complete vertex in $\{p_{j+2}, \dots, p_{n-1}\}$.

Suppose that v has a neighbour in $\{p_j, \dots, p_{n-1}\}$. Then there is a path Q from v to a Y'-complete vertex u say, with $V(Q) \subseteq \{v, p_j, \dots, p_{n-1}\}$, such that no internal vertex of Q is Y'-complete. The path p_i-$\cdots$-p_1-v-Q-u has both ends Y'-complete, and no internal vertex Y'-complete, and the Y'-complete vertex p_{i+1} has no neighbour in its interior; so this path is even, that is, Q is odd. Hence the path p_{i+1}-$\cdots$-p_j-v-Q-u is odd, and so by 13.6 has length 3; and hence $j = i+2$ and Q has length 1. Also, every Y'-green vertex is adjacent to one of p_j, v, by 2.2; and so p_i is adjacent to v, and so $i = 1, j = 3$; and v is adjacent to every Y'-complete vertex in C except p_2 and possibly p_4 (for no others are adjacent to p_3). In particular, there are two nonadjacent $Y' \cup \{v\}$-complete vertices in C, and so by 15.5 there are an even number of $Y' \cup \{v\}$-complete edges in C. But all Y'-complete edges of C are $Y' \cup \{v\}$-complete except p_1-p_2 and possibly p_3-p_4; and since there are also an even number of Y'-complete edges in C, it follows that p_4 is Y'-complete and nonadjacent to v. But then the vertices $v, p_1, p_2, p_3, p_4, p_5$ violate 15.1.

This proves that v has no neighbour in $\{p_j, \dots, p_{n-1}\}$. Choose k with $j \leq k \leq n$ minimum such that p_k is Y'-complete. From 15.5 it follows that the path p_{i+1}-$\cdots$-p_k is even, and so k is even. Suppose that v is not adjacent to p_{j+1}. Then by 16.1 applied to $(C, \{v\})$ it follows that v is not adjacent to p_n, so p_1, p_j are its only neighbours in C. But p_i-$\cdots$-p_1-v-p_j-$\cdots$-p_k is odd, and therefore has length 3 by 13.6; and by 2.2, every Y'-complete vertex in C is adjacent to v except possibly p_{j-1}, p_{j+1}, a contradiction since there is a Y'-complete vertex in $\{p_j + 2\}, \dots, p_{n-1}\}$. So v is adjacent to p_{i+1}. From 16.1 it is also adjacent to p_n, so it has exactly four neighbours in C. Choose m with $k \leq m \leq n$ maximum so that p_m is Y'-complete. It follows that $m \geq j+2$. Then 2.3 applied to the path p_m-$\cdots$-p_n-p_1-$\cdots$-p_i implies that m is odd, and therefore $m > k$. Suppose that $m > k+1$. Then p_m-$\cdots$-p_n-v-p_{j+1}-$\cdots$-p_k is an odd path, and p_{i+1} has no neighbour in its interior, contrary to 2.2. So $m = k+1$, and there is symmetry between the paths p_1-$\cdots$-p_j and p_{j+1}-$\cdots$-p_n. Both these paths have length ≥ 2; suppose they both have length 2. Then $n = 6$, and the only $y \cup \{v\}$-complete vertices in C are p_1, p_4, contrary to 15.5. So one of the paths has length > 2, and from the symmetry we may assume that $j \geq 4$. Hence the hole $H = v$-p_1-$\cdots$-p_j-v has length ≥ 6, and the only Y'-complete vertices in it are p_i, p_{i+1}. By 2.11, Y' contains a hat or a leap. But p_{k+1} has no neighbour in this hole, so the pair $(V(H), Y')$ is balanced by 2.6, and hence there is no leap. So there is a hat; that is, there exists $y \in Y'$ with no neighbours in H except p_i, p_{i+1}. From the minimality of Y' it follows that $Y' = \{y\}$. But then $G|(V(C) \cup \{v, y\}i)$ is the line graph of a bipartite subdivision of K_4, a contradiction. This proves (1).

From (1) the first assertion of the theorem follows. Now we prove the second assertion. Suppose that v has at least two neighbours in C of the same wheel-parity, and at least two others with the opposite wheel-parity. Then there are two disjoint paths as in (1), and therefore from (1) there are two disjoint $Y \cup \{v\}$-complete edges in C, and so $(C, Y \cup \{v\})$ is a wheel and the theorem holds. So we may assume that C has vertices $p_1, \dots, p_n$ in order, and v is adjacent to p_1, and v has no other neighbour in C with the same parity as p_1. Since v has at least one other neighbour, we may assume it has a neighbour in $V(C) \setminus \{p_1, p_n\}$ Choose $i > 1$ minimum so that v is adjacent to p_i; then $i < n$, so by (1), $i = 2$. So p_2 is $Y \cup \{v\}$-complete. If v has a third neighbour in C then similarly p_n is $Y \cup \{v\}$-complete and the theorem holds; and if not then again the theorem holds. This proves 16.2. ∎

16.3 *Let (C, Y) be a wheel in a nonconforming graph G. Let $F \subseteq V(G) \setminus (V(C) \cup Y)$ be connected, such that no vertex in F is Y-complete. Let $X \subseteq V(C)$ be the set of attachments of F in C. Suppose that there exist vertices in X with opposite wheel-parity, and there are two vertices in X that are nonadjacent. Then either:*

- *there is a vertex $v \in F$ so that $(C, Y \cup \{v\})$ is a wheel, or*
- *there is a 3-vertex path p_1-p_2-p_3 in C, all Y-complete, and a path p_1-f_1-$\cdots$-f_k-p_3 with interior in F, such that there no edges between $\{f_1, \dots, f_k\}$ and $\{p_4, \dots, p_n\}$.*

Proof. We may assume that F is minimal. If $|F| = 1$ then the result follows from 16.2, so we assume $|F| \geq 2$.

(1) *If there do not exist nonadjacent vertices in X with different wheel-parity, then the theorem holds.*

For there exist vertices in X with different wheel-parity, which are therefore adjacent; say p_1, p_2, where C has vertices $p_1, \dots, p_n$ in order. So p_1, p_2 are both Y-complete, since they have different wheel-parity. There is a third attachment of F, since there are two that are nonadjacent, say p_i where $3 \leq i \leq n$. Since p_1, p_2 have different wheel-parity, we may assume that p_2, p_i have different wheel-parity; and therefore p_2, p_i are adjacent, that is, $i = 3$, and p_3 is Y-complete. Suppose F has a fourth attachment p_j say, where $4 \leq j \leq n$. From the symmetry we may assume $J \neq n$; and so p_j is nonadjacent to both p_1, p_2, and one of these has different parity from p_j, a contradiction. So p_1, p_2, p_3 are the only attachments of F, and then the theorem holds.

From (1) we may assume there are nonadjacent vertices in X with opposite wheel-parity, say x_1, x_2, and therefore F is the interior of a path between x_1, x_2, from the minimality of F. Let C have vertices $p_1, \dots, p_n$ in order; then we may assume that p_1, p_k have opposite wheel-parity, and $3 \leq m \leq n-1$, and there is a path p_1-f_1-$\cdots$-f_k-p_m where $F = \{f_1, \dots, f_k\}$. Let X_1 be the set of attachments in C of $F \setminus f_k$, and X_2 the set of attachments of $F \setminus f_1$. From the minimality of F, for $i = 1, 2$ either all members of F_i have the same wheel-parity, or there are at most two members of F_1, adjacent if there are two. Since $k \geq 2$ it follows that $X_1 \cup X_2 = X$.

(2) *X_1 and X_k do not both have members of opposite wheel-parity.*

For suppose they do; then X_1, X_2 both consist of exactly two adjacent vertices of opposite wheel-parity, say $X_1 = \{p_1, p_2\}$ and $X_2 = \{p_m, p_{m+1}\}$. So p_1, p_2, p_m, p_{m+1} are all Y-complete, and all distinct since two of them are nonadjacent and of opposite wheel-parity. Then G contains a long prism since $n \geq 6$, a contradiction. This proves (2)

(3) *If X_1 has members of opposite wheel-parity then the theorem holds.*

For assume X_1 has members of opposite wheel-parity. Then we may assume its only members are p_1, p_2, and they are both Y-complete. From (1) we may assume that all members of Y_2 have the same wheel-parity as p_2. In particular, f_1 has no neighbour in $F \setminus f_1$. So the only edges between F and C are $f_1 p_1$, edges incident with p_2, and edges incident with f_k. Suppose that p_2 also has no neighbour in $F \setminus f_1$. If f_k has a unique neighbour x in C, then x can be linked onto the triangle $\{p_1, p_2, f_1\}$; if f_k has two nonadjacent neighbours in C then f_k can be linked onto the same triangle; and if it has exactly two neighbours and they are adjacent, then G contains a long prism, in each case a contradiction. So p_2 has a neighbour in $F \setminus f_1$. Let F_1 be the path p_1-f_1-$\cdots$-f_k, and let F_2 be the path from p_2 to f_k with interior in $F \setminus f_1$. Then p_1 has no neighbours in $F_2 \setminus p_2$. Let Q_1 be the path from f_k to p_n with interior in $C \setminus p_1$. Now p_1-F_1-f_k-Q_1-p_n-p_1 is a hole, so F_1 and Q_1 have the same parity; and since it contains an odd number of Y-complete edges (since all neighbours of f_k have wheel-parity opposite from that of p_1) it follows that it contains exactly one such edge and only two Y-complete vertices. Since p_1 is Y-complete, the other is therefore p_n. The path p_2-F_2-f_k-Q_1-p_n is between Y-complete vertices, and no internal vertex is Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior; so it is even by 2.2, that is, F_1, F_2 have opposite parity. Now there is a Y-complete vertex in $\{p_4\ p_{n-1}\}$; for there are two disjoint Y-complete edges in C, and an even number of Y-complete edges in C. Let p_s be such a vertex, where $4 \leq s \leq n-1$. We claim that f_k has a neighbour in $\{p_4, \dots, p_{n-1}\}$. For if not, then since $X \neq \{p_n, p_1, p_2\}$ (because there are nonadjacent vertices in X of opposite wheel-parity), it follows that f_k is adjacent to p_3. Since p_s is not in Q_1, it follows that p_3 is not in Q_1, and so f_k has another neighbour, which must be p_n; but then f_k-p_3-p_4-$\cdots$-p_n-f_k is an odd hole. So f_k has a neighbour in $\{p_4, \dots, p_{n-1}\}$; and therefore there is a path Q_2 from f_k to some x, such that x is the unique Y-complete vertex in Q_2, and $V(Q_2 \setminus f_k) \subseteq \{p_4, \dots, p_{n-1}\}$. Now the path p_2-F_2-f_k-Q_2 has both ends Y-complete, and no internal vertex Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior, so it is even by 2.2. Therefore the path p_1-F_1-f_k-Q_2 is odd, since F_1, F_2 have opposite parity; and again its ends are Y-complete and its internal vertices are not. So it has length 3, by 13.6, and so $k = 2$; and every Y-complete vertex is adjacent to one of f_1, f_2. Consequently there is no Y-complete vertex in C different from p_1 with the same wheel-parity as p_1, a contradiction. This proves (3).

From (3) we may assume that all members of X_1 have the same wheel-parity, and all members of X_2 have the opposite wheel-parity. It follows that $X_1 \cap X_2 = \emptyset$, and so there are no edges between the interior of F and C. So X_1 is the set of neighbours of f_1 in C, and X_2 is the set of neighbours of f_k in C.

(4) *Not both f_1, f_k have nonadjacent neighbours in C.*

For assume they do. Then there are paths Q, R of C, disjoint and with no edges between them, and both containing neighbours of both f_1, f_k. Choose Q, R minimal, and let Q have ends q_1, q_2; then from the minimality of Q, q_1 is the unique neighbour of one of f_1, f_k in Q, and q_2 is the unique neighbour of the other. Let f_1q_1 and f_kq_2 be edges say. Similarly let R have ends r_1, r_2, where f_1r_1, f_kr_2 are edges. Since q_1, q_2 have opposite wheel-parity, it follows that there are an odd number of Y-complete edges in the hole f_1-$\cdots$-f_k-q_2-Q-q_1-f_1; so by 16.1 there is exactly one, and just two Y-complete vertices. But this contradicts 15.1. This proves (4).

(5) *X_1 does not consist of just two adjacent vertices, and nor does X_2.*

For suppose $X_1 = \{p_1, p_2\}$ say. As before, if f_k has a unique neighbour x in C, then x can be linked onto the triangle $\{p_1, p_2, f_1\}$; if f_k has two nonadjacent neighbours in C then f_k can be linked onto the same triangle; and if it has exactly two neighbours and they are adjacent, then G contains a long prism, in each case a contradiction. This proves (5).

From (4) and (5) we may assume that X_1 has only one member, say p_1. Choose i, j with $2 \leq i, j \leq n$, such that p_i, p_j are adjacent to f_k, with i minimum and j maximum. From the hole p_1-f_1-$\cdots$-f_k-p_i-p_{i-1}-$\cdots$-p_1 ($= H_1$ say) we deduce that i, k have the same parity, and from the hole p_1-f_1-$\cdots$-f_k-p_i-p_{i+1}-$\cdots$-p_n-p_1 ($= H_2$ say) that j, k have the same parity. Since p_1, p_i have different wheel-parity, and so do p_1, p_j, there is an odd number of Y-complete edges in each of H_1, H_2; and therefore there is exactly one Y-complete edge and exactly two Y-complete vertices in each of the holes, by 16.1. Suppose that $i = j$. Then there are only two Y-complete edges in C, and therefore they are disjoint, and p_1, p_i are not Y-complete (since H_1, H_2 both have only two Y-complete vertices), contrary to 15.1. So $j > i$. If p_1 is not Y-complete, then the Y-complete edge in H_1 is disjoint from the path p_1-f_1-$\cdots$-f_k, and so is the one in H_2; but this contradicts 15.1. So p_1 is Y-complete. Since H_1 contains only two Y-complete vertices and they are adjacent, the other is p_2, and similarly p_n is Y-complete.

(6) *f_k has no neighbour in $\{p_3, \dots, p_{j-2}\}$.*

For assume it does. We claim there is also a Y-complete vertex in this set; for otherwise the only Y-complete vertices in C are p_n, p_1, p_2 and possibly p_{j-1}, which is impossible since there are two disjoint Y-complete edges and an even number of Y-complete edges in C. Hence there is a path P say from f_k to some x so that x is the unique Y-complete vertex in P and $V(P \setminus f_k) \subseteq \{p_3, \dots, p_{j-2}\}$. The path p_2-$\cdots$-p_i-f_k-P-x is even, since its ends are Y-complete, no internal vertex is Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior. The path p_1-f_1-$\cdots$-f_k-P-x is therefore odd (since k, i have the same parity), and also its ends are Y-complete and no internal vertex is Y-complete; so it has length 3 by 13.6, and hence $k = 2$ and every Y-complete vertex is adjacent to one of f_1, f_2, by 2.2. So there is no Y-complete vertex in $C \setminus p_1$ with the same wheel-parity as p_1, a contradiction. This proves (6).

Since f_k is adjacent to p_i, and $i < j$ and $j - i$ is even, it follows from (6) that $i = 2$, and similarly f_k has no neighbours in $\{p_{i+2}, \dots, p_{n-1}\}$ and $j = n$. So f_k has no neighbours in $\{p_3, \dots, p_{j-2}\} \cup \{p_{i+2}, \dots, p_{n-1}\} = \{p_3, \dots, p_{n-1}\}$, and therefore p_2, p_n are its only neighbours, contradicting that there are nonadjacent vertices in X of opposite wheel-parity. This proves 16.3. ∎

A maximal path in the rim whose vertices are all Y-complete is called a *segment* or *Y-segment* of the wheel; and a maximal path of the rim of length ≥ 2 such that all its internal vertices are not Y-complete is called a *gap* or *Y-gap* of the wheel. It follows that both ends of every gap are Y-complete, and that C is divided into alternating segments and gaps. By 15.5, every gap has even length. A wheel (C, Y) is *odd* if some segment has odd length. The main result of this section is the following.

16.4 *Let G be nonconforming; then G contains no odd wheel.*

Proof. Suppose it does, say (C, Y); and choose such an odd wheel with Y maximal, and subject to that, such that the number of Y-complete edges in C is minimum. (We refer to these conditions as the "optimality" of (C, Y).)

(1) *There is no vertex $v \in V(G) \setminus (V(C) \cup Y)$ such that v is not Y-complete and $(C, Y \cup \{v\})$ is a wheel.*

Suppose there is such a vertex v. Define a "line" to be a maximal subpath of C with no internal vertex adjacent to v. It follows that every edge of C is in a unique line. Let C have vertices $p_1, \dots, p_n$ in order, and let S be an odd Y-segment. By the optimality of Y, there are no odd $Y \cup \{v\}$-segments, and in particular, an even number of edges of S are $Y \cup \{v\}$-complete. Hence an odd number are not, and therefore there is a line L containing an odd number of edges of S that are not $Y \cup \{v\}$-complete. In particular it contains at least one edge that is Y-complete and not $Y \cup \{v\}$-complete, so L has length > 1. Let the ends of L be p, q. By 16.2, p and q have the same wheel-parity with respect to (C, Y), and so L contains an odd number of edges of some other Y-segment $S' \neq S$. In particular, there are two disjoint Y-complete edges in the hole v-p-L-q-v ($= H$ say); so H has length ≥ 6 (because v is not Y-complete) and so (H, Y) is a wheel. Moreover it is an odd wheel, for it contains an odd number of edges of S, and those edges form either one or two Y-segments in H, and one of these segments is odd. Since there is a $Y \cup \{v\}$-complete edge in C, which therefore does not belong to L, this contradicts the optimailty of (C, Y). This proves (1).

Since (C, Y) is an odd wheel, C has at least two segments, and therefore there are vertices u, v in C with different wheel-parity and neither of them Y-complete. Let X be the set of all Y-complete vertices in $V(G)$. Then $|X| > 1$, since there at least four in C; so by 14.5, $V(G) \setminus (X \cup Y)$ is nonempty and connected ($= Z$ say), and every vertex in X has a neighbour in it. In particular $p, q \in Z$, so there is a minimal connected subset F of Z such that there are two vertices of $C \setminus X$ of opposite wheel-parity, both with neighbours in F. From the minimality of F, F is a path, and no vertex if F is in C. By 16.3, either:

- there is a vertex $v \in F$ so that $(C, Y \cup \{v\})$ is a wheel, or
- there is a 3-vertex path p_1-p_2-p_3 in C, all Y-complete, and a path p_1-f_1-$\cdots$-f_k-p_3 with interior in F, such that there no edges between $\{f_1, \dots, f_k\}$ and $\{p_4, \dots, p_n\}$.

The first is impossible by (1), so the second holds. But then $C \setminus p_2$ can be completed to a hole C' say, via p_1-f_1-$\cdots$-f_k-p_3i; and C' has length ≥ 6. For every odd segment S of (C, Y), either it contained both or neither of the edges p_1p_2, p_2p_3; and so in either case an odd number of edges of S belong to C'. Since (C, Y) has an odd segment and there are an even number of Y-complete edges in C, it has at least two odd segments. It follows that there are two disjoint Y-complete edges in C', and so (C', Y) is a wheel. Since an odd number of edges of the odd segment S belong to C', it follows that (C', Y) is an odd wheel, contrary to the optimality of (C, Y). This proves 16.4. ∎

17 Another extension of the Roussel-Rubio lemma

Let $\{a_1, a_2, a_3\}$ be a triangle in G. A *reflection* of this triangle is another triangle $\{b_1, b_2, b_3\}$ of G, disjoint from the first, so that the only edges between it and the first triangle are a_1b_1, a_2b_2, a_3b_3. Hence these six vertices make a prism. A subset F of $V(G)$ is said to *catch* the triangle $\{a_1, a_2, a_3\}$ if it is connected and disjoint from that triangle and a_1, a_2, a_3 all have neighbours in F. We begin with the following extremely useful little fact.

17.1 *Let A be a triangle in a nonconforming graph G, and let $F \subseteq V(G)$ be connected and catch A. Then either F contains a reflection of A, or some vertex of F has ≥ 2 neighbours in A.*

Proof. Suppose not, and choose F minimal such that it catches A. Let $A = \{a_1, a_2, a_3\}$ say, and for $i = 1, 2, 3$, let B_i be the set of neighbours of a_i in F. This the three sets B_1, B_2, B_3 are pairwise disjoint and nonempty.

(1) *There is no path in F meeting all of B_1, B_2, B_3.*

For assume there is, and choose it minimal. So then we may assume there is a path P from $b_1 \in B_1$ to $b_2 \in B_2$, such that some vertex of P is in B_3, and for $i = 1, 2$, b_1 is the only vertex of P in B_i. Since B_3 is disjoint from $B_1 \cup B_2$, every vertex of B_3 in P is an internal vertex of P; and so P has length ≥ 2. But then $(C, \{b_3\})$ is an odd wheel, where C is the hole a_1-b_1-P-b_2-a_2-a_1, contrary to 16.4. This proves (1).

Choose $b_1 \in F$ so that $F \setminus b_1$ is connected; then from the minimality of F, $F \setminus x$ does not catch A, and so we may assume that $B_1 = \{b_1\}$. Since F is connected and $|F| \geq 2$, there is a second vertex $b_2 \neq b_1$ in F so that $F \setminus b_2$ is connected, and so similarly we may assume $B_2 = \{b_2\}$. Let P be a path in F between b_1, b_2. By (1) no vertex of P is in B_3, so F contains a connected subset F' including $V(P)$ which contains exactly one vertex of B_3. From the minimality of F, $|B_3| = 1$; let $B_3 = \{b_3\}$ say. Let Q be a minimal path in F such that $b_3 \in V(Q)$ and some vertex of P has a neighbour in Q. From the minimality of Q it follows that Q is vertex-disjoint from P, and Q has ends b_3, x say, where x is the unique vertex of Q with neighbours in P. From the minimality of F, x either has one neighbour in P, or just two neighbours and they are adjacent; for if it has two nonadjacent neighbours, and vertex of P between them could be deleted from F, contrary to the minimality of F. If x has just one neighbours y say in P, then y can be linked onto the traingle A, contrary to 2.4; so it has two adjacent. Since G does not contain a long prism it follows that Q has length 0 and P has length 1, and so F contains a reflection of A, as required. This proves 17.1. ▮

We did not use the full strength of 16.4 in proving 17.1; we just used that there were no odd wheels with hubs of cardinality 1. This suggest that there should be some generalization of 17.1 whose proof does use the full strength of 16.4, and that is true, but not easy - it will be a consequence of the main result of this section.

Before we start on that, let us give another easy application of 16.4, a strengthening of 2.11 for nonconforming graphs.

17.2 *Let G be nonfonforming, and let $F, Y \subseteq V(G)$ be disjoint, such that F is connected and Y is co-connected. Let $a, b \in V(G) \setminus (F \cup Y)$ and $a_0, b_0 \in F$ such that a-a_0-b_0-b is a 3-edge path in G. Suppose that:*

- *a_0, b_0 are both Y-complete, and a, b are not Y-complete,*
- *the only neighbours of a_0, b_0 in F are a and b respectively,*
- *$F \setminus a$ and $F \setminus b$ are both connected.*

Then either:

1. *there is a vertex in Y with no neighbour in F, or*
2. *there are two nonadjacent vertices $y_1, y_2 \in Y$, such that a is the only neighbour of y_1 in F, and b is the only neighbour of y_2 in F.*

Proof. We may assume that every vertex in Y has a neighbour in F, for otherwise statement 1 of the theorem holds.

(1) *There exist nonadjacent y_1, y_2 in Y, such that y_1 is adjacent to a and not b, and y_2 is adjacent to b and not a.*

For choose a path P between a and b. Then the hole a_0-a-P_0-b-b_0-a_0 ($= C$, say) has length ≥ 6. If there are any Y-complete vertices in P, then they belong to the interior of P since a, b are not Y-complete, and there is an odd number of Y-complete edges in P, by 16.1; but then (C, Y) is an odd wheel (the path a_0-b_0 is an odd Y-segment), a contradiction. So there are no Y-complete vertices in P. By 2.11 a pplied to C, Y contains either a hat or a leap. Suppose first it contains a hat, that is, there is a vertex $y \in Y$ with no neighbour in P. By the assumption above, y has a neighbour in F. Consequently F catches the triangle $\{a_0, b_0, y\}$. But y is not adjacent to a or b since it has no neighbour in P, and a is the only vertex in F adjacent to a_0, and the same for b, b_0; and a_0, b_0 are nonadjacent, so F contains no reflection of the triangle. This contradicts 17.1. So y has no neighbour in F, and therefore statement 1 of the theorem holds. Hence we may assume there is no such y, and so there is a leap. This proves (1).

(2) *There is no path in F between a and b such that y_1 or y_2 has a neighbour in its interior.*

For suppose there is such a path, P' say. Then the set $\{y_1, y_2\}$ contains neither a leap not a hat for the hole a_0-a-P'-b-b_0-a_0 ($= C$ say), and so by 2.11 there is a vertex in P adjacent to both y_1, y_2. By 16.1 there is an even number of $\{y_1, y_2\}$-complete edges in this hole, and since a, b are not $\{y_1, y_2\}$-complete, $(C, \{y_1, y_2\})$ is an odd wheel, a contradiction. This proves (2).

Now if neither of y_1, y_2 has any more neighbours in F then statement 2 of the theorem holds; so we assume at least one of them has another neighbour in F. Since $F \setminus a$, $F \setminus b$ are both connected, there is a connected subset F' of $F \setminus \{a, b\}$, so that both a and b have neighbours in F', and at least one of y_1, y_2 has a neighbour in F'. Choose F' minimal with these properties. At least one of y_1, y_2 has a neighbour (say x) in F'. We claim that $F' \setminus x$ is connected. For if not, let L be a component of it, and M the union of the other components. From the minimality of F, not both a, b have neighbours in $L \cup \{x\}$, and not both have neighbours in $M \cup X$; so we may assume all neighbours of a in F' are in L, and all neighbours of b are in M. But then there is a path from a to b with interior in F and with x in its interior, contrary to (2). This proves that $F' \setminus x$ is connected. There is a path from A to b with interior in F', and x is not in it, by (2), and it has length > 1 since A, b are nonadjacent. So a,b both have neighbours in $F' \setminus x$. From the minimality of F', y_1 and y_2 both have no neighbours in $F' \setminus x$. We claim that x is adjacent to both y_1 and y_2. For it is adjacent to at least one, say y_1; let Q be a path from x to b with interior in F'. Then y_1-x Q-b is a path, since y_1 has no more neighbours in F'. Since b_0-y_1-x-Q-b-b_0 is a hole it follows that Q is odd. Therefore a_0-y_1-x-Q-b-y_2-a_0 is not a hole, and so y_2 has neighbours in Q. Since it has no neighbours in $F' \setminus x$, this proves our claim that x is adjacent to both y_1, y_2.

There is a path Q from x to a with interior in F'. Since y_1, y_2 both have no more neighbours in F', a_0-y_2-x-Q-a-a_0 is a hole, and therefore Q is odd; and so y_1-x-Q-a-y_1 is not a hole, and therefore Q has length 1, that is, x is adjacent to a. Similarly x is adjacent to b; but then x-a-a_0-b_0-b-x is an odd hole, a contradiction. This proves 17.2. ∎

The following is a variant of 17.2, not so symmetrical, but more useful.

17.3 *Let G be nonfonforming, and let $F, Y \subseteq V(G)$ be disjoint, such that F is connected and Y is co-connected. Let $a, b \in V(G) \setminus (F \cup Y)$ and $a_0, b_0 \in F$ such that a-a_0-b_0-b is a 3-edge path in G. Suppose that:*

- *a_0, b_0 are both Y-complete, and a, b are not Y-complete,*
- *the only neighbours of a_0, b_0 in F are a and b respectively,*
- *$F \setminus a$ is connected.*

Then there is a vertex $y \in Y$ with no neighbour in $F \cup \{b\}$.

Proof. If $F \setminus b$ is connected, the result follows from 17.2. So assume it is not, and let F_1' be the component of $F \setminus b$ that contains a, and F_2' the union of all the other components. For $i = 1, 2$ let $F_i = F_i' \cup \{a\}$. Then $F_1 \setminus a$, $F_1 \setminus b$ are both connected, so by 17.2 either there exists $y \in Y$ with no neighbour in F_1, or there exist nonadjacent $y_1, y_2 \in Y$ with no neighbours in F_1 except a, b respectively. Suppose the first. If y has a neighbour in F_2 then b can be linked onto the triangle $\{y, a_0, b_0\}$, a contradiction to 2.4; and if not then y satsifies the theorem. Now suppose the second. If y_1 has neighbours in F_2 then $F \cup \{y_2\}$ catches the triangle $\{a, a_0, y_1\}$; the only neighbours of a, a_0, y_1 belong to the disjoint sets $F_1', \{y_2\}, F2'$; and there is no reflection since there are no edges between y_1 and F_1', contrary to 17.1. This proves 17.3. ∎

The next result is just a lemma for use in proving the main result of this section, which is 17.5.

17.4 *Let G be nonconforming and let P be a path in G with length > 1, with vertices $p_1, \dots, p_n$ in order. Let $X, Y \subseteq V(G) \setminus V(P)$ be co-connected sets, so that $X \cup Y$ is co-connected, p_1 is X-complete, and p_n is the unique Y-complete vertex in P. (Note that X, Y need not be disjoint.) Let $z \in V(G) \setminus (X \cup Y \cup V(P))$, complete to $X \cup Y$ and with no neighbours in P. Assume that p_n is not X-complete. Let p_n-x_1-$\cdots$-x_k-y be an antipath with interior in X from p_n to some $y \in Y$. Then p_{n-1} is nonadjacent to x_1.*

Proof. Let $F = \{p_{n-1}, x_1, \dots, x_k\} \cup Y$. Since p_{n-1} is not Y-complete it follows that F is co-connected, and both $F \setminus p_{n-1}$, $F \setminus x_1$ are co-connected. The only nonneighbour of z in Q is p_{n-1}, and the only nonneighbour of p_n is x_1; and we may assume that p_{n-1} is adjacent to x_1. Now p_{n-1}-z-p_n-x_1 is a path in $\overline{G}$, and F is connected in $\overline{G}$, and $\{p_1, \dots, p_{n-2}\}$ is co-connected in $\overline{G}$. Also, z and p_n are $\{p_1, \dots, p_{n-2}\}$-complete in $\overline{G}$, and p_{n-1}, x_1 are not. We may therefore apply 17.2 in $\overline{G}$, and deduce that there is a vertex in $\{p_1, \dots, p_{n-2}\}$ which is complete (in G) to $Q \setminus p_{n-1}$. But then this vertex is Y-complete, a contradiction. This proves 17.4. ∎

We gave in 2.10 an extension of the Roussel-Rubio lemma to two co-connected sets instead of one (we haven't used that theorem yet, but its time is coming.) In that extension the two sets had to be complete to each other. Now we prove a similar result where the two sets are not complete to each other. Incidentally, unlike 2.10, what we are going to prove here is not true for general Berge graphs - we need the hypothesis that G is nonconforming.

17.5 *Let G be nonconforming and let P be an odd path in G with length > 1, with vertices $p_1, \dots, p_n$ in order. Let $X, Y \subseteq V(G) \setminus V(P)$ be co-connected sets, so that $X \cup Y$ is co-connected, p_1 is X-complete, and p_n is the unique Y-complete vertex in P. (Note that X, Y need not be disjoint.) Let $z \in V(G) \setminus (X \cup Y \cup V(P))$, complete to $X \cup Y$ and with no neighbours in P. Then an odd number of edges of P are Y-complete.*

Proof. If possible choose a counterexample P, X, Y such that

1. P is minimal
2. subject to condition 1, $X \cup Y$ is minimal, and
3. subject to conditions 1 and 2, $|X| + |Y|$ is minimum.

We refer to this property as the "optimality" of P, X, Y.

(1) *No vertex of $P \setminus p_1$ is X-complete.*

If p_n is X-complete, then since P has odd length > 1, and the X-complete vertex z has no neighbour in P, it follows from 2.2 and 2.3 that there are an odd number of Y-complete edges in P, and the theorem holds, a contradiction. So p_n is not X-complete. By 17.4, p_{n-1} is not X-complete. Since p_1 is X-complete, we can choose i with $1 \leq i \leq n$ maximum such that p_i is X-complete. So $i \leq n-2$. Since z has no neighbour in the path p_1-$\cdots$-p_i, if i is even then there is an odd number of X-complete edges in this path and hence in P, by 2.2 and 2.3. So we may assume that i is odd. Hence the theorem is also false for X, Y and the path $p_i, \dots, p_n$.

From the optimality of P, X, Y it follows that $i = 1$. This proves (1).

(2) *Suppose that $x_1, x_2 \in X$ are distinct and such that $X \setminus x_i$ is co-connected for $i = 1, 2$. Then $X \cap Y = \emptyset$, and one of x_1, x_2 is the unique vertex of X with a nonneighbour in Y.*

For if $(X \setminus x_i) \cup Y$ is not co-connected for some i, then Y is disjoint from $X \setminus x_i$ (since both these sets are co-connected), and Y is complete to $X \setminus x_i$; and therefore $x_i \notin Y$ (since x_i has a nonneighbour in $X \setminus x_i$), so $X \cap Y = \emptyset$. But then the statement of (2) holds. So we may assume that $(X \setminus x_i) \cup Y$ is co-connected for $i = 1, 2$. From the optimality of P, X, Y it follows that the theorem holds for $X \setminus x_i, Y, P$; and so, since p_n is the unique Y-complete vertex in P, it follows that there are an odd number of $X \setminus x_i$-complete edges in P, for $i = 1, 2$. For $i = 1, 2$ let W_i be the set of $X \setminus x_i$-complete vertices in P. So $W_1 \cap W_2 = \{p_1\}$. It follows that there are nonadjacent vertices p_i, p_j of P, such that $p_i \in W_1 \setminus W_2$ and $p_j \in W_2 \setminus W_1$. Let Q be an antipath in X between x_1 and x_2. Since p_i-x_1-Q-x_2-p_j-p_i is an antihole it follows that Q is odd. Let us say a *line* is a minimal subpath of $P \setminus p_1$ meeting both W_1 and W_2. So every line has length ≥ 1, and has one end in W_1 and the other in W_2, and has no more vertices in either W_1 or W_2. If some line L has odd length > 1, then $(L, X \setminus x_1, X \setminus x_2)$ is another counterexample to the theorem, contrary to the optimality of P, X, Y; and if some line has length 1, say p_i-p_{i+1} where $p_i \in W_1$, then z-p_i-y_1-Q-y_2-p_{i+1}-z is an odd antihole, a contradiction. Hence every line is even. Choose i with $2 \leq i \leq n$ such that $\{p_2, \dots, p_i\}$ includes a line. (This is possible since both W_1, W_2 meet $P \setminus p_1$.) Since all lines have length ≥ 2 it follows that $i \geq 3$. From the minimality of i, $\{p_2, \dots, p_{i-1}\}$ does not include a line, and so for some $k \in \{1, 2\}$, the path p_1-$\cdots$-p_i has both ends $Y \setminus y_k$-complete and no internal vertex $Y \setminus y_k$-complete. But this path has length ≥ 2, and z has no neighbour in it, so by 2.2 it is even, that is, i is odd. Choose j with $j \geq 2$ maximum so that $\{p_j, \dots, p_n\}$ includes a line. Since every line has length ≥ 2 it follows that $2 \leq j \leq n - 2$. From the maximality of j it follows that for some $k \in \{1, 2\}$, $W_k \cap \{p_j, \dots, p_n\} = \{p_j\}$. If the path $p_j, \dots, p_n$ has odd length, then $p_j, \dots, p_n, X \setminus X_k, Y$ is a counterexample to the theorem, contrary to the optimality of P, X, Y. So $n - j$ is even, and hence j is even. Now i is odd, so if $i \geq j$ then p_j-$\cdots$-p_i is an odd line, a contradiction. Hence $i < j$, and $j - i$ is odd. Now the edges $p_{i-1}p_i, p_jp_{j+1}$ are in lines. Consequently we may choose r, s with $i \leq r < s \leq t$ such that the edges $p_{r-1}p_r, p_sp_{s+1}$ are in lines, and $s - r$ is odd, and therefore we may choose such r, s with $s - r$ minimum. From the minimality of $s - r$, none of the edges of the path p_r-$\cdots$-p_s is in a line; and since $p_{r-1}p_r, p_sp_{s+1}$ are in lines, there exist q, t with $2 \leq q < r < t < t \leq n$ such that p_q-$\cdots$-p_r and p_s-$\cdots$-p_t are lines. All lines are even, so $r - q$ and $t - s$ are even, and therefore $t - q$ is odd. Since there is no line in $\{p_{q+1}, \dots, p_{t-1}\}$, we may assume that none of these vertices is in W_1. Since L_1, L_2 are lines it follows that $p_q, p_t \in W_1$. But the path p_q-$\cdots$-p_t is odd, and z has no neighbour in it, contrary to 2.2. This proves (2).

(3) *There is an antipath x_1-$\cdots$-x_s-y_1-$\cdots$-y_t such that $s, t > 1$ and $X = \{x_1, \dots, x_s\}$, and $Y = \{y_1, \dots, y_t\}$.*

For if $|Y| = 1$, $Y = \{y\}$ say, then p_1-$\cdots$-p_n-y-z is an odd path of length ≥ 5 between X-complete vertices, and none of its internal vertices are X-complete, contrary to 13.6. So $|Y| \geq 2$, and similarly $|X| \geq 2$. Hence there are at least two vertices $y \in Y$ so that $Y \setminus y$ is co-connected, and from (2) , $X \cap Y = \emptyset$, and there is a unique vertex y_1 say in Y with nonneighbours in X. By 2, there do not exist two vertices $y \in Y \setminus y_1$ so that $Y \setminus y$ is co-connected; and therefore Y is an antipath with one end y_1, say y_1-$\cdots$-y_t. The same applies for X, and this proves (3).

Choose t' with $1 \leq t' \leq t$, minimum so that p_1 is nonadjacent to $y_{t'}$. (This is possible since p_1 is not Y-complete.) So x_1-$\cdots$-x_s-y_1-$\cdots$-$y_{t'}$-p_1 is an antipath. Define $W = (X \setminus x_1) \cup \{y_1, \dots, y_{t'-1}\}$.

(4) *For every subpath P' of P, if the ends of P' are adjacent to x_1, then there are an even number of W-complete edges in P'.*

For suppose not; then we may choose P' so that no internal vertex of P' is adjacent to x_1. Let P' be

p_h-$\cdots$-p_k say, where $1 \le h < k \le n$. Choose i, j with $h \le i \le j \le k$ such that p_i, p_j are W-complete, with i minimum and j maximum. Since p_k is not X-complete it follows that p_k is not W-complete (because it is adjacent to x_1), and so $j < k$. Since there are an odd number of W-complete edges in p_h-$\cdots$-p_k, it follows that $k \le h+2$, and x_1-p_h-$\cdots$-p_k-x_1 is a hole (so $k-h$ is even), containing an odd number of W-complete edges. By 16.1 it contains exactly one, and only two W-complete vertices; so $j = i+1$. The path p_i-p_{i-1}-$\cdots$-p_h-x_1-z has both ends W-complete, and no internal vertex W-complete, and the W-complete vertex p_j has no neighbour in its interior (since $j < k$); so it is even, by 2.2, and hence $i - h$ is even. Since $k - h$ is even, it follows that p_j-$\cdots$-p_k-x_1-z is an odd path; and again its ends are W-complete and its internal vertices are not. By 13.6 it has length 3, so $k = j + 1$; and by 2.2, every W-complete vertex is adjacent to one of p_k, x_1. But no W-complete vertex in P is adjacent to x_1 except p_1, since no other vertex of P is X-complete. So every W-complete vertex in $P \setminus p_1$ is adjacent to p_k, and so must be one of p_{k-1}, p_{k+1}. In particular, since $i < k-1$ it follows that $i = 1$, and so $j = 2, k = 3$, and the W-complete vertices in P are p_1, p_2 and possibly p_4.

By 17.4 (with X and Y exchanged), p_2 is nonadjacent to $y_{t'}$. Choose d with $1 \le d \le n$ minimum so that $y_{t'}$ is adjacent to p_d; then $d \ge 3$. Then the path p_1-$\cdots$-p_d-$y_{t'}$-z has length ≥ 4, and its ends are W-complete, and its internal vertices are not, so it is even by 13.6. Hence d is odd, and the path p_1-$\cdots$-p_d-$y_{t'}$ is odd. None of its internal vertices are X-complete, and the X-complete vertex z has no neighbour in its interior, and one end p_1 is X-complete, so the other end $y_{t'}$ is not; and hence $t' = 1$, since all other vertices of Y are X-complete. So $W = X \setminus x_1$. Let $V = X \setminus x_s$. Now the path p_1-$\cdots$-p_d-y_1 is between V-complete vertices, and is odd and has length > 1, and the V-complete vertex z has no neighbour in its interior; so by 2.2, there is a V-complete edge in its interior. Choose c with $2 \le c \le d$ minimum such that p_c is V-complete. Since p_2 is nonadjacent to x_1 it follows that $c \ge 3$. Since p_1-$\cdots$-p_c is between V-complete vertices and its internal vertices are not V-complete and z has no neighbour in it, it is even by 2.2, and so c is odd. We already saw that p_1, p_2 and possibly p_4 are W-complete, and $c \ge 3$, so we may choose b with $2 \le b \le c$ maximum such that p_b is W-complete. (Hence $b = 2$ or 4.) Then $b < c$ since p_c is not X-complete. The path p_b-$\cdots$-p_c is odd, and p_b is W-complete, and p_c is V-complete, and no other vertices of the path are either W- or V-complete. If $c - b > 1$ then p_b-$\cdots$-p_c, W, V is a counterexample to the theorem, contradicting the optimality of X, Y, P . So $c = b + 1$. Then the antipath z-p_b-x_1-$\cdots$-x_s-p_c-z is an antihole , so s is odd. But then p_2-x_1-$\cdots$-x_s-y_1-p_2 is an odd antihole, a contradiction. This proves (4).

Choose h with $1 \le h \le n$ maximum such that x_1 is adjacent to p_h. Since x_1-p_i-$\cdots$-p_n is between Y-complete vertices (since $s \ge 2$) and none of its internal vertices are Y-complete, and the Y-complete vertex z has no neighbour in its interior, this path either has length 1 or even length by 2.2. So either $h = n$ or h is odd. From the optimality of P, X, Y, it follows that P, W, Y is not a counterexample to the theorem, and so there are an odd number of W-complete edges in P. Since x_1 is adjacent to p_1, from (4) there are an even number of W-complete edges between p_1 and p_h, so there are an odd number in the path p_h-$\cdots$-p_n, and in particular $h < n$, so h is odd. Choose i, j with $h \le i \le j \le n$ such that p_i, p_j are W-complete, with i minimum and j maximum. Hence $j > i$. Since z-x_1-p_h-$\cdots$-p_i is a path of length ≥ 2 between W-complete vertices, and its internal vertices are not W-complete, and the W-complete vertex p_j has no neighbour in its interior, it follows from 2.2 that $i - h$ is even.

(5) $h > 1$.

For assume $h = 1$; so p_1 is the only neighbour of x_1 in P. Let S be the antipath x_1-$\cdots$-x_s-y_1-$\cdots$-y'_t-p_1. Now x_1-S-p_1-z is an antipath, of length ≥ 4; all its internal vertices have neighbours in $P \setminus p_1$, and its ends do not. By 13.6 applied in $\overline{G}$, it follows that this antipath has even length and so S has odd length. Its ends have no neighbours in $P \setminus \{p_1, p_2\}$, and z is complete to its interior and also has no neighbours in $P \setminus \{p_1, p_2\}$; so by 2.2 applied in $\overline{G}$, some internal vertex of S has no neighbour in $P \setminus \{p_1, p_2\}$. But they are all adjacent to p_j or to p_n, so $j = 2$. By 17.4, p_2 is nonadjacent to $y_{t'}$, and also to x_1 since it is not X-complete. Therefore p_2-x_1-$\cdots$-x_s-y_1-$\cdots$-$y_{t'}$-p_2 is an antihole D say. Choose d with $1 \le d \le n$ minimum so that $y_{t'}$ is adjacent to p_d; then $d \ge 3$, and so x_1-p_1-$\cdots$-p_d-$y_{t'}$-x_1 is a hole of length ≥ 6, with three vertices in common with D, namely $p_2, x_1, y_{t'}$. From 15.7, D has length 4, and so $t' = 1$ and $s = 2$. Since $W = \{x_2\}$ and $j = 2$, it follows that the only edges between x_1, x_2 and P are x_1p_1, x_2p_1, x_2p_2. But then the three paths p_1-x_1, x_2-z, p_2-$\cdots$-p_d-y_1 form a long prism, a contradiction. This proves (5).

From (5), since p_h is adjacent to x_1, it follows that p_h is not complete to $X \setminus x_1$, and therefore $h < i < j$. Choose s' with $1 \leq s' \leq s$ minimum such that p_h is nonadjacent to $x_{s'}$. So p_j-x_1-$\cdots$-$x_{s'}$-p_h-p_j is an antihole, and so s' is even. Hence x_1-$\cdots$-$x_{s'}$-p_h-z is an odd antipath; all its internal vertices have neighbours in $\{p_{h+1}, \ldots, p_n\}$, and its ends do not, so by 13.6 it has length 3, that is, $s' = 2$. The set $F = \{x_2, p_h, \ldots, p_n\}$ is connected; the only neighbour of x_1 in F is p_h; the only neighbour of z in F is x_2. Since x_1, z are $(X \setminus \{x_1, x_2\}) \cup Y$-complete, and p_h, x_2 are not (for p_h is not Y-complete), it follows from 17.2 that there is a vertex in $(X \setminus \{x_1, x_2\}) \cup Y$ with no neighbour in F except possibly x_2. But every vertex in $(X \setminus \{x_1, x_2\}) \cup Y$ is adjacent to either p_j or to p_n, a contradiction. This proves 17.5. ∎

18 Pseudowheels

Our next goal is to prove a version of 16.4 for "pseudowheels", in which one of the "vertices" is really a co-connected set. Fortunately we don't need to generalize 16.4 completely; it is enough to generalize the case when there is a segment of the wheel of length 1, and one of its vertices has blown up to become the co-connected set. (We did try to generalize 16.4 completely, but were unable to do it and it gave us a lot of trouble; so eventually we found a way to make do with this special case.)

It has to be admitted that pseudowheels are a little bizarre, but we seem to need them, unfortunately. The result we are proving here only has one application, several sections ahead, but at that point it seems to be crucial. (Incidentally, the only application of the main result of the last section is in the present one.)

We begin with an easy special case, a form of 15.1 when one vertex is replaced by a co-connected set.

18.1 *Let G be a graph, and let X, Y be disjoint nonempty co-connected subsets of $V(G)$, complete to each other. Let p_1-p_2-p_3-p_4-p_5 be a track in a nonconforming graph G, induced except possibly for the edge p_2p_5. Let X be complete to p_1, p_5 and not to p_2, p_3, p_4. Let Y be complete to p_1, p_3, p_4 and not to p_2, p_5. Then G is not nonconforming.*

Proof. Assume G is nonconforming. Then in $\overline{G}$, $\{p_1, p_3, p_5\}$ is a triangle, and the connected set $F = X \cup Y \cup \{p_2, p_4\}$ catches it. In $\overline{G}$, the only neighbours of p_5 in F are in $Y \cup \{p_2\}$, the only neighbours of p_3 in F are in X, and the only neighbour of p_1 in F is p_4. Hence no vertex of F has two neighbours in the triangle, so by 17.1, F contains a reflection of the triangle. So (back in G) there are vertices $b_1 \in X$ and $b_2 \in Y \cup \{p_2\}$ such that b_1, b_2, p_4 are pairwise nonadjacent, and b_1 is adjacent to p_1, p_5 and not p_3, and b_2 is adjacent to p_1, p_3 and not p_5. Since p_4 is Y-complete and b_2, p_4 are nonadjacent it follows that $b_2 \notin Y$, and so $b_2 = p_2$. If p_2, p_5 are not adjacent then Y and the six vertices $p_1, \ldots, p_5, b_1$ contradict 15.1. If they are adjacent, choose an antipath Q joining them with interior in Y; then the three antipaths p_1-p_4, p_3-b_1, p_5-Q-p_2 form a long prism in $\overline{G}$, a contradiction. This proves 18.1. ∎

We are finally going to need 2.10; but since we now only care about nonconforming graphs, it can be strengthened as follows.

18.2 *Let G be nonconforming, and let P be a path in G with even length > 0, with vertices $p_1, \ldots, p_n$ in order. Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be disjoint co-connected sets, so that p_1 is the unique X-complete vertex of P and p_n is the unique Y-complete vertex of P. Let X be complete to Y. Then P has length 2 and there is an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y, and exactly one of Q,R has odd length.*

Proof. Let us apply 2.10. We may therefore assume that P has length ≥ 4 and there are nonadjacent $x_1, x_2 \in X$ so that x_1-p_2-$\cdots$-p_n-x_2 is a path P' say, of odd length ≥ 5. But the ends of P' are $Y \cup \{p_1\}$-complete, and its internal vertices are not, contrary to 13.6. This proves 18.2. ∎

This has a reformulation that we sometimes need:

18.3 *Let G be nonconforming, and let P be a path in G with even length ≥ 2, with vertices $p_1, \dots, p_n$ in order. Let $X \subseteq V(G)$ and $Y \subseteq V(G)$ be disjoint co-connected sets, so that p_1 is X-complete, p_n is not X-complete and p_n is the unique Y-complete vertex of P. Let X be complete to Y. Suppose that there is a Y-complete vertex in G nonadjacent to both p_{n-1}, p_{n-2}. Then either:*

- *there is an odd number of X-complete edges in P, or*
- *$n = 2$ and there is an odd antipath joining p_{n-1} and p_n with interior in X.*

Proof. Choose an X-complete vertex p_i in P with i maximum. Suppose first that i is even. Then the path $p_1\text{-}\cdots\text{-}p_i$ is odd, and we may assume that an even number of its edges are X-complete. So it has length > 1; by 2.3, none of its internal vertices are X-complete; and by 13.6 it has length 3, and there is an odd antipath Q joining p_2, p_3 with interior in X. But there is another, R say, with interior in Y, which also must be odd. Hence R cannot be completed to an antihole via $p_3\text{-}p_n\text{-}p_2$, and so $n \leq 4$. Since n is odd it follows that $n = 3$; but then by hypothesis there is an Y-complete vertex v nonadjacent to p_1, p_2, and then $v\text{-}p_1\text{-}R\text{-}p_2\text{-}v$ is an odd antihole, a contradiction.

Now we assume that i is odd. Hence the path $p_i\text{-}\cdots\text{-}p_n$ is even, and by 18.2 it has length 2. Let Q be the antipath between p_{n-2}, p_{n-1} with interior in Y, and R the antipath between p_{n-1}, p_n with interior in X. By hypothesis there is a Y-complete vertex nonadjacent to p_{n-1}, p_{n-2}, and therefore Q is even, so R is odd by 18.2. Hence R cannot be completed to an antihole via $p_n\text{-}p_1\text{-}p_{n-1}$; and so $n = 3$ and the theorem holds. This proves 18.3. ∎

Let us say a *pseudowheel* in a graph G is a triple (X, Y, P), satisfying:

- X, Y are disjoint nonempty co-connected subsets of $V(G)$, complete to each other
- P is a path $p_1\text{-}\cdots\text{-}p_n$ of $G \setminus (X \cup Y)$, where $n \geq 5$
- p_1, p_n are the only X-complete vertices of P
- Y is complete to p_1 and to at least one other vertex of P, and not to p_2 or p_n.

18.4 *Let (X, Y, P) be a pseudowheel in a nonconforming graph G, where P is $p_1\text{-}\cdots\text{-}p_n$. Then*

- *P has even length ≥ 6,*
- *every maximal subpath of P of length ≥ 2 with no internal vertices Y-complete has even length, unless it contains p_n,*
- *the maximal subpath of P containing p_n with no internal vertices Y-complete has odd length, and*
- *P contains an odd number, at least 3, of Y-complete edges.*

Proof. Since P is a path of length ≥ 4, and its ends are X-complete and its internal vertices are not, it follows that P has even length. That proves half the first assertion; the other half is proved below. For the second, let P' be a maximal subpath of P of length ≥ 2 in which no internal vertex is Y-complete; and assume p_n is not in P'. Hence from the maximality of P', both ends of P' are Y-complete. Suppose P' has odd length, and let its ends be p_i, p_j where $i < j$. Then 13.6 implies that $j - i = 3$, and there is an odd antipath Q joining p_{i+1}, p_{i+2} with interior in Y. But p_{i+1}, p_{i+2} are not X-complete, so they are joined by an antipath Q' with interior in X. Since $Q \cup Q'$ is an antihole it follows that Q' is odd. But then $p_n\text{-}p_{i+1}\text{-}Q'\text{-}p_{i+2}\text{-}p_n$ is an odd antihole since $j < n$. This proves the second assertion. For the third assertion, let i be maximum with $1 \leq i \leq n$ such that p_i is Y-complete, and let P' be $p_i\text{-}\cdots\text{-}p_n$. Since p_i is the only Y-complete vertex of P' and there are at least two in P, it follows that $i > 1$, and so no vertex of P' is X-complete except p_1. Hence we may apply 18.2. We deduce that P' has length 2, and so $i = n - 2$. Now the antipath joining p_{n-2}, p_{n-1} with interior in

X is even since it can be completed to an antihole via p_{n-1}-p_1-p_{n-2}; and the antipath joining p_{n-1}, p_n with interior in Y is even since it can be completed to an antihole via p_n-p_1-p_{n-1}. But this contradicts 18.2. This proves the third assertion. By counting we see that P contains an odd number of Y-complete edges. To see that it contains at least 3, suppose it only contains one, say $p_a p_{a+1}$. There is an antipath joining p_a, p_{a+1} with interior in X, and by 15.2 applied to the path P, this antipath has length 2, that is, there exists $x \in X$ nonadjacent to both p_a, p_{a+1}. Let C be a hole containing x, p_a, p_{a+1} and with $C \setminus x \subseteq P$. Then (C, Y) is an odd wheel, since C contains the Y-complete vertices x, p_a, p_{a+1} and it also contains p_{a-1}, p_{a+2} which are not Y-complete. This is a contradiction, and therefore proves the fourth assertion. In particular, it follows that P has length ≥ 6, which completes the proof of the first assertion. This proves 18.4. ▮

A pseudowheel (X, Y, P) in a nonconforming graph G is *optimal* if

- there is no pseudowheel (X', Y', P') in G such that the number of Y'-complete vertices in P' is less than the number of Y-complete vertices in P, and
- there is no pseudowheel (X, Y', P) in G such that $Y \subset Y'$.

18.5 *Let (X, Y, P) be an optimal pseudowheel in a nonconforming graph G, where P is p_1-$\cdots$-p_n. Let $v \in V(G) \setminus (X \cup Y \cup V(P))$, not Y-complete. Then there is a subpath P' of P, containing all the neighbours of v in P, with no Y-complete vertex in its interior. Moreover, if v is X-complete, then $p_n \in V(P')$.*

Proof. Choose h, k with $1 \leq h \leq k \leq n$ such that v is adjacent to p_h, p_k, with h minimum and k maximum. (If this is impossible then the theorem holds.) Choose i, j with $2 \leq i \leq j \leq n$ such that p_i, p_j are Y-complete, with i minimum and j maximum. By 18.4 it follows that i is odd and j is even.

(1) *There is a path Q from v to some vertex q, such that q is the only Y-complete vertex in Q, and $V(Q \setminus v) \subseteq \{p_{i+1}, \dots, p_{j-1}\}$.*

For by 18.4 and the fact that there is a Y-complete edge in P, it follows that there is a Y-complete vertex in $\{p_{i+1}, \dots, p_{j-1}\}$. If v has a neighbour in this set then the claim holds, so suppose it does not. Hence there is a hole C containing v, with $C \setminus v \subseteq P$, such that p_i-$\cdots$-p_j is a path of C. Since all Y-complete edges in P belong to this path, and there are an odd number of them, it follows that there is an odd number (≥ 3) of Y-complete edges in C, contrary to 2.3. This proves (1).

(2) *If v is both adjacent to p_1 and X-complete then v has no more neighbours in P and the theorem holds.*

For assume that v has a neighbour in $p_2, \dots, pn\}$. From the optimality of (X, Y, P) it follows that $(X, Y \cup \{v\}, P)$ is not a pseudowheel, and so p_1 is the only $Y \cup \{v\}$-complete vertex in P. By 2.12 (with X, Y replaced by $Y \cup \{v\}, X$) we deduce that either there exists $y \in Y \cup \{v\}$ nonadjacent to all $p_2, \dots, p_n$, or there exist nonadjacent $y_1, y_2 \in Y \cup \{v\}$ so that y_1-p_2-$\cdots$-p_n-y_2 is a path. But p_i is Y-complete and $3 \leq i \leq n-1$, so every vertex in $Y \cup \{v\}$ has a neighbour in $p_2, \dots, p_n$, and the first statement does not hold; and if y_1, y_2 are as in the second statement, then one of y_1, y_2 is in Y and therefore adjacent to p_i, a contradiction. This proves (2).

(3) *If v is X-complete and nonadjacent to p_1 then the theorem holds.*

For then, if h is odd then p_1-$\cdots$-p_h-v is an odd path with ends X-complete and its internal vertices not, so it has length 3 by 13.6; but the X-complete vertex p_n has no neighbour in its interior (since $n \geq 5$ by 18.4), contrary to 2.2. So h is even. Suppose that one of $p_2, \dots, p_h$ is Y-complete. Then $h \neq 2$ since p_2 is not Y-complete, so $h \geq 4$. Hence $(X, Y, p_1$-$\cdots$-p_h-$v)$ is a pseudowheel, and from the optimality of (P, X, Y) it follows that there are no Y-complete vertices in $\{p_{h+1}, \dots, p_n\}$; but $\{p_h, \dots, p_n\}$ contains all the neighbours of v in P, and so the claim holds. We may therefore assume that there are no Y-complete vertices in $p_2, \dots, p_h$, and so $i > h$. Let Q, q be as in (1). Since the $X \cup Y$-complete vertex p_1 has no neighbours in Q, the pairs

$(V(Q), X), (V(Q), Y)$ are balanced by 2.6; so by 2.10, Q has odd length. Hence the path p_1-$\cdots$-p_h-v-Q-q has odd length, and its ends are Y-complete, and its internal vertices are not. By 13.6 it has length 3; so $h = 2$ and v is adjacent to q. Also every Y-complete vertex is adjacent to one of v, p_2, by 2.2, so they are all adjacent to v except p_1 and possibly p_3. Suppose p_i is adjacent to v, and is therefore $Y \cup \{v\}$-complete. Then the path p_1-$\cdots$-p_i has even length; the only X-complete vertex in it is p_1; and the only $Y \cup \{v\}$-complete vertex in it is p_i. By 18.2 it has length 2. But there is an X-complete vertex nonadjacent to both p_2, p_3, namely p_n, and there is a $Y \cup \{v\}$-complete vertex nonadjacent to both p_1, p_2, since there is a Y-complete vertex in $\{p_4, \ldots, p_n\}$ by 18.4, and it is necessarily adjacent to v as we already saw. Hence both pairs $\{p_1, p_2\}, Y \cup \{v\})$ and $\{p_2, p_3\}, X)$ are balanced by 2.6, contrary to 18.2. This proves that p_i is not adjacent to v, and therefore $i = 3$. Choose $h' > i$ minimum so that v is adjacent to $p_{h'}$. From the hole v-p_2-$\cdots$-$p_{h'}$-v it follows that h' is even. From 18.3 applied to the even path p_3-$\cdots$-$p_{h'}$-v, and using the fact that the $X \cup Y$-complete vertex p_1 has no neighbour in this path, we deduce that there is a Y-complete edge in p_3-$\cdots$-$p_{h'}$-v. Since v is adjacent to every Y-complete vertex in P except p_1, p_3, it follows that the only such edge is p_3p_4, and therefore $h' = 4$. But then the track p_1-$\cdots$-p_4-v violates 18.1. This proves (3).

(4) *If v is not X-complete and is not adjacent to p_1 then the theorem holds.*

Let P' be the path p_1-$\cdots$-p_h-v-p_k-$\cdots$-p_n. Suppose that any of $p_2, \ldots, p_h, p_k, \ldots, p_n$ is Y-complete. Then P' has has length ≥ 4, since $h > 1$ and p_2, p_n are not Y-complete, and so (X, Y, P') is a pseudowheel. By the optimality of (X, Y, P) it follows that there are no Y-complete vertices among $\{p_{h+1}, \ldots, p_{k-1}\}$; but then the claim holds. So we may assume that none of $p_2, \ldots, p_h, p_k, \ldots, p_n$ is Y-complete, and therefore $h \leq i \leq j \leq k$, and since $j - i \geq 3$ it follows that $k - h \geq 5$. Let Q, q be as in (1). Then q-Q-v-p_k-$\cdots$-p_n is a path; the only Y-complete vertex in it is q; the only X-complete vertex in it is p_n; and the $X \cup Y$-complete vertex p_1 has no neighbour in its interior. By 2.10 this path is odd. Therefore the paths p_1-$\cdots$-p_h-v-Q-q and p_1-$\cdots$-p_h-v-p_k-$\cdots$-p_n have lengths of opposite parity. For the first path, its ends are Y-complete and its internal vertices are not. For the second, its ends are X-complete and its internal vertices are not. So one of them has length 3, and so $h = 2$, and there is an odd antipath joining v, p_2 with interior in one of X, Y. Since v, p_2 are joined by an antipath with interior in X and by another with interior in Y, and all such pairs of antipaths have the same parity (since their union is an antihole), it follows that v, p_2 are joined by an odd antipath with interior in each of X,Y. Hence every X-complete vertex is adjacent to one of v, p_2, and so is every Y-complete vertex. In particular $k = n$, and v is adjacent to every Y-complete vertex in P except p_1 and possibly p_3. But then the path q-v-p_n has length 2, contradicting that it has odd length. This proves (4).

Henceforth then we assume that v is adjacent to p_1 and v is not X-complete.

(5) *p_{n-1} is not $Y \cup \{v\}$-complete.*

For suppose it is. Since $n \geq 7$, it follows from 13.6 applied to $P \backslash p_n$ and $Y \cup \{v\}$ that there is a $Y \cup \{v\}$-complete vertex in $\{p_2, \ldots, p_{n-2}\}$; choose such a vertex, $p_{j'}$ say, with j' maximum. If $j' < j - 1$ then $j - j'$ is even from 2.2 applied to p'_j-$\cdots$-p_j; but then the odd path $p_{j'}$-$\cdots$-p_n contains no $Y \cup \{v\}$-complete edges, contrary to 17.5. So $j' = j - 1$. But then let D be an antipath between p_{n-1} and p_n with interior in $X \cup Y \cup \{v\}$, and let $F = V(D \setminus p_n) \cup Y$. Then F is co-connected, and each of p_1, p_{n-2}, p_n has a nonneighbour in F; the only nonneighbour of p_1 in F is p_{n-1}; all nonneighbours of p_{n-2} in F belong to X; and all nonneighbours of p_n in F belong to $Y \cup \{v\}$. So in $\overline{G}$, the connected set F catches the triangle $\{p_1, p_{n-2}, p_n\}$, and by 17.1 it contains a reflection of the triangle, which is impossible since p_{n-1} is complete (in G) to $Y \cup \{v\}$. This proves (5).

(6) *v is not adjacent to p_n.*

For suppose it is. By (1) there is a Y-complete vertex p_a in P with $a \geq 3$, even and different from p_{n-1}. Thus $j - a$ is even, and so by 2.3 there is an even number of Y-complete edges in the even path p_a-$\cdots$-p_j, and hence in the odd path p_a-$\cdots$-p_n. But p_a is Y-complete, and p_n is the unique $X \cup \{v\}$-complete vertex in this path, contrary to 17.5. This proves (6).

(7) *There is no neighbour p_m of v in P with $1 \leq m \leq n$ so that v, p_m are joined by an odd antipath with interior in Y.*

For suppose such a neighbour exists. So $m > 1$, and if $m \leq n-2$ then there is an antipath joining v, p_m with interior in X, which therefore is also odd, since its union with the antipath through Y is an antihole; yet it can be completed to an odd antihole via p_m-p_n-v, a contradiction. So by (6), $m = n-1$, and in particular m is even. Since j is even it follows that either $p_j = p_m$ or p_j is nonadjacent to p_m; and in either case it follows that p_j is adjacent to v, since every Y-complete vertex is adjacent to one of v, p_m. By (5) $n - j \geq 3$ and odd, and this path (with co-connected sets X and $Y \cup \{v\}$) violates 17.5. This proves (9).

Suppose that $j \geq k$, and let P' be the path p_1-v-p_k-$\cdots$-p_n. Then P' has length ≥ 4, since p_{n-1} is not $Y \cup \{v\}$-complete, and so (X, Y, P') is a pseudowheel; and by the optimality of (X, Y, P) it follows that there are no Y-complete vertices in p_2-$\cdots$-p_{k_1}; but then the claim holds. So we assume $j < k$. Let Q, q be as in (1), and assume first that Q is even. Then the path p_1-v-Q-q has odd length; its ends are Y-complete, and its internal vertices are not, so by 13.6 it has length 3, and its internal vertices are joined by an odd antipath with interior in Y, contrary to (8). So Q is odd.

Next assume that k is even. Then the path p_1-v-p_k-$\cdots$-p_n is odd, and its ends are X-complete, and its internal vertices are not, so by 13.6 it has length 3, and $k = n-1$, and its internal vertices v, p_{n-1} are joined by an odd antipath with interior in X. Since p_{n-1} is not $Y \cup \{v\}$-complete, they are also joined by an odd antipath with interior in Y, contrary to (7). This proves that k is odd. Hence the path q-Q-v-p_k-$\cdots$-p_n is even, and by (6) it has length > 2 contrary to 18.2. This proves 18.5. ∎

18.6 *Let (X, Y, P) be an optimal pseudowheel in a nonconforming graph G, where P is p_1-$\cdots$-p_n. Let $F \subseteq V(G) \setminus (X \cup Y \cup V(P))$ be connected, such that no vertex in F is Y-complete. Then there is a subpath P' of P, containing all the attachments of F in P, with no Y-complete vertex in its interior. Moreover, if F contains an X-complete vertex, then $p_n \in V(P')$.*

Proof. Suppose the theorem is false, and choose a minimal counterexample F. From 18.5 $|F| \geq 2$.

(1) *Some vertex in F is C-complete.*

For suppose not. Since F is a counterexample, it has attachments p_a, p_c such that there is a Y-complete vertex p_b with $a < b < c$. From the minimality of F, F is the interior of a path p_a-f_1-$\cdots$-f_k-p_c. Let W_1 be the set of attachments of $F \setminus f_k$ in P, and W_2 the set of attachments of $F \setminus f_1$ in P. From the minimality of F, for $i = 1, 2$ there is a subpath p_{a_i}-$\cdots$-p_{b_i} of P ($= P_i$ say), so that no internal vertex of P_i is Y-complete, and $W_i \subseteq V(P_i)$. Choose P_1, P_2 minimal; then p_{a_1} is a neighbour of f_1, and p_{b_2} is a neighbour of f_k, and p_1-$\cdots$-p_{a_1}-f_1-$\cdots$-f_k-p_{b_2}-$\cdots$-p_n is a path P' say. Suppose that there is a Y-complete vertex in P' different from p_1. Then P' has length ≥ 4, and (X, Y, P') is a pseudowheel, contrary to the optimality of (X, Y, P). So there are no Y-complete vertices in P'. But also there are none in $\{p_{a_1+1}, \ldots, p_{b_1-1}\}$ and none in $\{p_{a_2+1}, \ldots, p_{b_2-1}\}$, so all the Y-complete vertices of P belong to $\{p_{b_1}, \ldots, p_{a_2}\}$, except for p_1. By 18.4 there are an odd number, at least 3, of Y-complete edges in this path. From the minimality of F, f_1-$\cdots$-f_k-p_{a_2}-p_{a_2-1}-$\cdots$-p_{b_1}-f_1 is a hole, which therefore also contains an odd number ≥ 3 of Y-complete edges. But this contradicts 2.3. This proves (1).

From (1), since F is a counterexample, there is a Y-complete vertex in F and there exists $a < b$ such that p_a is an attachment of F and p_b is Y-complete. From the minimality of F, there is a path f_1-$\cdots$-f_k-p_a such that $F = \{f_1, lf_k\}$ and f_k is the unique X-complete vertex in F. Let W_1 be the set of attachments of $F \setminus f_k$ in P, and W_2 the set of attachments of $F \setminus f_1$ in P. From the minimality of F, for $i = 1, 2$ there is a subpath p_{a_i}-$\cdots$-p_{b_i} of P ($= P_i$ say), so that no internal vertex of P_i is Y-complete, and $W_i \subseteq V(P_i)$, and $b_2 = n$. Choose P_1, P_2 minimal; then p_{a_1} is a neighbour of f_1, and p_1-$\cdots$-p_{a_1}-f_1-$\cdots$-f_k is a path P' say. Suppose that there is a Y-complete vertex in P' different from p_1. Then P' has length ≥ 4, and (X, Y, P')

is a pseudowheel, contrary to the optimality of (X, Y, P). So there are no Y-complete vertices in P'. But also there are none in $\{p_{a_1+1}, \ldots, p_{b_1-1}\}$ and none in $\{p_{a_2+1}, \ldots, p_{b_2-1}\}$, so all the Y-complete vertices of P belong to $\{p_{b_1}, \ldots, p_{a_2}\}$, except for p_1. By 18.4 there are an odd number, at least 3, of Y-complete edges in this path. From the minimality of F, f_1-$\cdots$-f_k-p_{a_2}-p_{a_2-1}-$\cdots$-p_{b_1}-f_1 is a hole, which therefore also contains an odd number ≥ 3 of Y-complete edges. But this contradicts 2.3. This proves 18.6. ▮

Now we come to the main result of this section, the following.

18.7 *Let G be nonconforming; then it contains no pseudowheel.*

Proof. Suppose G does contain a pseudowheel; then it contains an optimal pseudowheel, say (X, Y, P), where P is p_1-$\cdots$-p_n. Let Z be the set of all Y-complete vertices in G. So Y, Z are disjoint, nonempty, and complete to each other, and $|Z| \geq 2$. Let $F_0 = V(G) \setminus (Y \cup Z)$. By 14.5, F_0 is connected and every vertex in Z has a neighbour in F_0. Choose $i > 1$ so that $p_i p_{i+1}$ is Y-complete, and let A,B be the two components of $V(P \setminus p_i$. Since p_1, p_{i+1} both have neighbours in F_0, it follows that F_0 contains a minimal connected set so that there are vertices in A and in B with neighbours in F. From the minimality of F it is disjoint from $V(P)$; and disjoint from $X \cup Y$ since $X \subseteq Z$, contrary to 18.6. This proves 18.7. ▮

19 Wheel systems

The main part of the remainder of the proof of 13.5 is to show that there is no wheel in a nonconforming graph, and also that there is no "parachute" (a degenerate kind of wheel introduced by Conforti and Cornuejols). The proof is in two parts. First, given any parachute, we show there is a wheel with the same hub; and second, given any wheel, we show there is a parachute with a bigger hub. The first step is what we prove in this section. Starting with a parachute, we grow it to what we call a "wheel system", and make it maximal; and as usual, analyze how the remainder of the graph attaches to it. Since G does not admits a decomposition, we can show that there is a special 1-vertex jump with respect to this wheel system. Then we unravel the wheel system; we show that for any wheel system with such a special 1-vertex jump, either there is a smaller wheel system still with a special jump, or there is a wheel with this hub.

Let G be a graph. A *wheel system* in G of *height* $t \geq 1$ consists of a sequence $z, x_0, \ldots, x_t$ of distinct vertices of G, and a connected subset $A_0 \subseteq V(G) \setminus \{z, x_0, \ldots, x_t\}$ satisfying the following conditions:

1. A_0 contains neighbours of x_0 and of x_1, and no vertex in A_0 is $\{x_0, x_1\}$-complete, and A_0 contains no neighbour of z
2. for $2 \leq i \leq t$, there is a connected subset of $V(G)$ including A_0, containing a neighbour of x_i, containing no neighbour of z, and containing no $\{x_0, \ldots, x_{i-1}\}$-complete vertex
3. for $1 \leq i \leq t$, x_i is not $\{x_0, \ldots, x_{i-1}\}$-complete, and
4. z is adjacent to all of $x_0, \ldots, x_t$.

Note that this definition is symmetric between x_0, x_1. When we use a wheel system there will be a "hub", a co-connected set Y that is complete to $x_0, \ldots, x_{t-1}$ and sometimes to x_t, but it is convenient to introduce that later. Note that if (C, Y) is a wheel in a nonconforming graph, it gives a wheel system of height 1 as follows. Let C have vertices $p_1, \ldots, p_n$ in order. By 16.4 we may assume that p_1, p_2, p_3 are Y-complete. Define $x_0 = p_1, x_1 = p + 3, z = p_2$ and $A_0 = \{p_4, \ldots, p_n\}$; then this is a wheel system. However, we only used that there were two consecutive edges of C complete to Y, so something less than a wheel also gives rise to a wheel system.

Let $z, x_0, \ldots, x_t, A_0$ be a wheel system W of height t. For $1 \leq i \leq t$ we define $X_i = \{x_0, \ldots, x_i\}$, and we define A_i to be the maximal connected subset of $V(G)$ that includes A_0, contains no neighbour of z, and contains no X_i-complete vertex. We call A_i the ith *realm* of W, and the sequence $A_0, \ldots, A_t$ is called the *realm sequence* of W. So for each i, $A_{i-1} \subseteq A_i$. Note that condition 2 above just says that x_i has a neighbour in A_{i-1}.

We need three special kinds of wheel systems. Let $z, x_0, \ldots, x_t, A_0$ be a wheel system W of height t, and define X_i, A_i as above. Let $Y \subseteq V(G)$ be nonempty and co-connected, such that Y is disjoint from $\{z, x_0, \ldots, x_t\}$, and $x_0, \ldots, x_{t-1}$ are all Y-complete and x_t is not. (We do not require that z is Y-complete.) We say W is a Y-*diamond* if

- $t \geq 3$,
- x_t is X_{t-2}-complete, and
- x_t has a neighbour in A_{t-2}.

We say W is a Y-*square* if

- $t \geq 3$,
- x_t is adjacent to x_{t-1}
- x_t has no neighbour in A_{t-2}, and
- there is a vertex in A_{t-1} adjacent to x_t with a neighbour in A_{t-2}.

We say W is a Y*-diamond-square* if

- $t \geq 4$,
- x_t is X_{t-2}-complete, and x_{t-1} is not X_{t-3}-complete,
- x_t has no neighbour in A_{t-3}, and x_{t-1} has a neighbour in A_{t-3}, and
- there is a vertex in A_{t-2} adjacent to both x_t, x_{t-1} with a neighbour in A_{t-3}.

The main result of this section is the following - it is the "unravelling" step mentioned above.

19.1 *Let G be nonconforming, and let $Y \subseteq V(G)$ be nonempty and co-connected. Suppose that for some $t \geq 3$, either:*

1. *there is a Y-diamond in G of height t, or*
2. *there is a Y-square in G of height t, or*
3. *there is a Y-diamond-square of height $t+1$.*

Then G contains a wheel (C, Y).

The proof is by induction on t. We prove the case $t = 3$ below, and the subsequent three theorems are the inductive derivations of the three statements of 19.1. So first we need to show:

19.2 *If $t = 3$ then 19.1 holds.*

Proof. Let W be a wheel system in a nonconforming graph G, and let X_i, A_i be defined as before. Suppose first that W is a Y-square of height 3. (We do the diamond case last because it is the most difficult.) Hence x_3 is adjacent to x_2, x_3 has no neighbour in A_1, and there is a vertex q in A_2 adjacent to x_3 with a neighbour in A_1. From the maximality of A_1 it follows that q is X_1-complete, and therefore nonadjacent to x_2 (since it belongs to A_2 and so is not X_2-complete). Let Q be a path from q to x_2 with interior in A_1; so Q has length ≥ 2. But Q is even since it can be completed to a hole via x_2-x_3-q, and so q-Q-x_2-z is an odd path; its ends are X_1-complete, and its internal vertices are not. By 13.6 it has length 3, and there is an antipath with interior in X_1, joining its middle vertices (q and r say). This antipath can be completed via r-z-q-x_2 to an antihole of length ≥ 6, containing x_0, x_1 and z. But let P be a path from x_0 to x_1 with interior in A_0; then it has length ≥ 3 since A_0 contains no vertex adjacent to both x_0, x_1, and hence z-x_0-P-x_1-z is a hole of length ≥ 6 containing x_0, x_1 and z. But this contradicts 15.7, as required.

Now suppose W is a Y-diamond-square of height 4. Hence x_4 is X_2-complete, x_3 is not X_1-complete, x_4 has no neighbour in A_1, x_3 has a neighbour in A_1, and there is a vertex q in A_2 adjacent to both x_4, x_3 with a neighbour in A_1. As before q is X_1-complete, and therefore not adjacent to x_2; let Q be a path from q to x_2 with interior in A_1. The proof is completed exactly as in the previous paragraph.

So now we may assume that W is a Y-diamond of height 3. Hence x_3 is X_1-complete, and is therefore nonadjacent to x_2, and x_3 has a neighbour in A_1.

(1) *If x_0 is adjacent to x_2 and there is a path P from x_0 to x_1 with interior in A_1, so that its interior contains neighbours of both x_2 and x_3, then the theorem holds.*

For let P be x_0-p_1-$\cdots$-p_n-x_1; then z-x_0-P-x_1-z is a hole of length ≥ 6, say C_1. Suppose that x_2 is adjacent to p_n. Since x_0, x_1, p_n belong to C_1, there is no antihole of length ≥ 5 containing them by 15.7. If p_n is not $Y \cup \{x_2, x_3\}$-complete, then there is an antipath between p_n, x_1 with interior in this set, of length ≥ 3, and it can be completed to an antihole via x_1-x_0-p_n containing x_0, x_1, p_n, a contradiction. So p_n is $Y \cup \{x_2, x_3\}$-complete. If z is not Y-complete, then there is an antipath between z and x_1 with interior in

$Y \cup \{x_2, x_3\}$, that can be completed to an antihole via x_1-x_0-p_n-z, again a contradiction. So we may assume that z is Y-complete; but then (C_1, Y) is a wheel satisfying the theorem.

So we may assume that x_2 is not adjacent to p_n. By hypothesis, x_2 has a neighbour in the interior of P; choose i with $1 \leq i < n$ maximum such that x_2 is adjacent to p_i. Since the hole C_1 is even, it follows that n is odd. From the hole z-x_2-p_i-$\cdots$-p_n-x_1-z it follows that i is odd. Suppose first that $i > 1$. Then p_0-x_2-p_i-$\cdots$-p_n-x_1 is an odd path of length ≥ 5. Its ends are $Y \cup \{x_3, z\}$-complete, and its internal vertices are not, so by 13.6, $Y \cup \{x_3, z\}$ is not co-connected. Hence z is Y-complete, and therefore we may assume that p_n is not, since otherwise (C_1, Y) is a wheel satisfying the theorem. The ends of the same path p_0-x_2-p_i-$\cdots$-p_n-x_1 are both $Y \cup \{x_3\}$-complete, and this set is co-connected, and so by 13.6, some edge of the path is $Y \cup \{x_3\}$-complete, say $p_j p_{j+1}$i where $i \leq j \leq n-2$. (Note that the edge belongs to $P \setminus \{x_1, p_n\}$ since p_n, x_2 are not $Y \cup \{x_3\}$-complete.) Let C_2 be the hole z-x_2-p_i-$\cdots$-p_{n+1}-z. Then C_2 has length ≥ 6, and so $(C_2, Y \cup \{x_3\})$ is an odd wheel, since x_2, p_n are not $Y \cup \{x_3\}$-complete, contrary to 16.4. So $i = 1$.

Choose j with $1 \leq j \leq n$ minimum so that x_3 is adjacent to p_j (this is possible by hypothesis). From the hole z-x_2-p_1-$\cdots$-p_j-x_3-z it follows that it follows that j is odd; and so x_0-p_1-$\cdots$-p_j-x_3-x_0 is not a hole, that is, $j = 1$, and hence p_1 is adjacent to x_3. If p_1 is not Y-complete, then an antipath between p_1 and x_3 with interior in Y can be extended to an antihole via x_3-x_2-x_1-p_1, and this antihole shares the vertices p_1, x_1, x_2 with the hole z-x_2-p_1-$\cdots$-p_n-x_1-z, contrary to 15.7. So p_1 is Y-complete. If z is not Y-complete, an antipath between z and x_3 with interior in Y can be extended to an antihole via x_3-x_2-x_1-p_1-z, and this shares the vertices z, x_1, p_1 with the hole C_1, contrary to 15.7. So z is Y-complete; but then (C_1, Y) is a wheel satisfying the theorem. This proves (1).

(2) *If there is a path P from x_0 to x_1 with interior in A_1, so that its interior contains neighbours of both x_2 and x_3, then the theorem holds.*

By (1) we may assume that x_0 is nonadjacent to x_2. Now the definition of a wheel system, and of a Y-diamond, were both symmetric between x_0, x_1. In other words, $z, x_1, x_0, x_2, \ldots, x_t, A_0$ is also a Y-diamond; so we may therefore also assume that x_1 is nonadjacent to x_2. Let P be x_0-p_1-$\cdots$-p_n-x_1, so n is odd as before. Now $\{x_2, p_1, \ldots, p_n\}$ is connected and catches the triangle $\{z, x_1, x_3\}$. If it contains a reflection of the triangle, then so does $\{p_1, \ldots, p_n\}$ since x_2 has no neighbours in the triangle, which is impossible since P is a path. So by 17.1, there is a vertex in $\{x_2, p_1, \ldots, p_n\}$ with two neighbours in the triangle. The only neighbour of z in it is x_2, which is nonadjacent to both x_1, x_3. The only neighbour of x_1 in it is p_n, and therefore p_n is adjacent to x_3. From the symmetry between x_0 and x_1 it follows that x_3 is adjacent to p_1. Choose $i \leq n$ maximum such that x_2 is adjacent to p_i. From the hole z-x_2-p_i-$\cdots$-p_n-x_1-z it follows that i is odd. Since x_3 is adjacent to p_n, we may choose j with $i \leq j \leq n$ minimum such that x_3 is adjacent to p_j. From the hole z-x_2-p_i-$\cdots$-p_j-x_3-z we see that j is odd. Suppose $j \neq i$. Then the path x_2-p_i-$\cdots$-p_j-x_3 is even and has length ≥ 4. By 18.2 with co-connected sets $X_1, Y \cup \{z\}$ we deduce that $Y \cup \{z\}$ is not co-connected, and hence z is Y-complete. By 18.3 with sets X_1, Y , since the X-complete vertex z has no neighbours in the interior of P, it follows that there is a Y-complete edge in x_2-p_i-$\cdots$-p_j. If p_i is Y-complete then z-x_2-p_i-$\cdots$-p_j-x_3-z is the rim of a wheel with hub Y, so the theorem is satisfied. If p_i is not Y-complete, there is a Y-complete edge in the path p_{i+1}-$\cdots$-p_n, so the hole z-x_0-P-x_1-z is the rim of a wheel with hub Y, satisfying the theorem. Hence we may assume that $j = i$, that is, x_3 is adjacent to p_i.

Now suppose $i < n$. If p_i is not Y-complete then an antipath between p_i and x_3 with interior in Y can be extened via x_3-x_2-x_1-p_i to an antihole sharing the vertices p_i, x_1, x_2 with the hole z-x_2-p_i-p_n-x_1-z ($= C$ say), contrary to 15.7. So p_i is Y-complete. If z is not Y-complete then an antipath between z and x_3 with interior in Y can be extened to an antihole via x_3-x_2-p_i-z meeting C in four vertices, a contradiction. So z is Y-complete; but then (C, Y) is a wheel satisfying the theorem. So we may assume $i = n$, and hence p_n is adjacent to both x_2, x_3. From the symmetry between x_0, x_1 we may also assume that p_1 is adjacent to both x_2, x_3. Suppose that p_1, p_n are both Y-complete. If z is Y-complete then the hole z-x_0-P-x_1-z is the rim of a wheel with hub Y, as required, so we assume z is not Y-complete. But then the cycle z-x_0-p_1-x_2-p_n-x_1-z and Y violate 15.1. Hence we may assume that not both p_1, p_n are Y-complete. So in $\overline{G}$, the connected set

$Y \cup \{x_3, p_1, p_n\}$ catches the triangle $\{x_0, x_1, x_2\}$; x_0, x_1, x_2 all have unique neighbours in it, namely p_n, p_1, x_3 respectivelt; and these three vertices do not form a triangle since x_3p_1 is not an edge, contrary to 17.1 (in $\overline{G}$). This proves (2).

Choose a minimal connected subgraph F of A_1 containing neighbours of each of x_0, x_1, x_2, x_3.

(3) *x_2, x_3 have a common neighbour in F.*

For by (2), F is not minimal containing neighbours of x_0, x_1, and so there exists $f \in F$ such that $F \setminus f$ is connected and contains neighbours of x_0, x_1. From the minimality of F it does not contain neighbours of both x_2, x_3; so f is the unique neighbour in f of one of x_2, x_3. Suppose first that it is the unique neighbour of x_3. We may assume f is not adjacent to f_2. Now z,x_3 both have unique neighbours (x_2, f respectively) in the connected graph $F \cup \{x_2\} = F'$ say; $F' \setminus x_2$ is connected; and z, x_3 are both X_1-complete, and x_2, f are not. By 17.3, there exists $x \in \{x_0, x_1\}$ with no neighbours in $F' \setminus x_2 = F$, a contradiction. So f is not the unique neighbour of x_3. Now assume f is the unique neighbour of x_2. Let x_2-p_1-$\cdots$-p_n-x_3 be a minimal path from x_2 to x_3 with interior in F. Then $n \geq 3$ and n is odd, and $p_1 = f$. The only X_1-complete vertices in the hole z-x_2-p_1-$\cdots$-p_n-x_3-z are z, x_3, so by 2.11, X_1 contains a leap or a hat for this hole. Suppose it contains a leap; then x_2-p_1-$\cdots$-p_n is the interior of an odd path P between x_0, x_1. Since x_0, x_1 are $Y \cup \{x_3\}$-complete, and P has length ≥ 5, it follows from 13.6 that some vertex of x_2-p_1-$\cdots$-p_n is $Y \cup \{x_3\}$-complete, and this must be p_n, since no other vertex is adjacent to x_3. The ends of P are also $Y \cup \{x_3, z\}$-complete, and since no internal vertex is $Y \cup \{x_3, z\}$-complete it follows that $Y \cup \{x_3, z\}$ is not co-connected, that is, z is Y-complete. But then the hole z-x_2-p_1-$\cdots$-p_n-x_3-z is the rim of a wheel satisfying the theorem. Next assume that X_1 contains a hat for the hole z-x_2-p_1-$\cdots$-p_n-x_3-z; then (by exchanging x_0, x_1 if necessary) we may assume that x_1 has no neighbours in x_2-p_1-$\cdots$-p_n. But then the path x_2-p_1-$\cdots$-p_n-x_3-x_1 has length ≥ 5 and odd, and its ends are $Y \cup \{z\}$-complete, and its internal vertices are not; so z is Y-complete by 13.6. The same path has both ends Y-complete, and so by 13.6 it contains a Y-complete edge. Since (C, Y) is not an odd wheel, where C is the hole z-x_2-p_1-$\cdots$-p_n-x_3-z, it follows that the segment of its rim containing z-x_2 has length > 1, and therefore p_1 is Y-complete. But then let Q be a path between x_1, x_2 with interior in A_1; it has length ≥ 4 since x_1 is nonadjacent to f, and therefore the hole z-x_1-Q-x_2-z is the rim of a wheel satisfying the theorem. This proves (3).

(4) *If x_2 is adjacent to x_0, then the theorem holds.*

For $F \cup \{x_1\}$ catches the triangle $\{z, x_0, x_2\}$; it contains no reflection of this triangle, since x_0, x_1 have no common neighbour in A_1; and the unique neighbour of z in this set is nonadjacent to both x_0, x_2. So by 17.1 it follows that there is a vertex in F adjacent to both x_0, x_2.

Now $F \cup x_2$ catches the triangle $\{z, x_1, x_3\}$. Suppose that $F \cup \{x_2\}$ contains a reflection of this triangle; then there exist $f \in F$ adjacent to x_1, x_2 and not to x_3. Since $f \in A_1$ it follows that f is nonadjacent to x_0; but then f-x_2-x_0-x_3-x_1-f is an odd hole, a contradiction. Hence by 17.1 there is a vertex in F adjacent to both x_1, x_3.

Consequently from the minimality of F, F is the vertex set of a path f_1-$\cdots$-f_n, where f_1 is adjacent to x_0, x_2, and f_n to x_1, x_3. Since $f_1 \in A_1$ it follows that f_1 is not adjacent to x_1.

Now assume that f_1 is the unique neighbour of x_0 in F. By (2), we may assume that f_n is not the unique neighbour of x_1, and so from the minimality of F it is the unique neighbour of x_3 in F. In particular x_3 is not adjacent to f_1. Both x_0, z have unique neighbours in $F \cup \{x_1\} = F'$ say, namely f_1, x_1 respectively. Now x_0, z are both $\{x_2, x_3\}$-complete, and f_1, x_1 are not. Since $F' \setminus x_1$ is connected, this contradicts 17.3. So we may assume that f_1 is not the unique neighbour of x_0 in F. It follows that f_1 is the unique neighbour of x_2 in F. From (3), f_1 is adjacent to x_3, and so from the minimality of F, f_n is the unique neighbour of x_1 in F. Let C be the hole z-x_2-f_1-$\cdots$-f_n-x_1-z; then this has length ≥ 6. If f_1 is not Y-complete, an antipath betwen f_1 and x_3 with interior in Y can be completed to an antihole via x_3-x_2-x_1-f_1, which shares the three vertices x_1, x_2, f_1 with C, contrary to 15.7. So f_1 is Y-complete. If z is not Y-complete, an antipath between z, x_3

with interior in Y can be completed to an antihole via x_3-x_2-x_1-f_1-z, which has four vertices in common with C, again a contradiction. So z is Y-complete; but then the hole C is the rim of a wheel satisfying the theorem. This proves (4).

Henceforth then we assume that x_2 is nonadjacent to x_1 and similarly to x_0.

(5) *If x_3 has a unique neighbour in F then the theorem holds.*

For let f be its unique neighbour. By (3), f is adjacent to x_2. Since f is not adjacent to both x_0, x_1 we may assume from the symmetry that it is not adjacent to x_1. Now $F \cup \{x_2\}$ catches the triangle $\{z, x_1, x_3\}$. The only neighbour of z in $F \cup \{x_2\}$ is x_2, and that is nonadjacent to both x_1, x_3. Also no vertex of $F \cup \{x_2\}$ is adjacent to both x_1, x_3, because the only vertex adjacent to x_3 is f. So $F \cup \{x_2\}$ contains a reflection of the triangle. In particular there is a vertex $f_1 \in F \setminus f$, adjacent to f, x_1, x_2 and not to x_3. Since every path between x_0, x_1 with interior in A_1 has length ≥ 4, it follows that x_0 is nonadjacent to f, f_1; and so we can apply the same argument to x_0, to prove that there is a vertex f_0 say in $F \setminus f$, adjacent to f, x_0, x_2 and not to x_1, x_3. Hence f_0 is different from f_1. If they are adjacent then z-x_0-f_0-f_1-x_1-z is an odd hole; and it they are nonadjacent then the eight vertices $\{z, x_0, x_1, x_2, x_3, f, f_1, f_2\}$ induce a double diamond, contrary to 14.3. This proves (5).

From (2) it follows that F is not minimal subject to containing neighbours of x_0, x_1, and so there is a vertex $f \in F$ such that $F \setminus f$ is connected and contains neighbours of x_0, x_1. From (5) and the minimality of F it follows that f is the unique neighbour in F of x_2. By (3), f is adjacent to x_3. Since $f \in A_1$ we may assume from the symmetry between x_0, x_1 that f is nonadjacent to x_1. Let P be a path from x_1 to x_2 with interior in F; then $f \in V(P)$, and P has length ≥ 4 and even. Let C be the hole z-x_1-P-x_2-z. If f is not Y-complete, an antipath between f, x_3 with interior in Y can be completed to an antihole via x_3-x_2-x_1-f, and this shares the three vertices x_1, x_2, f with C, contrary to 15.7. So f is Y-complete. If z is not Y-complete, an antipath between z, x_3 with interior in Y can be completed to an antihole via x_3-x_2-x_1-f-z, sharing four vertices with C, again a contradiction. So z is Y-complete; but then (C, Y) is a wheel satisfying the theorem. This proves 19.2. ∎

19.3 *Let G be nonconforming, and let $Y \subseteq V(G)$ be nonempty and co-connected. Suppose that for some $t \geq 4$, there is a Y-diamond in G of height t. Suppose that there is no co-connected set Y' with $Y \subseteq Y' \subseteq V(G)$ such that either:*

- *there is a Y'-diamond in G of height $t-1$, or*
- *there is a Y'-square in G of height $t-1$, or*
- *there is a Y'-diamond-square in G of height t.*

Then G contains a wheel (C, Y).

Proof. Assume G contains no wheel (C, Y). Let $z, x_0, \dots, x_t, A_0$ be a Y-diamond in G, and define X_i, A_i as usual. So x_t is X_{t-2}-complete, and x_t has a neighbour in A_{t-2}.

(1) *Not both x_t and x_{t-1} have neighbours in A_{t-3}.*

For suppose they do. If x_{t-1} is X_{t-3}-complete, then

$$z, x_0, \dots, x_{t-1}, A_0$$

is a $Y \cup \{x_t\}$-diamond of height $t-1$, while if x_{t-1} is not X_{t-3}-complete, then

$$z, x_0, \dots, x_{t-3}, x_{t-1}, x_t, A_0$$

is a Y-diamond of height $t-1$, in both cases a contradiction. This proves (1).

(2) *There is a vertex q in A_{t-2} adjacent to both x_t and x_{t-1}, and a path R in A_{t-2} from q to A_{t-3} so that not both x_t and x_{t-1} have neighbours in $A_{t-3} \cup V(R \setminus q)$.*

For let F be a minimal connected subgraph of A_{t-2} including A_{t-3} and containing neighbours of both x_t and x_{t-1}. If x_t, x_{t-1} have a common neighbour in F, then the claim is satisfied (from the minimality of F), so we assume not. Let P be a path between x_t and x_{t-1} with interior in F. Hence P has length > 2, and the hole z-x_1-P-x_2-z ($= C$ say) it follows that P is even. The only X_{t-2}-complete vertices in C are z and x_t, so by 2.11, X_{t-2} contains a leap or a hat for C. Suppose it contains a leap; then there are nonadjacent $x_i, x_j \in X_{t-2}$ so that x_i-p_1-$\cdots$-p_n-x_{t-1}-x_j is an odd path. Since x_i, x_j are $Y \cup \{x_t\}$-complete, it follows from 13.6 that this path contains another $Y \cup \{x_t\}$-complete vertex, which must be p_1 since no others are adjacent to x_t. Its ends are also $Y \cup \{x_t, z\}$-complete, and no internal vertex is $Y \cup \{x_t, z\}$-complete, so by 13.6, $Y \cup \{x_t, z\}$ is not co-connected, that is, z is Y-complete. But then (C, Y) is a wheel, a contradiction.

So X_{t-2} contains a hat for C; that is, there exists $x_i \in X_{t-2}$ with no neighbours in C except x_t, z. Hence the path x_i-x_t-p_1-$\cdots$-p_n-x_{t-1} is odd and has length ≥ 5, and its ends are $Y \cup \{z\}$-complete, and no internal vertex is $Y \cup \{z\}$-complete, so by 13.6, z is Y-complete. Let R be a path between x_i and x_{t-1} with interior in F. Then $V(R \cup P) \setminus \{x_i, x_t\}$ ($= F'$ say) is connected and catches the triangle $\{z, x_i, x_t\}$. The only neighbour of z in F' is x_{t-1}, which is nonadjacent to both x_i, x_t. If F' contains a reflection of the triangle, there is an antihole of length 6 containing z, x_{t-1}, x_t, which is impossible by 15.7 since these three vertices belong to C. So by 17.1, there is a vertex in F' adjacent to both x_i, x_t. Since x_i has no neighbour in $P \setminus x_t$, it follows that both x_t, x_{t-1} have neighbours in the interior of R, and so there is a path P' between x_t, x_{t-1} with $P' \setminus x_t$ a subpath of $R \setminus x_i$. As before P' has length ≥ 4, and so R has length ≥ 4. Since the X_{t-2}-complete vertex z has no neighbours in the interior of P, from 18.2 applied to P with co-connected sets Y and X_{t-2}, it follows that there is a Y-complete edge in P, and since x_t is not Y-complete, there is therefore one in R. But since the edges zx_{t-1}, zx_i are also Y-complete, we deduce that there are at least three Y-complete edges in the hole z-x_i-R-x_{t-1}-z, and so that hole is the rim of a wheel with hub Y, a contradiction. This proves (2).

Choose q, R as in (2) with R minimal, and let R be r_1-$\cdots$-r_n, where $r_1 = q$ and r_n is the only vertex of R in A_{t-3} .

(3) *x_{t-1} has neighbours in A_{t-3}.*

For assume not. Since x_{t-1} has no neighbours in A_{t-3} it follows that $q \notin A_{t-2}$, and so R has length > 0. Suppose first that every antipath between x_{t-1} and q with interior in X_{t-2} is odd, and let Q be such an antipath. Since all internal vertices of Q have neighbours in A_{t-3}, and z is complete to ite interior and co-complete to A_{t-3}, it follows from 2.2 applied in $\overline{G}$ one end of Q has a neighbour in A_{t-3}. By hypothesis, x_{t-1} does not, so q does. From the maximality of A_{t-3} it follows that q is X_{t-3}-complete; and since $q \in A_{t-2}$ and is therefore not X_{t-2}-complete, q is nonadjacent to x_{t-2}. Now by assumption, every every antipath between x_{t-1} and q with interior in X_{t-2} is odd, and so x_{t-2} is adjacent to x_{t-1}. But then

$$z, x_0, \ldots, x_{t-1}, A_0$$

is a $Y \cup \{x_t\}$-square of height $t-1$, a contradiction. So we may assume some antipath Q between between x_{t-1} and q with interior in X_{t-2} is even.

From (2), not both x_t, x_{t-1} have neighbours in $A_{t-3} \cup V(R \setminus q)$. Suppose that x_{t-1} has such a neighbour, and so x_t does not. Since by assumption x_{t-1} has no neighbours in A_{t-3}, it follows that all neighbours of x_{t-1} in $A_{t-3} \cup V(R \setminus q)$ lie in the interior of R, and in particular R has length ≥ 2. The antipath x_t-x_{t-1}-Q-q is odd, and its ends have no neighbours in the connected set $A_{t-3} \cup \{r_3, \ldots, r_n\}$. Since z is complete to its interior and co-complete to $A_{t-3} \cup \{r_3, \ldots, r_n\}$, it follows from 2.2 applied in $\overline{G}$ that some internal vertex of this antipath has no neighbours in $A_{t-3} \cup \{r_3, \ldots, r_n\}$. But all internal vertices of Q lie in X_{t-2} and therefore have neighbours in A_{t-3}; so x_{t-1} has no neighbour in $A_{t-3} \cup \{r_3, \ldots, r_n\}$. Hence r_2 is its only neighbour in

$A_{t-3} \cup V(R \setminus q)$. Suppose that every antipath between x_{t-1} and r_2 with interior in X_{t-2} is odd, and let Q' be such an antipath. All internal vertices of Q' have neighbours in the connected set A_{t-3}, and z is complete to the interior of Q' and co-complete to A_{t-3}; so by 2.2 applied in $\overline{G}$, it follows that r_2 has neighbours in A_{t-3}. From the maximality of A_{t-3}, r is X_{t-3}-complete, and therefore not adjacent to x_2. Since by assumption every antipath between x_{t-1} and r_2 with interior in X_{t-2} is odd, it follows that x_{t-1} is adjacent to x_{t-2}. But then

$$z, x_0, \dots, x_{t-1}, A_0$$

is a $Y \cup \{x_t\}$-square of height $t-1$, a contradiction. So some antipath Q' between x_{t-1} and r_2 with interior in X_{t-2} is even. Hence the antipath x_{t-1}-Q'-r_2-z is odd. All its internal vertices have neighbours in the connected set $A_{t-3} \cup \{r-3, \dots, r_n\}$ and its ends do not, so by 13.6 this antipath has length 3, that is, Q' has length 2. Let x_i be its middle vertex. Then the connected set $A_{t-3} \cup V(R \setminus \{r_1, r_2\}) \cup \{x_i, x_t, z\}$ ($= F$ say) catches the triangle $\{r_1, r_2, x_{t-1}\}$; the only neighbours of r_1 in F are x_t and possibly x_i; the neighbours of r_2 in F lie in $A_{t-3} \cup \{r_3\}$; and the only neighbour of x_{t-1} in F is z. This contradicts 17.1, since z has no neighbour in $A_{t-3} \cup \{r_3\}$.

So x_{t-1} has no neighbours in $A_{t-3} \cup V(R \setminus q)$. Now the antipath z-q-Q-x_{t-1} is odd, and all its internal vertices have neighbours in $A_{t-3} \cup V(R \setminus q)$, and its ends do not, so by 13.6 it has length 3, that is, Q has length 2 (let its middle vertex be x_i); and there is an odd path P between q, x_i with interior in $A_{t-3} \cup V(R \setminus q)$. Let C be the hole z-x_{t-1}-q-P-x_i-z; then C has length ≥ 6. By 15.7 there is no antihole of length ≥ 6 containing q, x_i, x_{t-1}. If q is not Y-complete then an antipath between q, x_t can be completed to such an antihole via x_t-x_{t-1}-x_i-q, so q is Y-complete; and if z is not Y-complete, an antipath between z and x_t can be extended to such an antihole, via x_t-x_{t-1}-x_{t-2}-q-z. So z is also Y-complete. Now both x_i, q are $Y \cup \{x_t\}$-complete, and therefore by 2.2 applied to P, it follows that P contains a $Y \cup \{x_t\}$-complete edge. Hence the hole C contains at least three y-complete edges, a contradiction. This proves (3).

From (3) and the choice of R it follows that x_t has no neighbours in $A_{t-3} \cup V(R \setminus q)$. Let Q be an antipath between q and x_{t-1} with interior in X_{t-2}. Then z-q-Q-x_{t-1}-x_t is an antipath of length ≥ 4, and its ends have no neighbours in the connected set $A_{t-3} \cup V(R \setminus q)$, and its internal vertices do, so by 13.6 it has even length, that is, Q is even. The antipath x_t-x_{t-1}-Q-q is therefore odd, and its internal vertices have neighbours in A_{t-3}, and z is complete to its interior and co-complete to A_{t-3}, so by 2.2 applied in $\overline{G}$, it follows that one of its ends, and hence q has a neighbour in A_{t-3}. From the maximality of A_{t-3} it follows that q is X_{t-3}-complete and therefore nonadjacent to x_{t-2}. If x_{t-1} is not X_{t-3}-complete, then

$$z, x_0, \dots, x_t, A_0$$

is a Y-diamond-square of height t; while if x_{t-1} is X_{t-3}-complete, then

$$z, x_0, \dots, x_{t-1}, A_0$$

is a $Y \cup \{x_t\}$-diamond of height $t-1$, in both cases a contradiction. This proves 19.3. ∎