

Problem Set 11: Linear Programming

Due: Thursday, April 17, 2014, at the beginning of class

1. Find conditions on s and t to make the following LP problem:

$$\begin{aligned} &\text{minimize } x_1 + x_2 \\ &\text{such that } sx_1 + tx_2 \geq 1 \\ &\quad x_1 \geq 0 \\ &\quad x_2 \text{ is unconstrained} \end{aligned}$$

- (a) infeasible
 (b) feasible
 (c) have a unique optimal solution
 (d) have multiple optimal solutions
 (e) unbounded
2. Work exercise 29.3-5 (page 878) by hand, showing your work.
3. In a *Boolean linear program*, the variable x is constrained to have components equal to 0 or 1:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } aX \leq b \\ &\quad x_i \in \{0,1\}, \quad i = 1, \dots, n \end{aligned}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } aX \leq b \\ &\quad 0 \leq x_i \leq 1, \quad i = 1, \dots, n \end{aligned}$$

We refer to this problem as the LP *relaxation* of the Boolean LP. The LP relaxation is far easier to solve than the original Boolean LP.

- (a) Show that the optimal value of the LP relaxation is a lower bound on the optimal value of the Boolean LP. What can you say about the Boolean LP if the LP relaxation is infeasible?
- (b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

4. One of the most important problems in the field of statistics is the *linear regression problem*. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ on a graph. If we denote the line by $y = a + bx$, the objective is to choose the constants a and b to provide the best fit according to some criterion. The criterion usually used is the method of least squares, but there are other interesting criteria where linear programming can be used to solve for the optimal values of a and b . Formulate a linear programming model for this problem under the following criterion:

Minimize the sum of the absolute deviations of the data from the line; that is,

$$\text{minimize } \sum_{i=1}^n |y_i - (a + bx_i)|$$

5. Consider the following LP problem:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & -x_1 + 2x_2 \leq 6 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Draw the feasible region of this LP in (x_1, x_2) space and label the constraints.
- Notice that including non-negativity, we have 5 constraints. What is the solution corresponding to each extreme point in the feasible region?

You will notice that one of the extreme points in the feasible region is the intersection of 3 constraints, and any 2 of them will uniquely specify that extreme point. Such an extreme point is said to be *degenerate*.
- Solve the problem graphically and verify that the optimal point is a degenerate extreme point.
- Note that in the simplex method, the way that ties are broken can affect the number of iterations required to find a solution. Solve this problem by the simplex method, but select the entering and leaving variables by breaking ties in such a way as to maximize the number of iterations needed. Show your work. What is this maximum number of iterations?
- From part c, identify the constraint that causes degeneracy and solve the problem using simplex without this constraint, again breaking ties (when you have a choice of the entering and leaving variables) in such a way as to maximize the number of iterations needed. Show your work. Note that the degeneracy disappears and the same optimal solution is obtained. What is the maximum number of iterations needed?
- Is it true in general that degenerate extreme points can be made nondegenerate by eliminating some (properly selected) constraints without affecting the feasible region? Explain.