Today:
– Max Flow Proofs

COSC 581, Algorithms

March 4, 2014

Many of these slides are adapted from several online sources

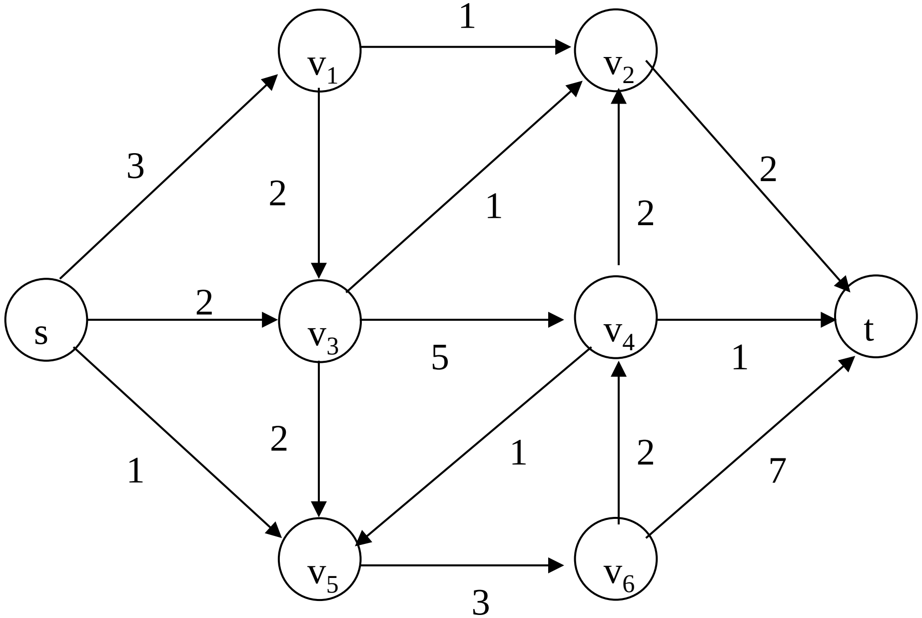
Reading Assignments

- Today's class:
  - Chapter 26

- Reading assignment for next class:
  - Chapter 17 (Amortized analysis)

In-Class Exercise #1

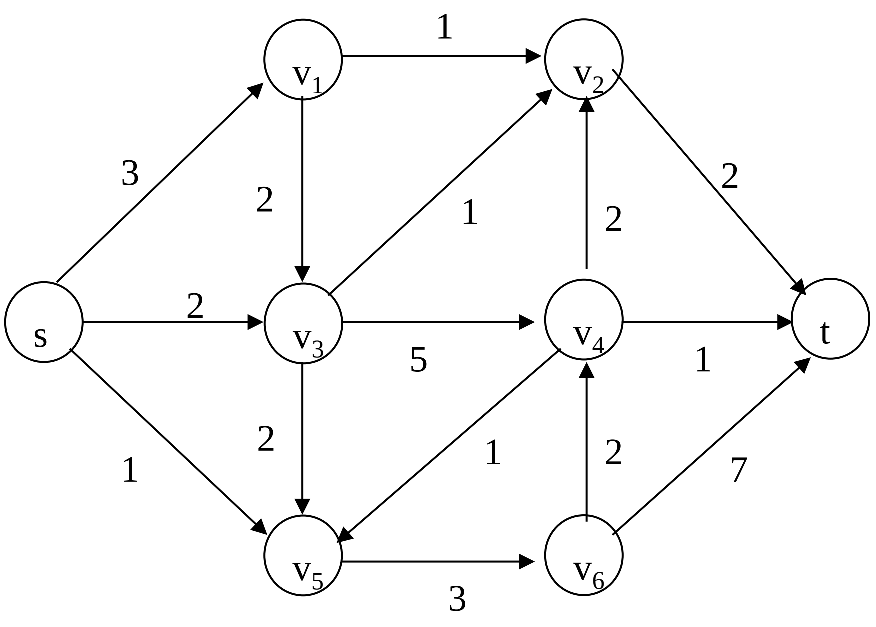
- Find the maximum flow in the following flow network:



In-Class Exercise #1

- Find the maximum flow in the following flow network:

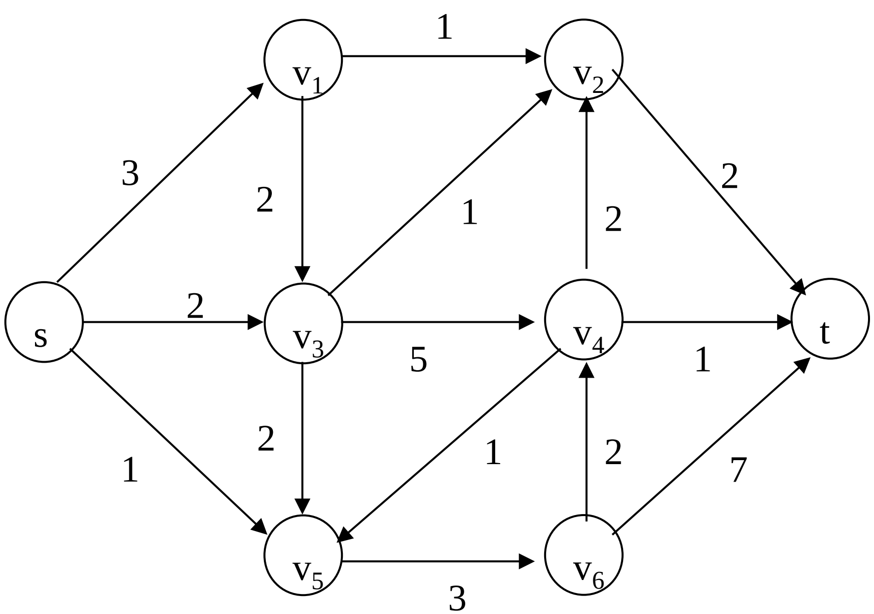
What does residual graph look like (at the beginning) after updating flow from $s \rightarrow v_1 \rightarrow v_2 \rightarrow t$?

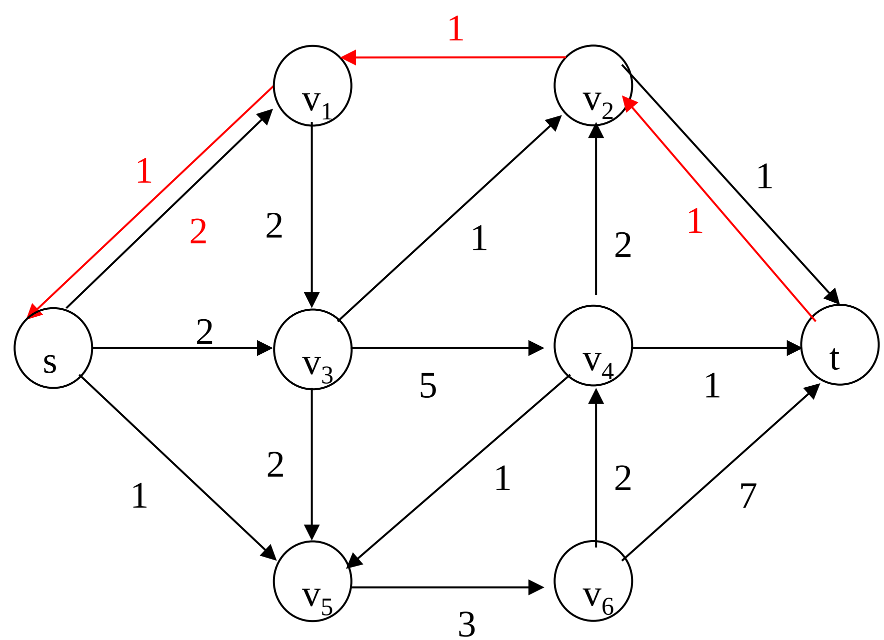


In-Class Exercise #1

- Find the maximum flow in the following flow network:

What does residual graph look like (at the beginning) after updating flow along augmenting path from $s \rightarrow v_1 \rightarrow v_2 \rightarrow t$?



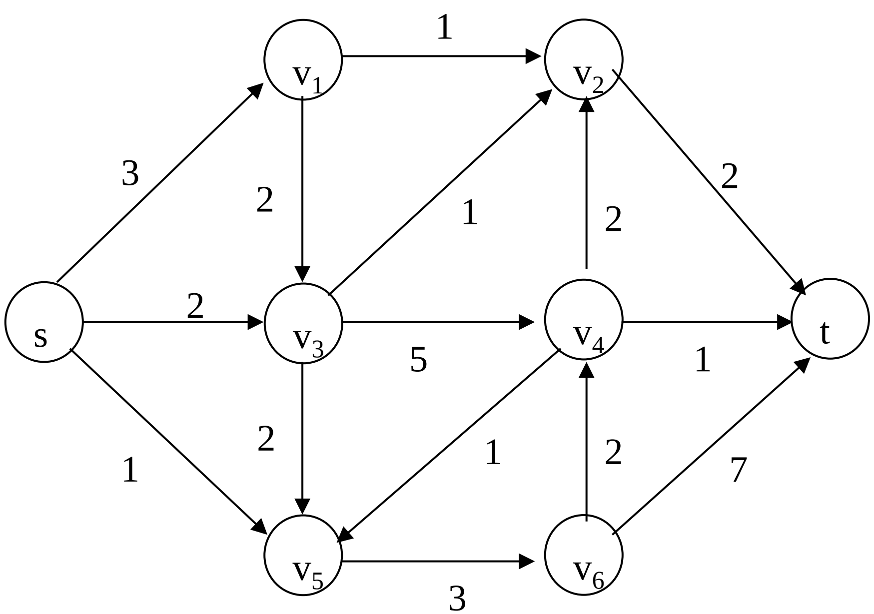


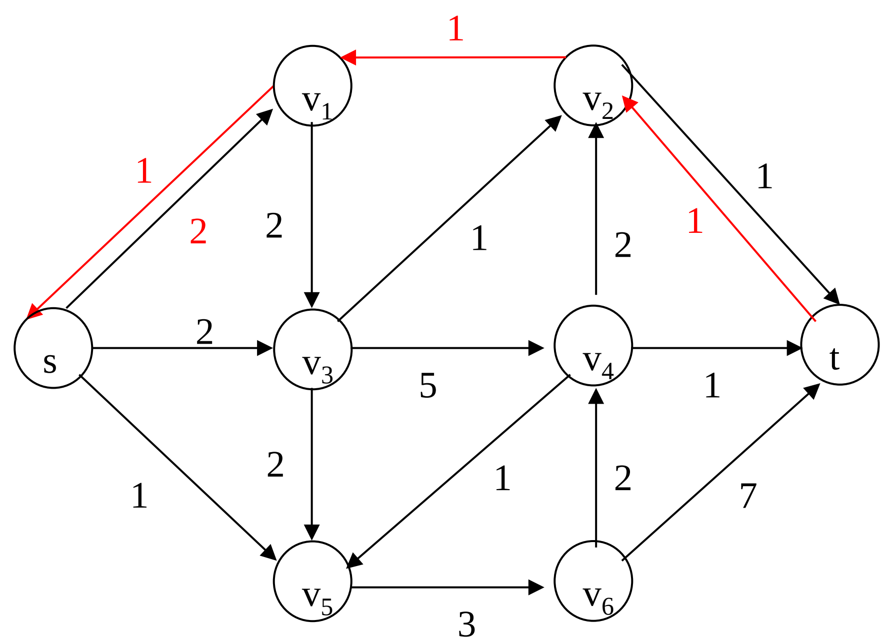
What happened to distance from s to v_2?

In-Class Exercise #1

- Find the maximum flow in the following flow network:

What does residual graph look like (at the beginning) after updating flow along augmenting path from $s \rightarrow v_1 \rightarrow v_2 \rightarrow t$?





What happened to distance from s to v_2? It increased by at least a distance of 1. (Remember this for next proof!)

Edmonds Karp = Ford-Fulkerson Method implemented to select shortest augmenting path

```
FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p
      do augment flow f along p
return f
```

- Take **shortest path** (in terms of number of edges) as an augmenting path – Edmonds-Karp algorithm
  - How do we find such a shortest path?
  - Running time $O(VE^2)$, because the number of augmentations is $O(VE)$
  - To prove this we need to prove that:
    - *The length of the shortest path does not decrease*
    - *Each edge can become* ***critical*** *at most ~ V/2 times.* Edge (u,v) on an augmenting path p is critical if it has the minimum residual capacity in the path*:*

$$c_f(u,v) = c_f(p)$$

Non-decreasing shortest paths

- *Why does the length of a shortest path from s to any v does not decrease?*
  - Observation: Augmentation may add some edges to residual network or remove some.
  - Only the added edges ("shortcuts") may potentially decrease the length of a shortest path.
  - Let's suppose (*s,…,v*) is the shortest decreased-length path and let's derive a contradiction

Number of augmentations

- *Why does each edge become critical at most* ~V/2 *times*?
  - Scenario for edge (*u,v*), for *u, v* ∉ {*s, t*}:
    - Critical the first time: (*u,v*) on an augmenting path
    - Disappears from the network
    - Reappears on the network: (*v,u*) has to be on an augmenting path
    - We can show that in between these events the distance from *s* to *u* increased by at least 2.
    - This can happen at most V/2 times, because the distance from *s* to *u* can never be more than |V|-2
    - Since there are |E| pairs of vertices with edge in residual network, the total # of critical edges during alg. execution is O(VE) (i.e., O(VE) iterations).
    - Each iteration requires |E| work to find shortest path.
- Thus, we have proved that the running time of Edmonds-Karp is $O(VE^2)$.

Defining Flow, Capacity Across Cuts

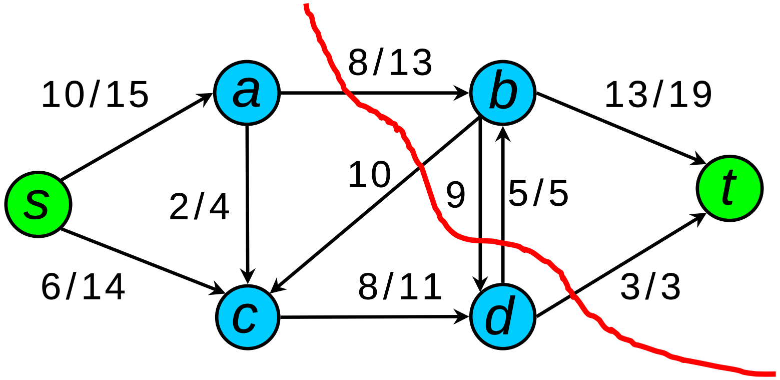
- If f is a flow, then the net flow f(S,T) across the cut (S,T) is defined to be:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- The capacity of the cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

Example:



Flow across this cut?

Capacity of this cut?

Defining Flow, Capacity Across Cuts

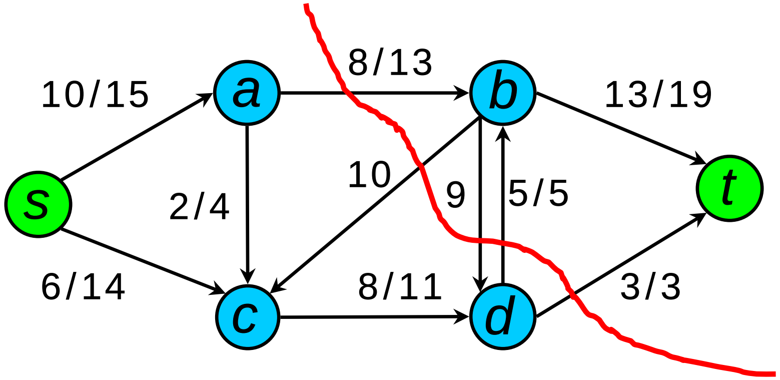
- If f is a flow, then the net flow f(S,T) across the cut (S,T) is defined to be:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- The capacity of the cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

Example:



Flow across this cut? 16

Capacity of this cut?

Defining Flow, Capacity Across Cuts

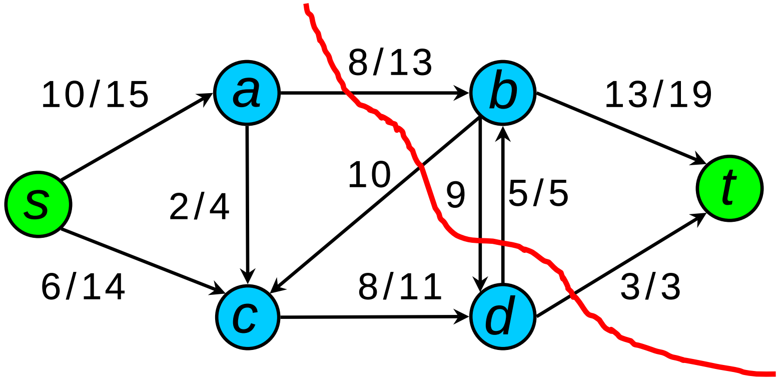
- If f is a flow, then the net flow f(S,T) across the cut (S,T) is defined to be:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- The capacity of the cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

Example:



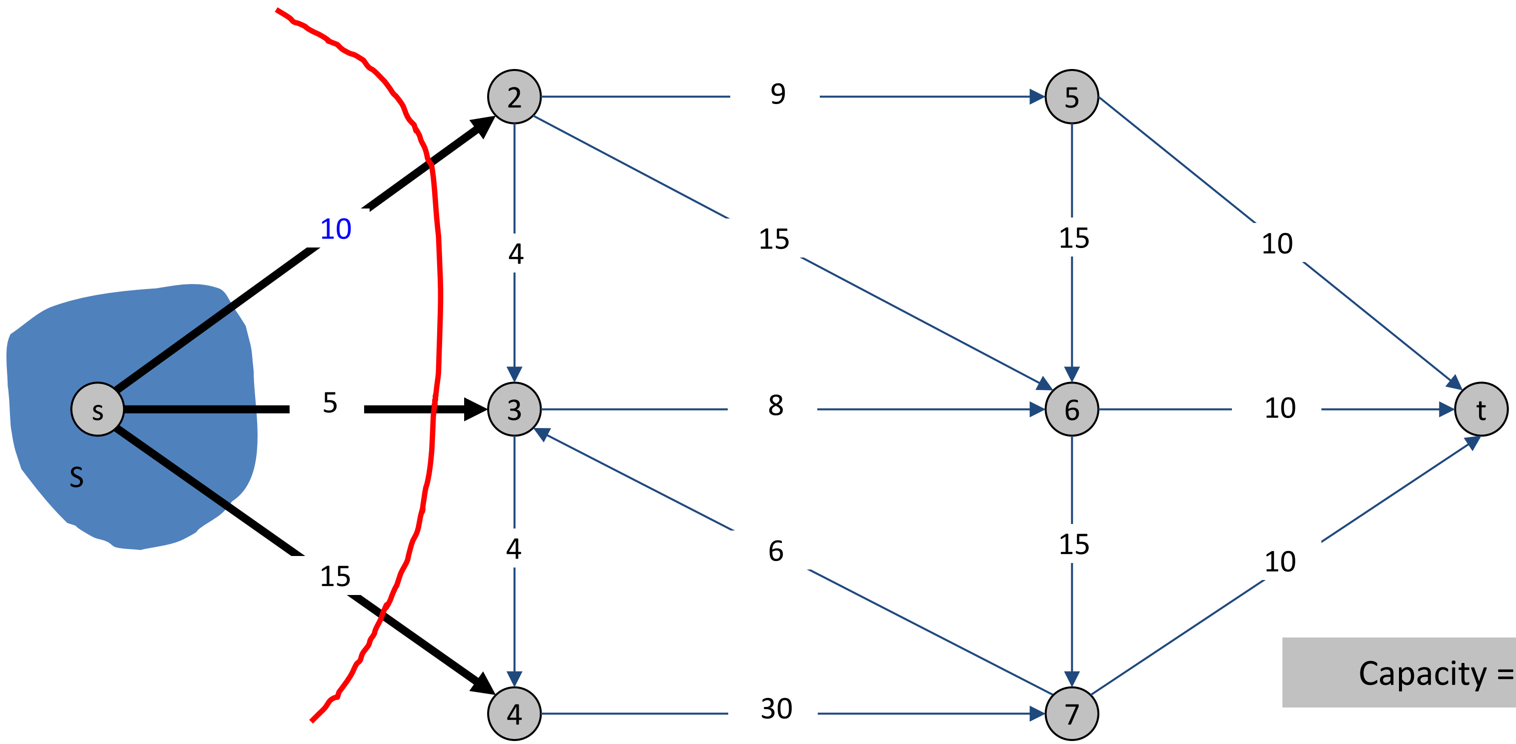
Flow across this cut? 16

Capacity of this cut? 21

Proving Max Flow

- **Weak duality lemma**: Let *f* be any flow, and let (S, T) be any *s*-*t* cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30



Capacity = 30

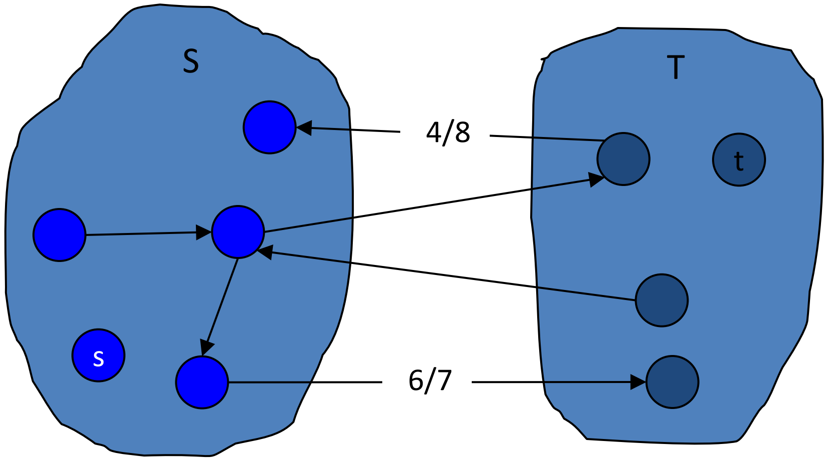
Flows and Cuts

Weak duality lemma: Let f be any flow. Then, for any s-t cut (S, T) we have

$$|f| = f(S,T) \le \text{cap}(S, T).$$

- Proof:

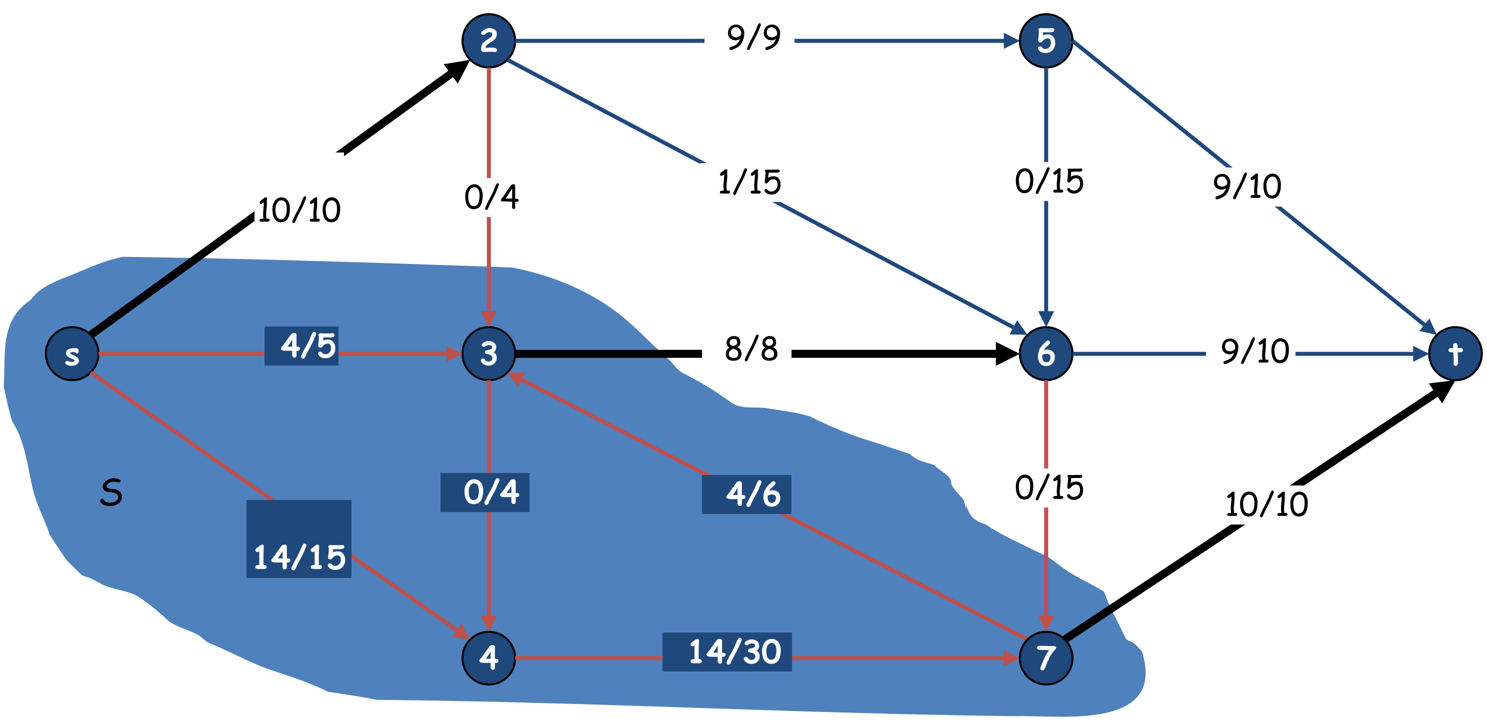
$$
\begin{aligned}
|f| &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) \\
&\le \sum_{e \text{ out of } S} f(e) \\
&\le \sum_{e \text{ out of } S} c(e) \\
&= \text{cap}(S, T)
\end{aligned}
$$



Certificate of Optimality

- Corollary: Let f be any flow, and let (S, T) be any cut.
  If $|f|$ = cap(S, T), then f is a max flow and (S, T) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



Max-Flow Min-Cut Theorem

- Augmenting path theorem: Flow f is a max flow iff there are no augmenting paths.

- Max-flow min-cut theorem: [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

- Proof strategy: We prove both simultaneously by showing the following are equal:
  (i) There exists a cut (S, T) such that $|f|$ = cap(S, T).
  (ii) Flow f is a max flow.
  (iii) There is no augmenting path relative to f.

- (i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

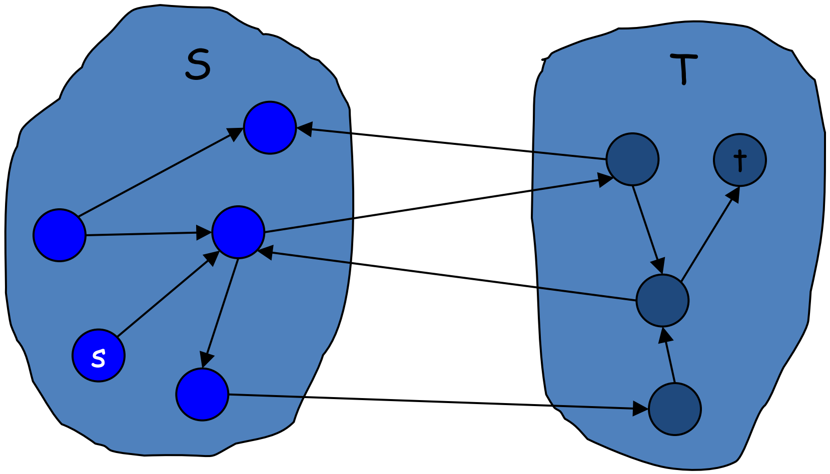
- (ii) $\Rightarrow$ (iii) We show contrapositive.
  - Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

- (iii) $\Rightarrow$ (i)
  - Let *f* be a flow with no augmenting paths.
  - Let A be set of vertices reachable from s in residual graph.
  - By definition of S, s $\in$ S.
  - By definition of *f*, t $\notin$ S.

$$
\begin{aligned}
|f| &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) \\
&= \sum_{e \text{ out of } S} c(e) \\
&= cap(S, T)
\end{aligned}
$$

□



original network

In-Class Exercise #2

You are given a standard flow network G = (V, E) with source *s* and sink *t*, in which each edge (*u*, *v*) $\in$ E has a positive integral capacity *c*(*u, v*). We define an edge in this flow network to be *sensitive* if it crosses some minimum cut (S, T) of the network. We define an edge in this flow network to be *saturated* if its flow equals its capacity.

Are all saturated edges also (necessarily) sensitive? Explain your answer.

In-Class Exercise #3

Updating Maximum Flow: Let G = (V, E) be a flow network with source *s*, sink *t*, and integer capacities. Suppose that we are given a maximum flow in G, and the capacity of a single edge $(u, v) \in E$ is decreased by 1. Give an $O(V + E)$-time algorithm to update the maximum flow.

Reading Assignments

- Reading assignment for next class:
  - Chapter 17 (Amortized analysis)