Merge Sort and Recurrences

COSC 581, Algorithms

January 14, 2014

Reading Assignments

- Today's class:
  - Chapter 2, 4.0, 4.4

- Reading assignment for next class:
  - Chapter 4.2, 4.5

3 Common Algorithmic Techniques

- Divide and Conquer
- Dynamic Programming
- Greedy Algorithms

Divide and Conquer

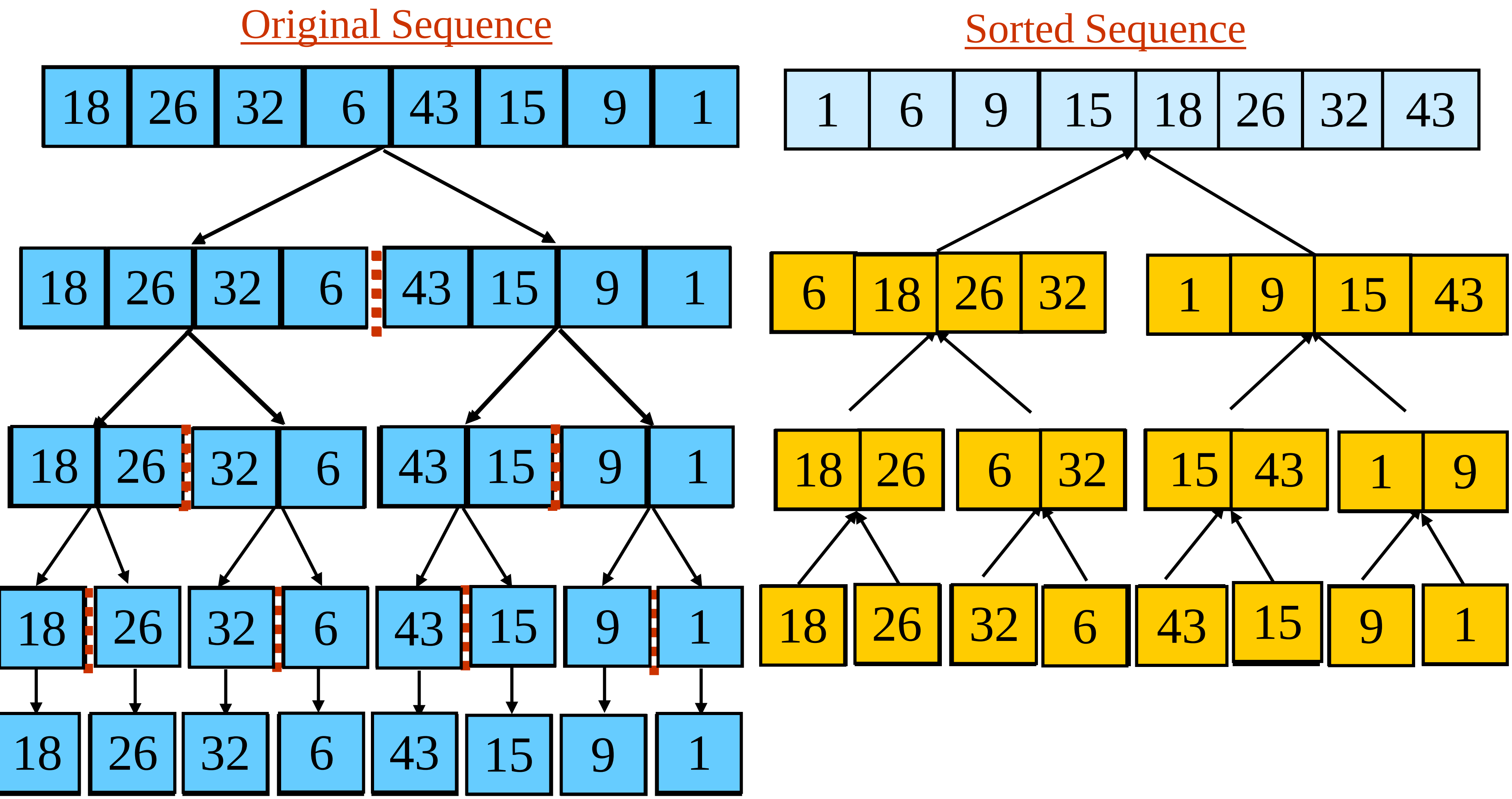
- Recursive in structure
  - ***Divide*** the problem into sub-problems that are similar to the original but smaller in size
  - ***Conquer*** the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.
  - ***Combine*** the solutions to create a solution to the original problem

An Example: Merge Sort

***Sorting Problem*:** Sort a sequence of n elements into non-decreasing order.

- ***Divide*:** Divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each
- ***Conquer:*** Sort the two subsequences recursively using merge sort.
- ***Combine*:** Merge the two sorted subsequences to produce the sorted answer.

Merge Sort – Example



Merge-Sort (A, p, r)

INPUT: a sequence of *n* numbers stored in array A

OUTPUT: an ordered sequence of *n* numbers

***MergeSort* (*A*, *p*, *r*)** // sort $A[p..r]$ by divide & conquer

if $p < r$

  then $q \leftarrow \lfloor (p+r)/2 \rfloor$

    MergeSort (A, p, q)

    MergeSort (A, q+1, r)

    Merge (A, p, q, r) // merges $A[p..q]$ with $A[q+1..r]$

Initial Call: MergeSort(A, 1, n)

Procedure Merge

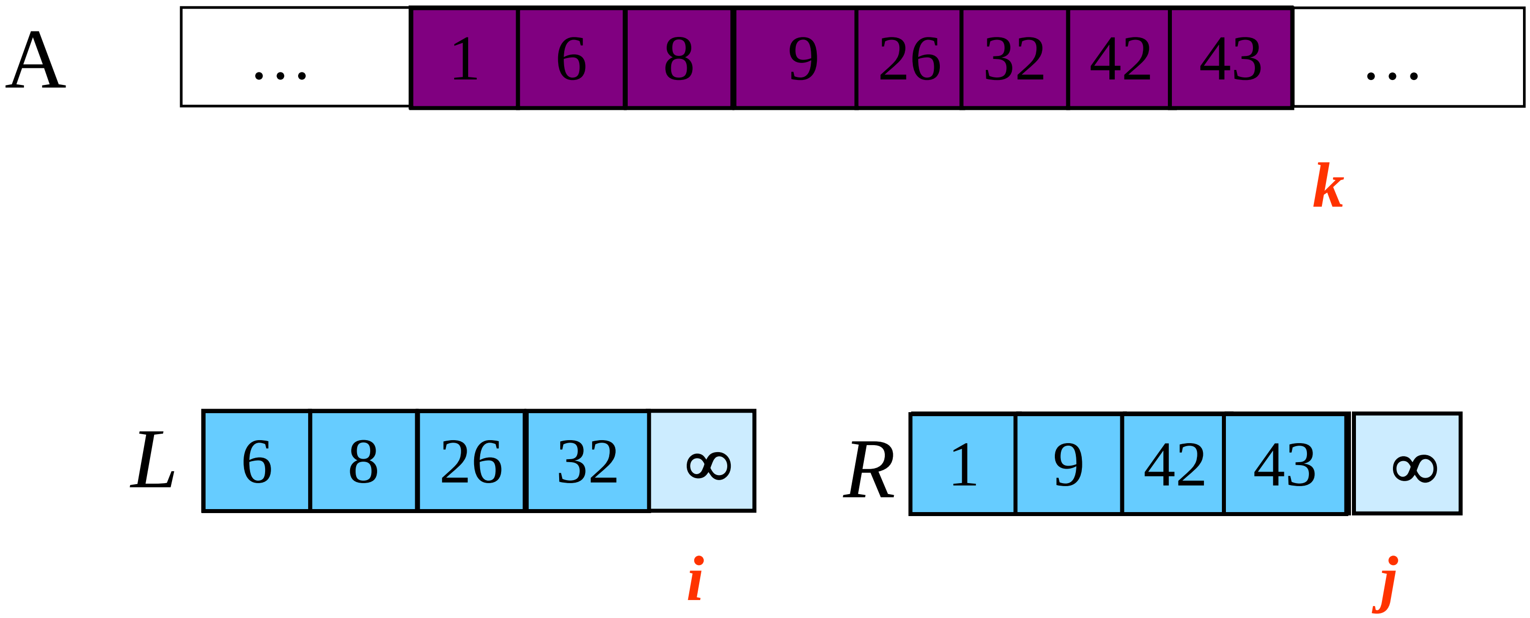
```
Merge(A, p, q, r)
n1 ← q − p + 1
n2 ← r − q
    for i ← 1 to n1
        do L[i] ← A[p + i − 1]
    for j ← 1 to n2
        do R[j] ← A[q + j]
    L[n1+1] ← ∞
    R[n2+1] ← ∞
    i ← 1
    j ← 1
    for k ←p to r
        do if L[i] ≤ R[j]
            then A[k] ← L[i]
                i ← i + 1
            else A[k] ← R[j]
                j ← j + 1
```

Input: Array containing sorted subarrays $A[p..q]$ and $A[q+1..r]$.

Output: Merged sorted subarray in $A[p..r]$.

Sentinels, to avoid having to check if either subarray is fully copied at each step.

Merge – Example



Correctness of Merge

```
Merge(A, p, q, r)
n1 ← q − p + 1
n2 ← r − q
    for i ← 1 to n1
        do L[i] ← A[p + i − 1]
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    L[n1+1] ← ∞
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    i ← 1
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    for k ← p to r
        do if L[i] ≤ R[j]
            then A[k] ← L[i]
                i ← i + 1
            else A[k] ← R[j]
                j ← j + 1
```

Loop Invariant for the *for* loop

At the start of each iteration of the for loop:

Subarray $A[p..k-1]$ contains the $k-p$ smallest elements of L and R in sorted order.

$L[i]$ and $R[j]$ are the smallest elements of L and R that have not been copied back into A.

Initialization:

Before the first iteration:

- $A[p..k-1]$ is empty.
- $i = j = 1$.
- $L[1]$ and $R[1]$ are the smallest elements of L and R not copied to A.

Correctness of Merge

Merge(*A, p, q, r*)

```
1  n1 ← q − p + 1
2  n2 ← r − q
3      for i ← 1 to n1
4          do L[i] ← A[p + i − 1]
5      for j ← 1 to n2
6          do R[j] ← A[q + j]
7      L[n1+1] ← ∞
8      R[n2+1] ← ∞
9      i ← 1
10     j ← 1
11     for k ← p to r
12         do if L[i] ≤ R[j]
13             then A[k] ← L[i]
14                 i ← i + 1
15             else A[k] ← R[j]
16                 j ← j + 1
```

Maintenance:

Case 1: $L[i] \le R[j]$

- By LI, A contains $p - k$ smallest elements of L and R in sorted order.
- By LI, $L[i]$ and $R[j]$ are the smallest elements of L and R not yet copied into A.
- Line 13 results in A containing $p - k + 1$ smallest elements (again in sorted order). Incrementing i and k reestablishes the LI for the next iteration.

Similarly for $L[i] > R[j]$.

Termination:

- On termination, $k = r + 1$.
- By LI, A contains $r - p + 1$ smallest elements of L and R in sorted order.
- L and R together contain $r - p + 3$ elements. All but the two sentinels have been copied back into A.

Analysis of Merge Sort

- Running time $T(n)$ of Merge Sort:
- Divide: computing the middle takes $\Theta(1)$
- Conquer: solving 2 subproblems takes $2T(n/2)$
- Combine: merging n elements takes $\Theta(n)$
- Total:

$$
\begin{aligned}
T(n) &= \Theta(1) && \text{if } n = 1 \\
T(n) &= 2T(n/2) + \Theta(n) && \text{if } n > 1
\end{aligned}
$$

$\Rightarrow T(n) = \Theta(n \lg n)$ (CLRS, Chapter 4)

Recurrences

- Recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs
- Often used to define a recursive algorithm's runtime
- Example: $T(n) = 2T(n/2) + n$

Recurrence Relations

- Equation or an inequality that characterizes a function by its values on smaller inputs.
- **Solution Methods** (Chapter 4)
  - Substitution Method.
  - Recursion-tree Method.
  - Master Method.
- Recurrence relations **arise when we analyze the running time of iterative or recursive algorithms**.
  - **Ex:** Divide and Conquer.

$$T(n) = \Theta(1) \qquad \text{if } n \leq c$$

$$T(n) = a\,T(n/b) + D(n) + C(n) \qquad \text{otherwise}$$

Substitution Method

- **Guess** the form of the solution, then **use mathematical induction** to show it correct.
  - Substitute guessed answer for the function when the inductive hypothesis is applied to smaller values – hence, the name.
- Works well when the solution is easy to guess.
- No general way to guess the correct solution.

Example 1 – Exact Function

Recurrence: $T(n) = 1$ if $n = 1$

$T(n) = 2T(n/2) + n$ if $n > 1$

- Guess: $T(n) = n \lg n + n$.
- Induction:
  - **Basis:** $n = 1 \Rightarrow n \lg n + n = 1 = T(n)$.
  - **Hypothesis:** $T(k) = k \lg k + k$ for all $k < n$.
  - **Inductive Step:** $T(n) = 2\,T(n/2) + n$

$$= 2\,((n/2)\lg(n/2) + (n/2)) + n$$

$$= n\,(\lg(n/2)) + 2n$$

$$= n \lg n - n + 2n$$

$$= n \lg n + n$$

Recursion-tree Method

- Making a good guess is sometimes difficult with the substitution method.
- Use **recursion trees** to devise good guesses.
- Recursion Trees
  - Show successive expansions of recurrences using trees.
  - Keep track of the time spent on the subproblems of a divide and conquer algorithm.
  - Help organize the algebraic bookkeeping necessary to solve a recurrence.

Recursion Tree – Example

- Running time of Merge Sort:

$$T(n) = \Theta(1) \qquad \text{if } n = 1$$
$$T(n) = 2T(n/2) + \Theta(n) \qquad \text{if } n > 1$$

- Rewrite the recurrence as

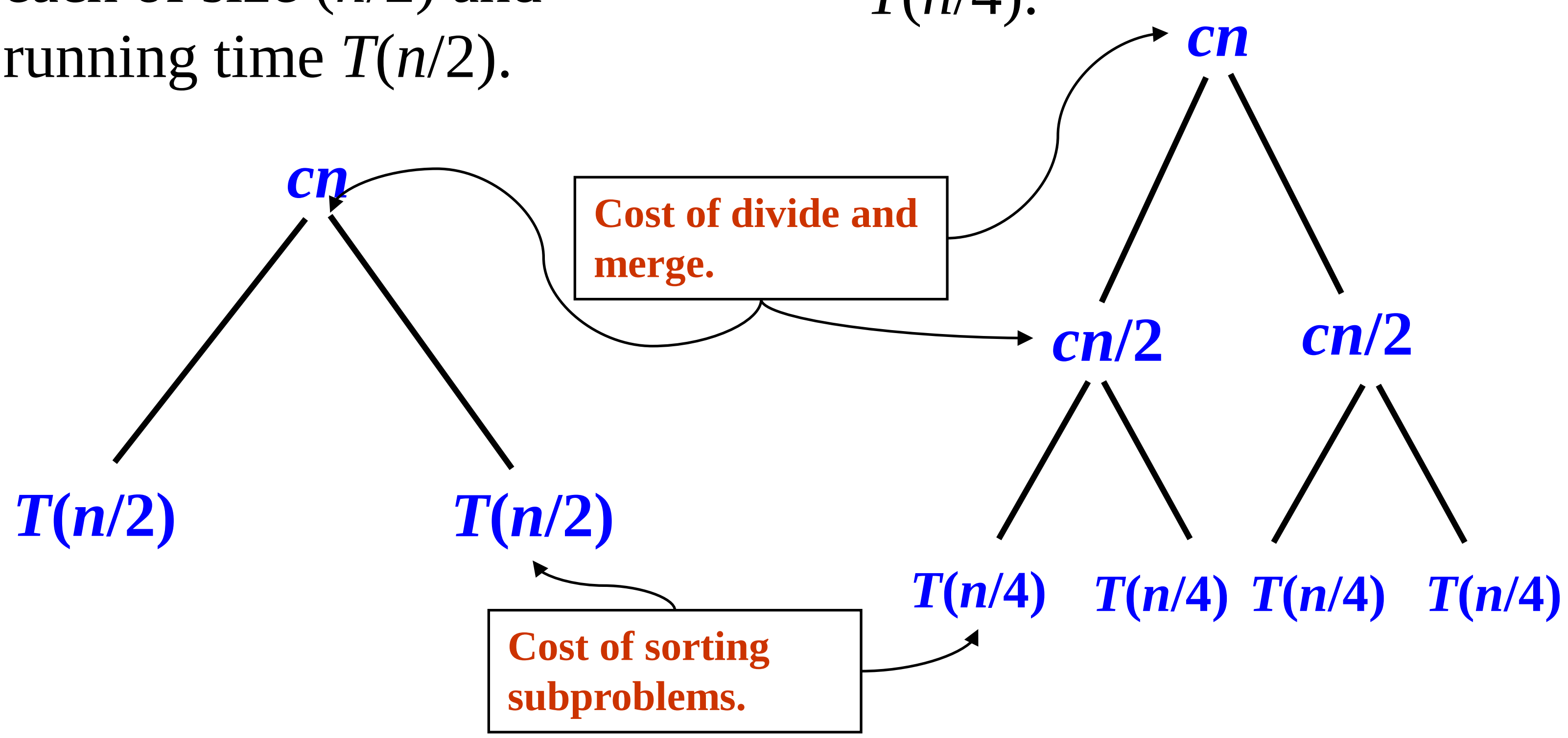
$$T(n) = \boldsymbol{c} \qquad \text{if } n = 1$$
$$T(n) = 2T(n/2) + \boldsymbol{cn} \qquad \text{if } n > 1$$

$\boldsymbol{c > 0}$: Running time for the base case and time per array element for the divide and combine steps.

Recursion Tree for Merge Sort

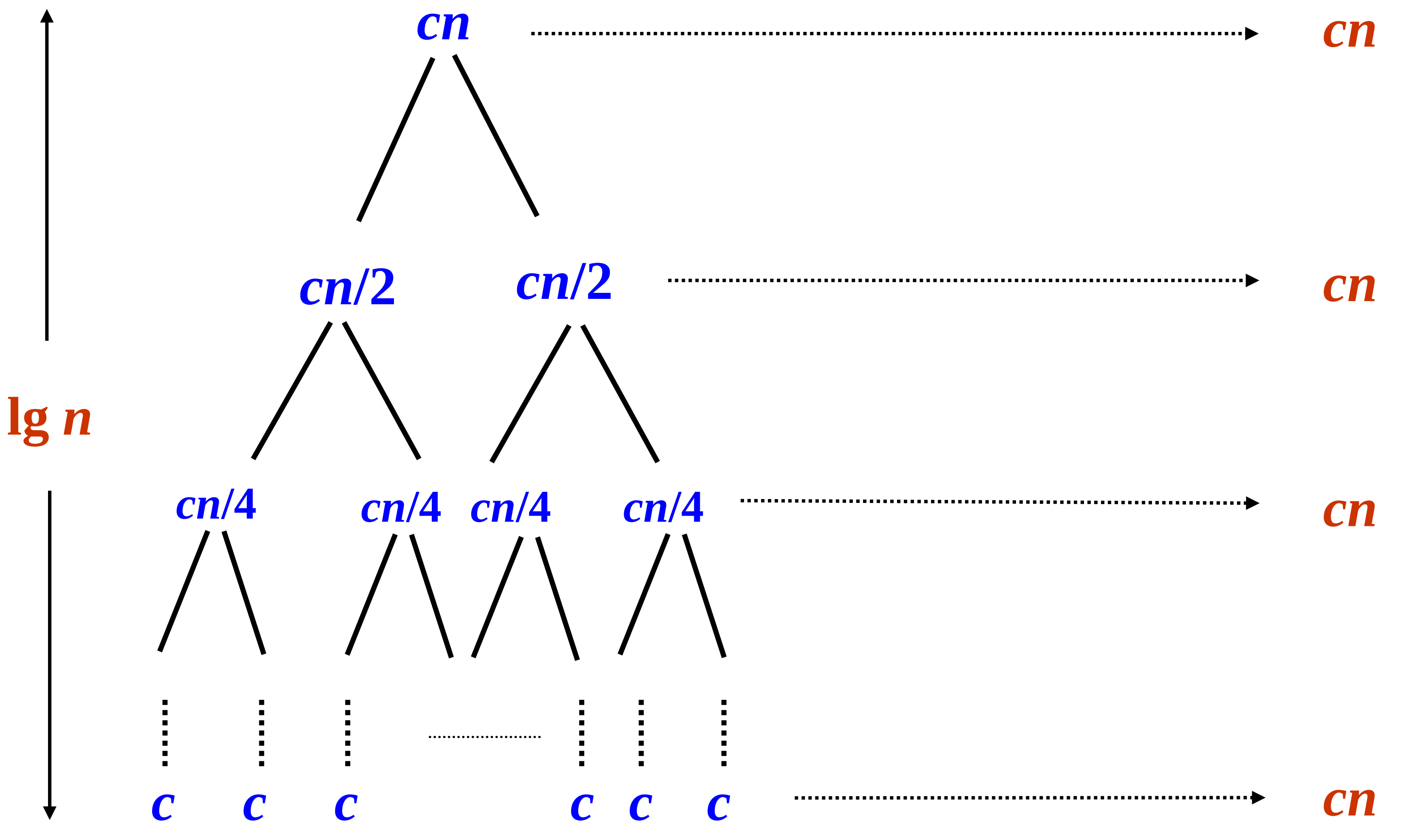
For the original problem, we have a cost of *cn*, plus two subproblems each of size (*n*/2) and running time *T*(*n*/2).

Each of the size *n*/2 problems has a cost of *cn*/2 plus two subproblems, each costing *T*(*n*/4).



Recursion Tree for Merge Sort

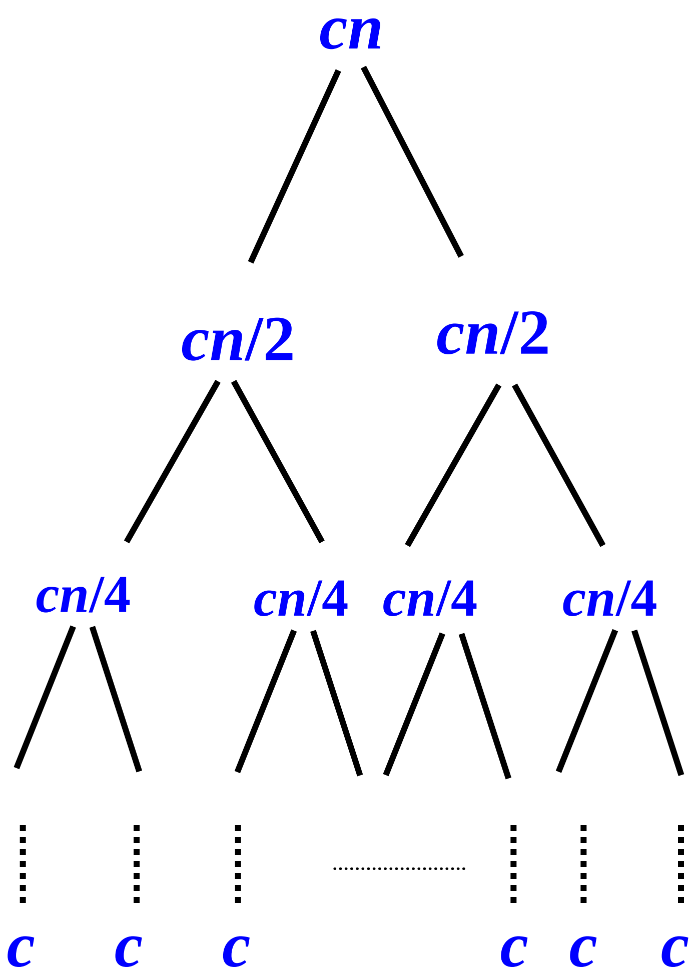
Continue expanding until the problem size reduces to 1.



Total : $cn\lg n + cn$

Recursion Tree for Merge Sort

Continue expanding until the problem size reduces to 1.



- Each level has total cost ***cn***.
- Each time we go down one level, the number of subproblems doubles, but the cost per subproblem halves $\Rightarrow$ *cost per level remains the same*.
- There are lg n + 1 levels, height is lg n. (Assuming n is a power of 2.)
- Can be proved by induction.
- Total cost = sum of costs at each level = $(\lg n + 1)cn = cn\lg n + cn = \Theta(n \lg n)$.

Recursion Trees – Caution Note

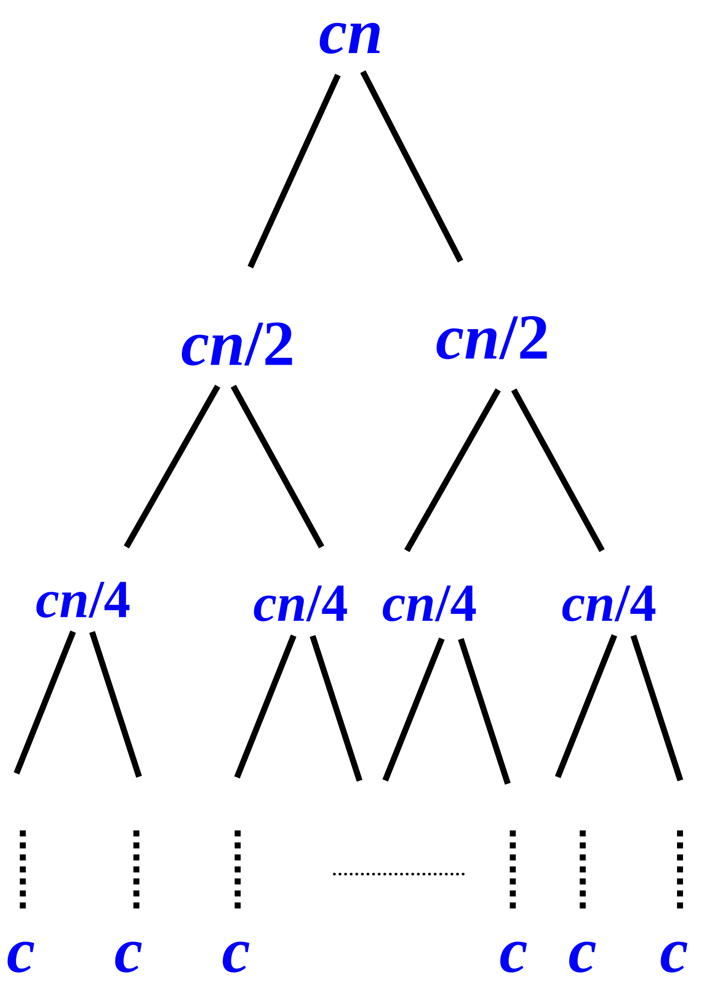
- Recursion trees only generate guesses.
  - Verify guesses using substitution method.
- A small amount of "sloppiness" can be tolerated. Why?
- If careful when drawing out a recursion tree and summing the costs, can be used as direct proof.

Summing up Cost of Recursion Trees

- Evaluate:
  - Cost of individual node at depth *i*
  - Number of nodes at depth *i*
  - Total height of tree

Recursion Tree for Merge Sort

Continue expanding until the problem size reduces to 1.



- Cost of node at depth $i = \frac{cn}{2^i}$
- Number of nodes at depth $i = 2^i$
- Depth of tree
  = # times can divide cn by 2^i until we get value of 1
  $= \lg n + 1$

- Putting together:

$$\sum_{i=0}^{\lg n+1} 2^i \frac{cn}{2^i} = \sum_{i=0}^{\lg n+1} cn$$

$$= \Theta(n \lg n)$$

Can also write out algebraically…

$$T(n) = cn + 2T\left(\frac{n}{2}\right)$$

$$= cn + 2\left(cn/2 + 2T\left(\frac{n}{4}\right)\right)$$

$$= cn + 2cn/2 + 2\left(2cn/4 + 2T\left(\frac{n}{8}\right)\right)$$

$$= \ldots$$

$$= \sum_{i=0}^{\lg n+1} 2^i \frac{cn}{2^i} = \sum_{i=0}^{\lg n+1} cn$$

$$= \Theta(n \lg n)$$

Example 2

- Formulate (and solve) recursion tree for:

$$T(n) = 2T(n-1) + c$$

Example #3

- Insertion sort can be expressed as a recursive procedure as follows:
  - In order to sort $A[1..n]$, we recursively sort $A[1..n-1]$ and then insert $A[n]$ into the sorted array $A[1..n-1]$. Write a recurrence for the running time of this recursive version of insertion sort.

Example #4

- Argue that the solution to the recurrence:

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + cn$$

where c is constant,

is $\Omega(n \lg n)$ by appealing to a recursion tree.

Recall 3 Methods for Solving Recurrence Relations

- **Solution Methods** (Chapter 4)
  - Substitution Method -- Today
  - Recursion-tree Method -- Today
  - Master Method -- Next time

Next Time: The Master Method

- Based on the Master theorem.
- "Cookbook" approach for solving recurrences of the form

  $T(n) = aT(n/b) + f(n)$

  - $a \geq 1$, $b > 1$ are constants.
  - $f(n)$ is asymptotically positive.
  - n/b may not be an integer, but we ignore floors and ceilings. Why?
- Requires memorization of three cases.

Reading Assignments

- Reading assignment for next class:
  - Chapter 4.2, 4.5