Today:

- Linear Programming (con’t.)

COSC 581, Algorithms

April 3, 2014

Many of these slides are adapted from several online sources

Reading Assignments

- Today's class:
  - Chapter 29.2
- Reading assignment for next Thursday's class:
  - Chapter 29.3-4

First, a bit of review…

The General LP Problem

maximize $c_1 x_1 + c_2 x_2 + \cdots + c_d x_d$ ← Linear objective function

subject to:

Linear constraints (stated as inequalities)

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1d} x_d \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2d} x_d \leq b_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nd} x_d \leq b_n$$

General Steps of LP

Step 1: Determine the decision variables

Step 2: Determine the objective function

Step 3: Determine the constraints

Step 4: Convert into standard or slack form

Step 5: Solve

Two Canonical Forms for LP: Standard and Slack

- An LP is in standard form if it is the maximization of a linear function subject to linear inequalities

- An LP is in slack form if it is the maximization of a linear function subject to linear equalities

Equivalence of Linear Programs

- Two maximization LPs, L and L', are equivalent if for each feasible solution $\mathbf{x}$ to L with objective value z there is a corresponding feasible solution $\mathbf{x}'$ to L' with objective value z, and vice versa.

- A maximization LP, L, and a minimization LP, L', are equivalent if for each feasible solution $\mathbf{x}$ to L with objective value z there is a corresponding feasible solution $\mathbf{x}'$ to L' with objective value $-z$, and vice versa.

Standard Form

- We're given:
  n real numbers $c_1, c_2, \ldots\ c_n$
  m real numbers $b_1, b_2, \ldots\ b_m$
  mn real numbers a_{ij}, for i = 1, 2, …, m and j = 1, 2, … n

- We want to find:
  n real numbers $x_1, x_2, \ldots\ x_n$ that:

Maximize: $\sum_{j=1}^{n} c_j x_j$
Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n$$

Compact Version of Standard Form

- Let: $A = (a_{ij})$ be $m \times n$ matrix

  $b = (b_i)$ be an m-vector

  $c = (c_j)$ be an n-vector

  $x = (x_j)$ be an n-vector

- Rewrite LP as:

  Maximize: $c^T x$

  Subject to:

  $$Ax \leq b$$
  $$x \geq 0$$

- Now, we can concisely specify LP in standard form as (A, b, c)

Slack Form – Useful for Simplex

- In slack form, the only inequality constraints are the non-negativity constraints
  - All other constraints are equality constraints
- Let:

  $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$

  be an inequality constraint

- Introduce new variable *s*, and rewrite as:

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$s \geq 0$$

- *s* is a slack variable; it represents difference between left-hand and right-hand sides

Slack Form (con't.)

- In general, we'll use x_{n+i} (instead of s) to denote the slack variable associated with the ith inequality.

- The ith constraint is therefore:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_i$$

along with the non-negativity constraint $x_{n+i} \geq 0$

Example

Standard form:

Maximize $2x_1 - 3x_2 + 3x_3$

subject to:

$$x_1 + x_2 - x_3 \leq 7$$
$$-x_1 - x_2 + x_3 \leq -7$$
$$x_1 - 2x_2 + 2x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

Slack form:

Maximize $2x_1 - 3x_2 + 3x_3$

subject to:

$$x_4 = 7 - x_1 - x_2 + x_3$$
$$x_5 = -7 + x_1 + x_2 - x_3$$
$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Concise Representation of Slack Form

- Can eliminate “maximize”, “subject to”, and non-negativity constraints (all are implicit)
- And, introduce *z* as value of objective function:

$$
\begin{aligned}
z &= 2x_1 - 3x_2 + 3x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3
\end{aligned}
$$

- Then, define slack form of LP as tuple (*N, B, A, b, c, v*)

  where *N* = indices of nonbasic variables

  B = indices of basic variables

- We can rewrite LP as:

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \text{ for } i \in B$$

Formatting problems as LPs

- Single Source Shortest Path :
  - Input: A weighted direct graph G=<V,E> with weighted function w: E$\rightarrow$R, a source s and a destination t, compute d which is the weight of the shortest path from s to t.
  - Formulate as a LP:
    - For each vertex v, introduce a variable d_v: the weight of the shortest path from s to v.
    - LP:

maximize d_t

subject to:

$d_v \leq d_u + w(u,v)$ for each edge $(u,v) \in E$

$d_s = 0$

Q: Why is this a maximization?

Q: How many variables?

Q: How many constraints?

Formatting problems as LPs

- Single Source Shortest Path :
  - Input: A weighted direct graph G=<V,E> with weighted function w: E$\rightarrow$R, a source s and a destination t, compute d which is the weight of the shortest path from s to t.
  - Formulate as a LP:
    - For each vertex v, introduce a variable d_v: the weight of the shortest path from s to v.
    - LP:

maximize d_t

subject to:

$d_v \leq d_u + w(u,v)$ for each edge $(u,v) \in E$

$d_s = 0$

Q: Why is this a maximization?

Q: How many variables? |V|

Q: How many constraints?

Formatting problems as LPs – SSSP

- Single Source Shortest Path :
  - Input: A weighted direct graph G=<V,E> with weighted function w: E→R, a source s and a destination t, compute d which is the weight of the shortest path from s to t.
  - Formulate as a LP:
    - For each vertex v, introduce a variable d_v: the weight of the shortest path from s to v.
    - LP:

maximize d_t

subject to:

$d_v \le d_u + w(u,v)$ for each edge $(u,v) \in E$

$d_s = 0$

Q: Why is this a maximization?

Q: How many variables? |V|

Q: How many constraints? |E|+1

Formatting problems as LPs – Max Flow

- Recall (how could you forget?) Max-flow problem:
  - A directed graph G=<V,E>, a capacity function on each edge $c(u,v) \geq 0$ and a source s and a sink t. A flow is a function f : $V \times V \rightarrow R$ that satisfies:
    - Capacity constraints: for all $u,v \in V$, $f(u,v) \leq c(u,v)$.
    - Skew symmetry: for all $u,v \in V$, $f(u,v) = -f(v,u)$.
    - Flow conservation: for all $u \in V - \{s,t\}$, $\sum_{v \in V} f(u,v) = 0$
  - Find a maximum flow from s to t.

Formatting Max-flow problem as LP

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$	for all $u, v \in V$	//capacity constraints
$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$	for all $u \in V - \{s,t\}$	//flow conservation
$f_{uv} \geq 0$	for all $u, v \in V$	//non-negativity constraints

Q: How many variables?

Q: How many constraints?

Formatting Max-flow problem as LP

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$	for all $u, v \in V$	//capacity constraints
$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$	for all $u \in V - \{s,t\}$	//flow conservation
$f_{uv} \geq 0$	for all $u, v \in V$	//non-negativity constraints

Q: How many variables? $|V|^2$

Q: How many constraints?

Formatting Max-flow problem as LP

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$ for all $u, v \in V$ //capacity constraints

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$ //flow conservation

$f_{uv} \geq 0$ for all $u, v \in V$ //non-negativity constraints

Q: How many variables? $|V|^2$

Q: How many constraints? $2|V|^2 + |V| - 2$

Lots of “standard” problems can be formulated as LPs

- Question:

  When do you use specialized algorithms (like Dijkstra for SSSP), and when do you use LP (like the LP formulation we just made for SSSP)?

Lots of “standard” problems can be formulated as LPs

- Question:

  When do you use specialized algorithms (like Dijkstra for SSSP), and when do you use LP (like the LP formulation we just made for SSSP)?

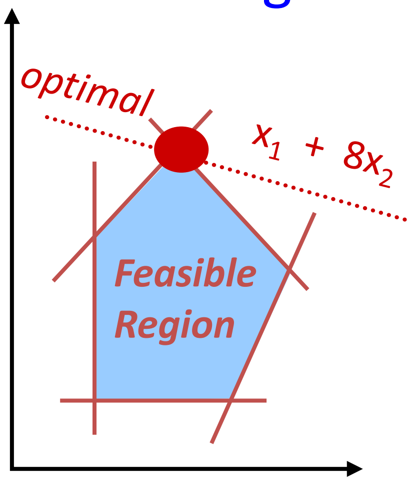
- Answer:
  - Specialized solutions often provide better runtime performance
  - But, when specialized solutions aren’t available, LP gives a “generic” approach applicable to many types of problems

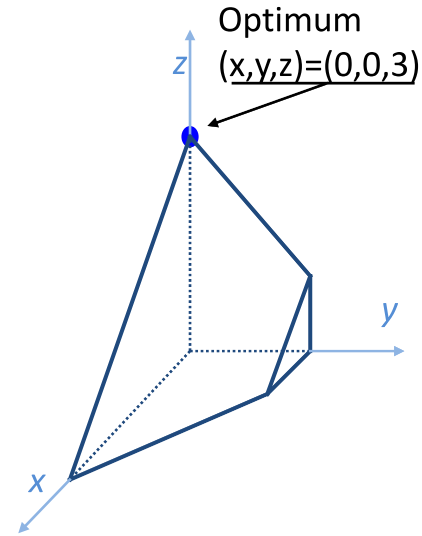
The Simplex algorithm for LP

- Classical method for solving LP problems
- Very simple
- Worst case run time is *not polynomial*
- But, often very fast in practice

Recall Important Observation: Optimal Solutions are at a Vertex or Line Segment

- Intersection of objective function and feasible region is either vertex or line segment





- Feasible region is *convex* – makes optimization much easier!
- Simplex algorithm finds LP solution by:
  - Starting at some vertex
  - Moving along edge of simplex to neighbor vertex whose value is at least as large
  - Terminates when it finds local maximum
- *Convexity ensures this local maximum is globally optimal*

Example for Simplex algorithm

Maximize $3x_1+x_2+2x_3$

Subject to:

$x_1+x_2+3x_3 \leq 30$

$2x_1+2x_2+5x_3 \leq 24$

$4x_1+x_2+2x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

Change to slack form:

$z= 3x_1+x_2+2x_3$

$x_4=30- x_1-x_2-3x_3$

$x_5=24- 2x_1-2x_2-5x_3$

$x_6=36- 4x_1-x_2-2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Recall, regarding Slack Form…

Slack form:

Maximize $2x_1 - 3x_2 + 3x_3$

subject to:

$$x_4 = 7 - x_1 - x_2 + x_3$$
$$x_5 = -7 + x_1 + x_2 - x_3$$
$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Basic variables – variables on left-hand side

Non-basic variables – variables on right-hand side

Simplex algorithm steps

$z = 3x_1 + x_2 + 2x_3$
$x_4 = 30 - x_1 - x_2 - 3x_3$
$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
$x_6 = 36 - 4x_1 - x_2 - 2x_3$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Recall: "Feasible solutions" (infinite number of them):
  - A feasible solution is any whose values satisfy constraints
  - In previous example, solution is feasible as long as all of x_1, x_2, x_3, x_4, x_5, x_6 are nonnegative
- Basic solution:
  - set all nonbasic variables to 0 and compute all basic variable values
- Iteratively rewrite the set of equations such that:
  - There is no change to the underlying LP problem (i.e., new form is equivalent to old)
  - Feasible solutions stay the same
  - The basic solution is changed, to result in a greater objective value:
    - Select a nonbasic variable x_e whose coefficient in the objective function is positive
    - Increase value of x_e as much as possible without violating any of the constraints
    - Make x_e a basic variable
    - Select some other variable to become nonbasic

Example

$z = 3x_1 + x_2 + 2x_3$
$x_4 = 30 - x_1 - x_2 - 3x_3$
$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
$x_6 = 36 - 4x_1 - x_2 - 2x_3$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Basic solution: $(x_1,x_2,x_3,x_4,x_5,x_6) = (0,0,0,30,24,36)$
  - The objective value is $z = 3 \cdot 0 + 0 + 2 \cdot 0 = 0$ (Not a maximum)

- Try to increase the value of nonbasic variable x_1 while maintaining constraints:

  Increase x_1 to 30: means that x_4 will be OK (i.e., non-negative)

  Increase x_1 to 12 means that x_5 will be OK 9:

  Increase x_1 to 9 means that x_6 will be OK.

  We have to choose most constraining value → x_1 is most constrained by x_6, so we switch the roles of x_1 and x_6

- Change x_1 to basic variable by rewriting last constraint to:

  $x_1 = 9 - x_2/4 - x_3/2 - x_6/4$

  - Note: x_6 becomes nonbasic.
  - Replace x_1 with above formula in all equations to get…

Example (con't.)

$z=27+x_2/4 + x_3/2 - 3x_6/4$
$x_1=9-x_2/4 - x_3/2 - x_6/4$
$x_4=21-3x_2/4 - 5x_3/2 + x_6/4$
$x_5=6-3x_2/2 - 4x_3 + x_6/2$

- This operation is called pivot
  - A pivot chooses a nonbasic variable, called entering variable, and a basic variable, called leaving variable, and changes their roles.
  - The pivot operation results in an equivalent LP.
  - Reality check: original solution (0,0,0,30,24,36) satisfies the new equations.

- In the example,
  - x_1 is entering variable, and x_6 is leaving variable.
  - x_2, x_3, x_6 are nonbasic, and x_1, x_4, x_5 becomes basic.
  - The basic solution for this new LP form is (9,0,0,21,6,0), with z=27.

    (Yippee → z = 27 is better than z = 0!)

Example (con't.)

$z=27+x_2/4 + x_3/2 - 3x_6/4$
$x_1=9-x_2/4 - x_3/2 - x_6/4$
$x_4=21-3x_2/4 - 5x_3/2 + x_6/4$
$x_5=6-3x_2/2 - 4x_3 + x_6/2$

- We iterate again –try to find a new variable whose value may increase.
  - x_6 will not work, since z will decrease.
  - x_2 and x_3 are OK. Suppose we select x_3.
- How far can we increase x_3?
  - First constraint limits it to 18
  - Second constraint limits it to 42/5
  - Third constraint limits it to 3/2 – most constraining ➔ swap roles of x_3 and x_5
- So rewrite last constraint to:

$$x_3=3/2 - 3x_2/8 - x_5/4 + x_6/8$$

- Replace x_3 with the above in all the equations to get…

Example (con't.)

- The new LP equations:
  - z=111/4+x_2/16 –x_5/8 - 11x_6/16
  - x_1=33/2- x_2/16 +x_5/8 - 5x_6/16
  - x_3=3/2-3x_2/8 –x_5/4+x_6/8
  - x_4=69/4+3x_2/16 +5x_5/8-x_6/16
- The basic solution is (33/4,0,3/2,69/4,0,0) with z=111/4.

- Now we can only increase x_2.
  - First constraint limits x_2 to 132
  - Second to 4
  - Third to ∞
- So rewrite second constraint to:

  x_2= 4 – 8x_3/3 – 2x_5/3 + x_6/3

- Replace in all equations to get…

Example (con't.)

- Rewritten LP equations:

  $z=28-x_3/6 - x_5/6-2x_6/3$

  $x_1=8+x_3/6 + x_5/6-x_6/3$

  $x_2=4-8x_3/3 - 2x_5/3+x_6/3$

  $x_4=18-x_3/2 + x_5/2$
- At this point, all coefficients in objective functions are negative.
- So, no further rewrite is possible.

- Means that we've found the optimal solution.
- The basic solution is (8,4,0,18,0,0) with objective value z=28.
- The original variables are x_1, x_2, x_3 , with values (8,4,0)

Next time…

- More details on the correctness and optimality of SIMPLEX

Reading Assignments

- Reading assignment for next Thursday's class:
  - Chapter 29.3-4