Today:
– Linear Programming (con’t.)

COSC 581, Algorithms

April 8, 2014

Many of these slides are adapted from several online sources

Reading Assignments

- Today's class:
  - Chapter 29.3, 29.5

- Reading assignment for next Thursday's class:
  - Chapter 29.4

Recall: Formatting problems as LPs – SSSP

- Single Source Shortest Path :
  - Input: A weighted direct graph G=<V,E> with weighted function w: $E \rightarrow R$, a source s and a destination t, compute d which is the weight of the shortest path from s to t.
  - Formulate as a LP:
    - For each vertex v, introduce a variable d_v: the weight of the shortest path from s to v.
    - LP:

maximize d_t

subject to:

$d_v \leq d_u + w(u,v)$ for each edge $(u,v) \in E$

$d_s = 0$

Q: Why is this a maximization?

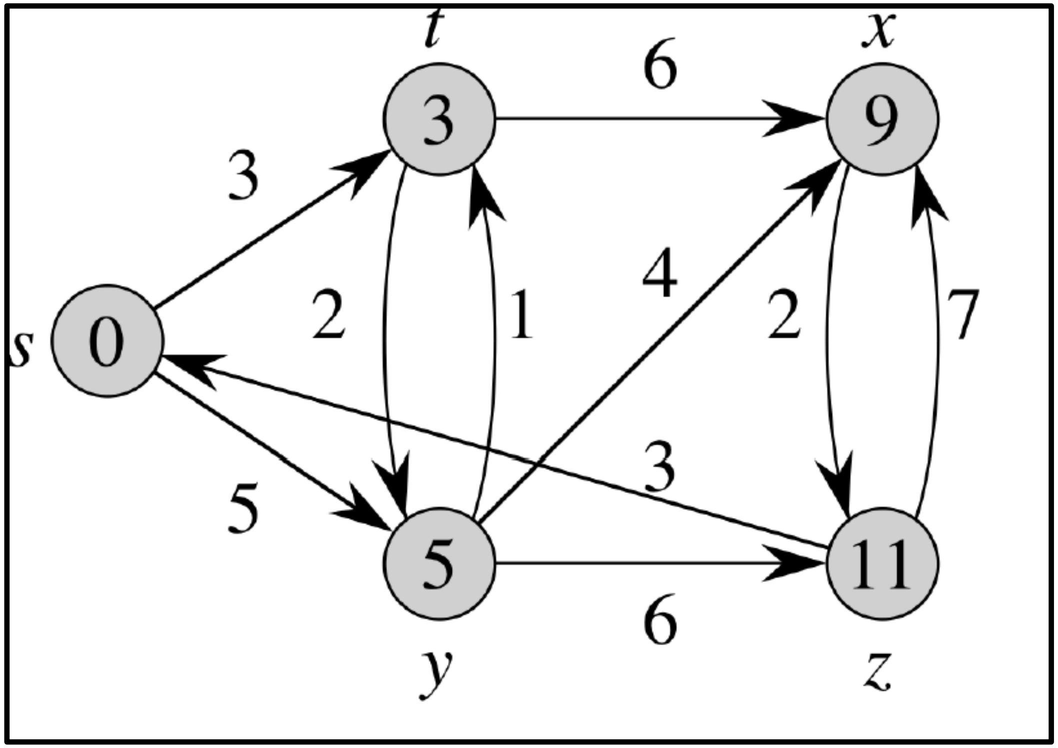
Q: How many variables? |V|

Q: How many constraints? |E|+1

In-Class Exercise #1

maximize d_t
subject to:
$d_v \leq d_u + w(u,v)$ for each edge $(u,v) \in E$
$d_s = 0$

Write out explicitly the linear program corresponding to finding the shortest path from node *s* to node *y* in the figure below:



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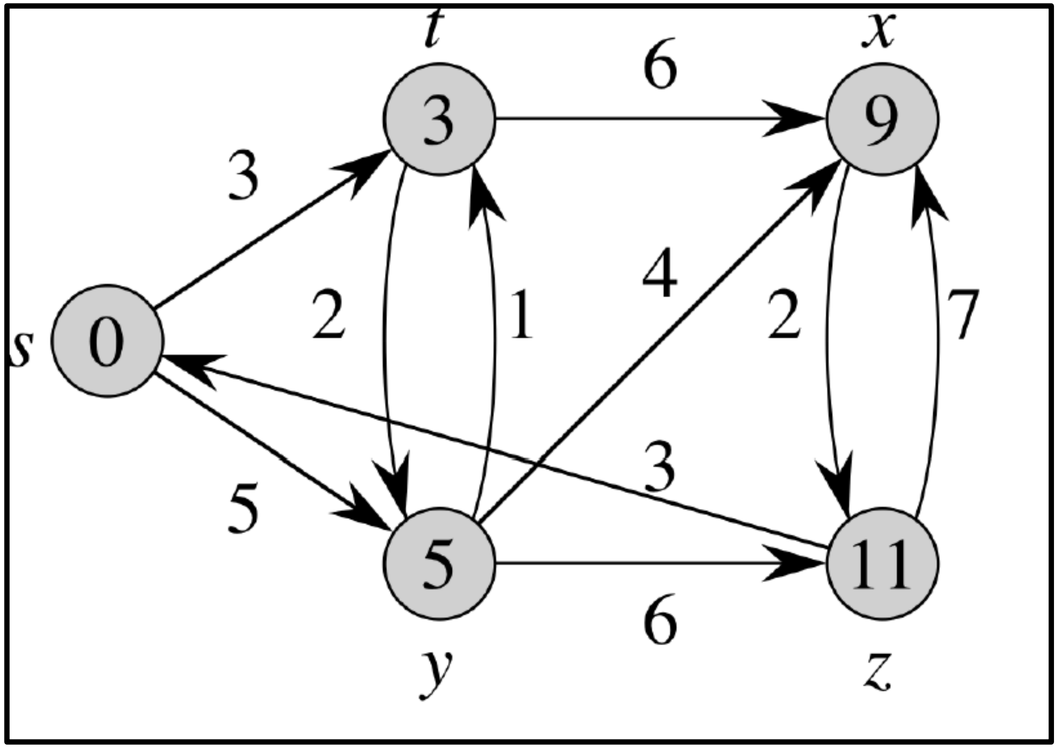


maximize d_y
subject to:

$$d_t \leq d_s + 3$$
$$d_y \leq d_s + 5$$
$$d_x \leq d_t + 6$$
$$d_y \leq d_t + 2$$
$$d_z \leq d_x + 2$$
$$d_t \leq d_y + 1$$
$$d_x \leq d_y + 4$$
$$d_z \leq d_y + 6$$
$$d_x \leq d_z + 7$$
$$d_s \leq d_z + 3$$
$$d_s = 0$$

Recall: Formatting Max-flow problem as LP

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to:

$f_{uv} \leq c(u,v)$ for all $u, v \in V$ //capacity constraints

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$ //flow conservation

$f_{uv} \geq 0$ for all $u, v \in V$ //non-negativity constraints

In-Class Exercise #2

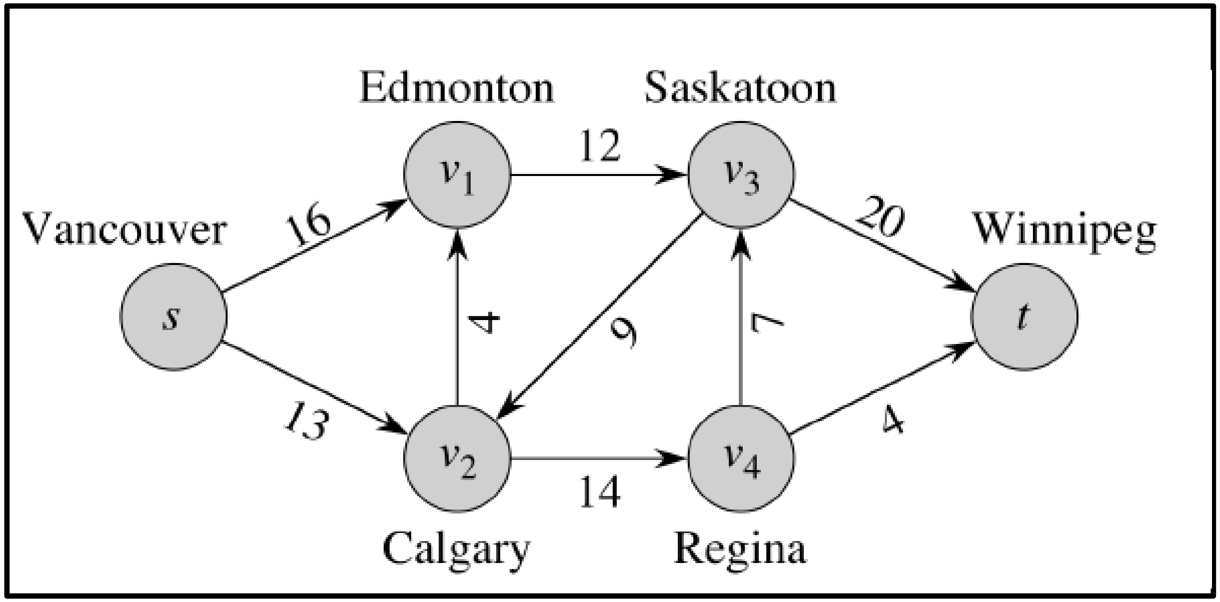
maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$
subject to:

$f_{uv} \leq c(u,v)$ for all $u, v \in V$

$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$

$f_{uv} \geq 0$ for all $u, v \in V$

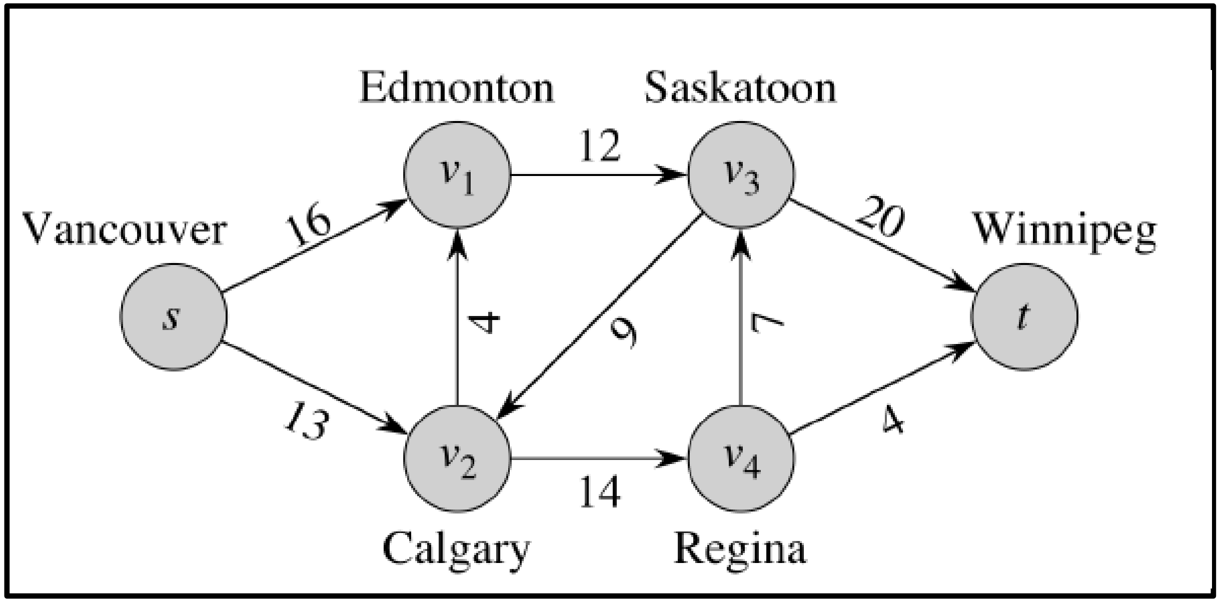
Write out explicitly the linear program corresponding to finding the maximum flow in the figure below:



In-Class Exercise #2

maximize $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$
subject to:
$f_{uv} \leq c(u,v)$ for all $u, v \in V$
$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}$ for all $u \in V - \{s,t\}$
$f_{uv} \geq 0$ for all $u, v \in V$

Write out explicitly the linear program corresponding to finding the maximum flow in the figure below:



maximize $f_{ss} + f_{sv_1} + f_{sv_2} + f_{sv_3} + f_{sv_4} + f_{st}$
subject to:

$f_{sv_1} \leq 16$	$f_{v_1v_3} \leq 12$	$f_{v_4v_3} \leq 7$	$f_{ss} \leq 0$	$f_{st} \leq 0$
$f_{sv_2} \leq 13$	$f_{v_3v_2} \leq 9$	$f_{v_4t} \leq 4$	$f_{sv_3} \leq 0$	$f_{v_1s} \leq 0$
$f_{v_2v_1} \leq 4$	$f_{v_2v_4} \leq 14$	$f_{v_3t} \leq 20$	$f_{sv_4} \leq 0$	$f_{v_2s} \leq 0$

…

Solving LPs using SIMPLEX…

- First, another recap (via example) to remember how SIMPLEX works…

Example for Simplex algorithm

Maximize $3x_1+x_2+2x_3$

Subject to:

$x_1+x_2+3x_3 \leq 30$

$2x_1+2x_2+5x_3 \leq 24$

$4x_1+x_2+2x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

Change to slack form:

$z= 3x_1+x_2+2x_3$

$x_4=30- x_1-x_2-3x_3$

$x_5=24- 2x_1-2x_2-5x_3$

$x_6=36- 4x_1-x_2-2x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Simplex algorithm steps

$z = 3x_1 + x_2 + 2x_3$
$x_4 = 30 - x_1 - x_2 - 3x_3$
$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
$x_6 = 36 - 4x_1 - x_2 - 2x_3$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Recall: "Feasible solutions" (infinite number of them):
  - A feasible solution is any whose values satisfy constraints
  - In previous example, solution is feasible as long as all of x_1, x_2, x_3, x_4, x_5, x_6 are nonnegative
- Basic solution:
  - set all nonbasic variables to 0 and compute all basic variable values
- Iteratively rewrite the set of equations such that:
  - There is no change to the underlying LP problem (i.e., new form is equivalent to old)
  - Feasible solutions stay the same
  - The basic solution is changed, to result in a greater objective value:
    - Select a nonbasic variable x_e whose coefficient in the objective function is positive
    - Increase value of x_e as much as possible without violating any of the constraints
    - Make x_e a basic variable
    - Select some other variable to become nonbasic

Example

$z = 3x_1 + x_2 + 2x_3$
$x_4 = 30 - x_1 - x_2 - 3x_3$
$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
$x_6 = 36 - 4x_1 - x_2 - 2x_3$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

- Basic solution: $(x_1,x_2,x_3,x_4,x_5,x_6) = (0,0,0,30,24,36)$
  - The objective value is $z = 3\cdot 0 + 0 + 2\cdot 0 = 0$ (Not a maximum)

- Try to increase the value of nonbasic variable x_1 while maintaining constraints:

  Increase x_1 to 30: means that x_4 will be OK (i.e., non-negative)

  Increase x_1 to 12 means that x_5 will be OK 9:

  Increase x_1 to 9 means that x_6 will be OK.

  We have to choose most constraining value ➔ x_1 is most constrained by x_6, so we switch the roles of x_1 and x_6

- Change x_1 to basic variable by rewriting last constraint to:

  $x_1 = 9 - x_2/4 - x_3/2 - x_6/4$

  - Note: x_6 becomes nonbasic.
  - Replace x_1 with above formula in all equations to get…

Example (con't.)

$z=27+x_2/4+x_3/2-3x_6/4$
$x_1=9-x_2/4-x_3/2-x_6/4$
$x_4=21-3x_2/4-5x_3/2+x_6/4$
$x_5=6-3x_2/2-4x_3+x_6/2$

- This operation is called pivot
  - A pivot chooses a nonbasic variable, called entering variable, and a basic variable, called leaving variable, and changes their roles.
  - The pivot operation results in an equivalent LP.
  - Reality check: original solution (0,0,0,30,24,36) satisfies the new equations.

- In the example,
  - x_1 is entering variable, and x_6 is leaving variable.
  - x_2, x_3, x_6 are nonbasic, and x_1, x_4, x_5 becomes basic.
  - The basic solution for this new LP form is (9,0,0,21,6,0), with z=27.

    (Yippee → z = 27 is better than z = 0!)

Example (con't.)

$z=27+x_2/4 + x_3/2 - 3x_6/4$
$x_1=9-x_2/4 - x_3/2 - x_6/4$
$x_4=21-3x_2/4 - 5x_3/2 + x_6/4$
$x_5=6-3x_2/2 - 4x_3 + x_6/2$

- We iterate again –try to find a new variable whose value may increase.
  - x_6 will not work, since z will decrease.
  - x_2 and x_3 are OK. Suppose we select x_3.
- How far can we increase x_3?
  - First constraint limits it to 18
  - Second constraint limits it to 42/5
  - Third constraint limits it to 3/2 – most constraining ➔ swap roles of x_3 and x_5
- So rewrite last constraint to:

$$x_3=3/2 - 3x_2/8 - x_5/4 + x_6/8$$

- Replace x_3 with the above in all the equations to get…

Example (con't.)

- The new LP equations:
  - $z=111/4+x_2/16 - x_5/8 - 11x_6/16$
  - $x_1=33/2- x_2/16 + x_5/8 - 5x_6/16$
  - $x_3=3/2-3x_2/8 - x_5/4+x_6/8$
  - $x_4=69/4+3x_2/16 +5x_5/8-x_6/16$
- The basic solution is (33/4,0,3/2,69/4,0,0) with z=111/4.

- Now we can only increase x_2.
  - First constraint limits x_2 to 132
  - Second to 4
  - Third to ∞
- So rewrite second constraint to:

$$x_2= 4 - 8x_3/3 - 2x_5/3 + x_6/3$$

- Replace in all equations to get…

Example (con't.)

- Rewritten LP equations:

  $z=28-x_3/6 - x_5/6-2x_6/3$

  $x_1=8+x_3/6 + x_5/6-x_6/3$

  $x_2=4-8x_3/3 - 2x_5/3+x_6/3$

  $x_4=18-x_3/2 + x_5/2$
- At this point, all coefficients in objective functions are negative.
- So, no further rewrite is possible.

- Means that we've found the optimal solution.
- The basic solution is (8,4,0,18,0,0) with objective value z=28.
- The original variables are x_1, x_2, x_3 , with values (8,4,0)

Simplex algorithm --Pivot

PIVOT(N, B, A, b, c, v, l, e)

▷ Compute the coefficients of the equation for new basic variable x_e.
$\widehat{b}_e \leftarrow b_l / a_{le}$
for each $j \in N - \{e\}$
    do $\widehat{a}_{ej} \leftarrow a_{lj} / a_{le}$
$\widehat{a}_{el} \leftarrow 1/a_{le}$
▷ Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    do $\widehat{b}_i \leftarrow b_i - a_{ie}\widehat{b}_e$
        for each $j \in N - \{e\}$
            do $\widehat{a}_{ij} \leftarrow a_{ij} - a_{ie}\widehat{a}_{ej}$
        $\widehat{a}_{il} \leftarrow -a_{ie}\widehat{a}_{el}$
▷ Compute the objective function.
$\widehat{v} \leftarrow v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    do $\widehat{c}_j \leftarrow c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l \leftarrow -c_e \widehat{a}_{el}$
▷ Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

N: indices set of nonbasic variables
B: indices set of basic variables
A: a_{ij}
b: b_i
c: c_i
v: constant coefficient.
e: index of entering variable
l: index of leaving variable

$z = v + \sum_{j \in N} c_j x_j$
$x_i = b_i - \sum_{j \in N} a_{ij} x_j$ for $i \in B$

Issues in Solving LP

- How to determine if LP is feasible?
- What if LP is feasible, but initial basic solution is not feasible?
  - Presume we have procedure, INITIALIZE-SIMPLEX, that takes LP in standard form and returns slack form for which initial basic solution is feasible (or states that the problem is infeasible)
- How to determine whether LP is unbounded?
  - If none of the constraints limits the amount by which the entering variable can increase, the LP is unbounded
- How to choose entering and leaving variables?
  - By selecting variable that limits entering variables the most
  - Break ties using *Bland's rule,* which always chooses variable with smallest index

Formal Simplex algorithm

SIMPLEX(A, b, c)

$(N, B, A, b, c, v) \leftarrow$ INITIALIZE-SIMPLEX(A, b, c)
while some index $j \in N$ has $c_j > 0$
    do choose an index $e \in N$ for which $c_e > 0$
        for each index $i \in B$
            do if $a_{ie} > 0$
                then $\Delta_i \leftarrow b_i / a_{ie}$
                else $\Delta_i \leftarrow \infty$
        choose an index $l \in B$ that minimizes Δ_i
        if $\Delta_l = \infty$
            then return "unbounded"
            else $(N, B, A, b, c, v) \leftarrow$ PIVOT(N, B, A, b, c, v, l, e)
for $i \leftarrow 1$ **to** n
    do if $i \in B_{+m}$
        then $\bar{x}_i \leftarrow b_i$
        else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Correctness of SIMPLEX

(Presume INITIALIZE-SIMPLEX is correct, for now.)

- First:
  - Show that if solution is returned, then that solution is feasible
  - Show that if SIMPLEX says “unbounded”, then the LP is indeed unbounded
- Sketch of this part of proof:
  - 3-part invariant (at the beginning of the while loop):
    - The slack form is equivalent to that returned by INITIALIZE-SIMPLEX
    - For each $i \in B, b_i \geq 0$
    - The basic solution associated with slack form is feasible
  - Show that this invariant is true:
    - At the beginning (easy to show)
    - During each iteration (show via correctness of pivot)
    - At termination (look at 2 cases of when SIMPLEX terminates, and show true for each case)

Correctness of SIMPLEX (con't.)

- Next, show that SIMPLEX does indeed terminate
- Reason why it might not terminate?
  - Cycling:
    - Would occur if SIMPLEX oscillates between solutions that leave objective value unchanged ("degeneracy")
- Helpful lemma:
  - The slack form of a LP is uniquely determined by the set of basic variables
    - Proof:
      - By contradiction. Assume there are 2 different slack forms, then work through the algebra to show that the 2 forms must be identical.

Correctness of SIMPLEX (con’t.)

- How to prevent cycling?
  - Break ties for choosing entering and leaving variables, using *Bland’s rule:*
    - Choose entering variable with smallest index (which also has positive coefficient in objective function)
    - After having chosen entering variable, if there are now ties for choosing leaving variable, chose the leaving variable with smallest index
    - Proof is tedious, so omitted here ☺

Running time of Simplex

- *Lemma:*
  - Assuming that INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that a linear program is unbounded, or it terminates with a feasible solution in at most $\binom{n+m}{m}$ iterations
    (where *n* = # non-basic variables and *m* = # basic variables)

- Idea:
  - There are at most $\binom{n+m}{m}$ ways to choose the basic variables.
  - The set of basic variables defines a unique slack from.
  - Thus, there at most $\binom{n+m}{m}$ unique slack forms.
  - If S SIMPLEX runs for more than $\binom{n+m}{m}$ iterations, it cycles.

    (Thus, need to ensure there isn't cycling. Can do this using *Bland's rule,* which always chooses variable with smallest index. Proof omitted)

How to find an initial basic feasible solution?

- A LP might be feasible, but the initial basic solution might not be feasible
- To address, formulate an auxiliary LP
- Given an LP in standard form, introduce new variable x_0 and formulate auxiliary LP as:

Maximize: $-x_0$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad \text{for } i = 1, 2, \dots, m$$

$$x_j \geq 0 \quad \text{for } j = 0, 1, \dots, n$$

- Then original LP is feasible iff the optimal objective value of auxiliary LP is 0.
- Proof is based on original solution and the fact that x_0 = 0 must be an optimal solution to the auxiliary LP.

Design of INITIALIZE-SIMPLEX

- Check original slack form; if feasible, then done
- Otherwise
  - Form auxiliary LP, as defined previously
  - Perform a single pivot of auxiliary LP, selecting leaving variable as that with most negative value
    - In this form, the basic solution is feasible
  - Repeatedly call PIVOT (i.e., while loop of SIMPLEX) to solve auxiliary LP
  - If solution to auxiliary LP is 0, then original LP is feasible
    - Rewrite the auxiliary LP, to eliminate x_0

Proof of correctness of INITIALIZE-SIMPLEX is based on algebraic argument, correctness of Pivot, etc.

Optimality of SIMPLEX

- Duality is a way to prove that a solution is optimal
- Can you think of an example of duality we've already seen this semester?
  - Max Flow, Min Cut

- This is an example of duality: given a maximization problem, we define a related minimization problem s.t. the two problems have the same optimal objective value

Duality in LP

- Given an LP, we'll show how to formulate a dual LP in which the objective is to minimize, and whose optimal value is identical to that of the original LP (now called primal LP)

Primal Dual LPs:

Primal:

maximize $c^T x$

subject to: $Ax \leq b$

$x \geq 0$

(standard form)

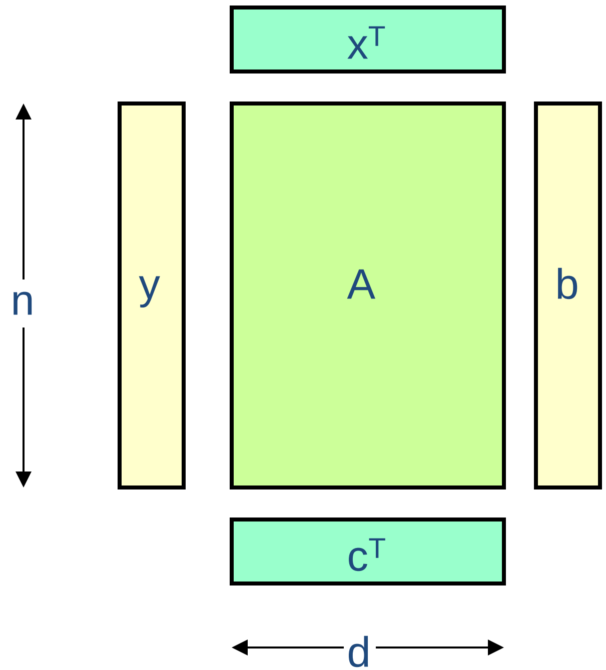
Dual:

minimize $y^T b$

subject to: $y^T A \geq c^T$

$y \geq 0$

(standard form)



Forming dual

- Change maximization to minimization
- Exchange roles of coefficients on RHSs and the objective function
- Replace each $\leq$ with $\geq$

- Each of the *m* constraints in primal has associated variable y_i in the dual
- Each of the *n* constraints in the dual as associated variable x_i in the primal

Example : Primal-Dual

PRIMAL:

$$\max \quad 16\,x_1 - 23\,x_2 + 43\,x_3 + 82\,x_4$$

subject to:

$$3\,x_1 + 6\,x_2 - 9\,x_3 + 4\,x_4 \leq 239$$
$$-9\,x_1 + 8\,x_2 + 17\,x_3 - 14\,x_4 = 582$$
$$5\,x_1 + 12\,x_2 + 21\,x_3 + 26\,x_4 \geq -364$$

$$x_1 \geq 0, \quad x_2 \leq 0, \quad x_4 \geq 0$$

DUAL:

$$\min \quad 239\,y_1 + 582\,y_2 - 364\,y_3$$

subject to:

$$3\,y_1 - 9\,y_2 + 5\,y_3 \geq 16$$
$$6\,y_1 + 8\,y_2 + 12\,y_3 \leq -23$$
$$-9\,y_1 + 17\,y_2 + 21\,y_3 = 43$$
$$4\,y_1 - 14\,y_2 + 26\,y_3 \geq 82$$

$$y_1 \geq 0, \quad y_3 \leq 0$$

Next time…

- We'll look at how to use dual to prove optimality

Reading Assignments

- Reading assignment for Thursday’s class:
  - Chapter 29.4