Today:
- Linear Programming (con't.)

COSC 581, Algorithms

April 10, 2014

Many of these slides are adapted from several online sources

Reading Assignments

- Today's class:
  - Chapter 29.4

- Reading assignment for next class:
  - Chapter 9.3 (Selection in Linear Time)
  - Chapter 34 (NP Completeness)

Optimality of SIMPLEX

- Duality is a way to prove that a solution is optimal

- Max-Flow, Min-Cut is an example of duality
- Duality: given a maximization problem, we define a related minimization problem s.t. the two problems have the same optimal objective value

Duality in LP

- Given an LP, we'll show how to formulate a dual LP in which the objective is to minimize, and whose optimal value is identical to that of the original LP (now called primal LP)

Primal Dual LPs:

Primal:

maximize $c^T x$

subject to: $Ax \leq b$

$x \geq 0$

(standard form)

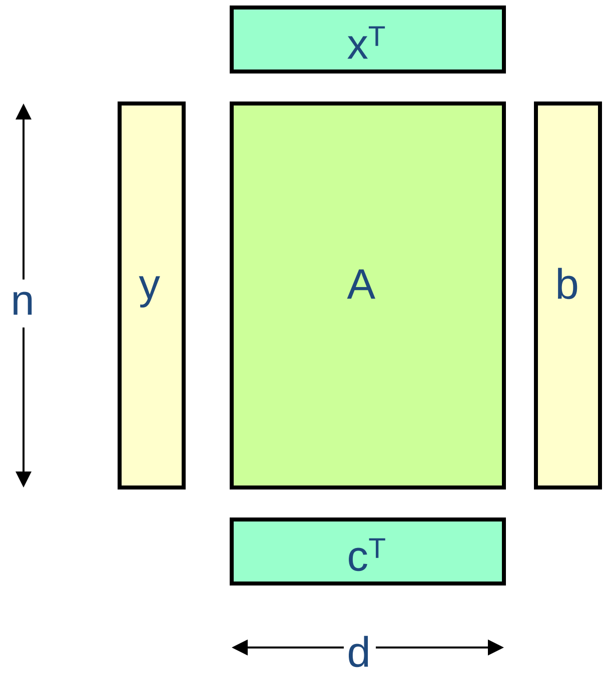
Dual:

minimize $y^T b$

subject to: $y^T A \geq c^T$

$y \geq 0$

(standard form)



Forming dual

- Change maximization to minimization
- Exchange roles of coefficients on RHSs and the objective function
- Replace each $\leq$ with $\geq$

- Each of the *m* constraints in primal has associated variable y_i in the dual
- Each of the *n* constraints in the dual as associated variable x_i in the primal

Example : Primal-Dual

PRIMAL:

$$\max \quad 16x_1 - 23x_2 + 43x_3 + 82x_4$$

subject to:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 + 4x_4 &\leq 239 \\ -9x_1 + 8x_2 + 17x_3 - 14x_4 &= 582 \\ 5x_1 + 12x_2 + 21x_3 + 26x_4 &\geq -364 \end{aligned}$$

$x_1 \geq 0, \quad x_2 \leq 0, \quad x_4 \geq 0$

DUAL:

$$\min \quad 239y_1 + 582y_2 - 364y_3$$

subject to:

$$\begin{aligned} 3y_1 - 9y_2 + 5y_3 &\geq 16 \\ 6y_1 + 8y_2 + 12y_3 &\leq -23 \\ -9y_1 + 17y_2 + 21y_3 &= 43 \\ 4y_1 - 14y_2 + 26y_3 &\geq 82 \end{aligned}$$

$y_1 \geq 0, \quad y_3 \leq 0$

Think about bounding optimal solution…

$$\min \quad 7x_1 + x_2 + 5x_3$$

$$x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

$$x_3 \geq 1$$

$$-x_2 \geq -1$$

$$x_1, x_2 \geq 0$$

Is optimal solution ≤ 30?

Yes, consider (2,1,3)

Think about bounding optimal solution…

$$\min \quad 7x_1 + x_2 + 5x_3$$

$$x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

$$x_3 \geq 1$$

$$-x_2 \geq -1$$

$$x_1, x_2 \geq 0$$

Is optimal solution ≥ 5?

Yes, because x3 ≥ 1.

Is optimal solution ≥ 6?

Yes, because 5x1 + x2 ≥ 6.

Is optimal solution ≥ 16?

Yes, because 6x1 + x2 +2x3 ≥ 16.

Strategy for bounding solution?

$$\min \quad 7x_1 + x_2 + 5x_3$$

$$x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

$$x_3 \geq 1$$

$$-x_2 \geq -1$$

$$x_1, x_2 \geq 0$$

What is the strategy we're using to prove lower bounds?

Take a linear combination of constraints!

Strategy for bounding solution?

min $\quad 7x_1 + x_2 + 5x_3$

y_1: $x_1 - x_2 + 3x_3 \geq 10$

y_2: $5x_1 + 2x_2 - x_3 \geq 6$

y_3: $x_3 \geq 1$

y_4: $-x_2 \geq -1$

$x_1, x_2 \geq 0$

max $\quad 10y_1 + 6y_2 + y_3 - y_4$

$y_1 + 5y_2 \leq 7$

$-y_1 + 2y_2 - y_4 \leq 1$

$3y_1 - y_2 + y_3 \leq 5$

$y_1, y_2, y_3, y_4 \geq 0$

Don't reverse inequalities.

$$x = \left(\frac{7}{4}, 0, \frac{11}{4}\right) \quad y = (2, 1, 0, 0)$$

What's the objective??

To maximize the lower bound.

Optimal solution = 26

Note: Use of primal as *minimization*

- Just to show you something a bit different from the text, the following discussion assumes the primal is a minimization problem, and thus the dual is a maximization problem

- Doesn't change the meaning (compared to text)

Primal-Dual Programs

$$\min \sum_{j=1}^{n} c_j x_j$$

y_i →

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_j$$

$$x_j \geq 0$$

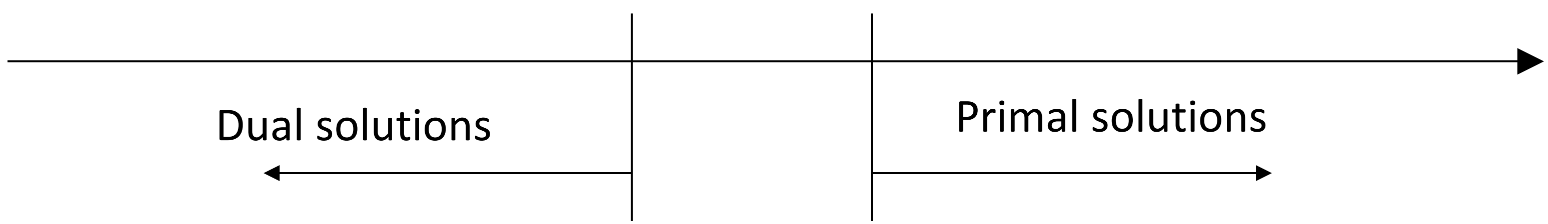
Primal Program

$$\max \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} y_i a_{ij} \leq c_j$$

$$y_i \geq 0$$

Dual Program



Weak Duality

Primal	**Dual**
$\min \sum_{j=1}^{n} c_j x_j$	$\max \sum_{i=1}^{m} b_i y_i$
$\sum_{j=1}^{n} a_{ij} x_j \geq b_j$	$\sum_{i=1}^{m} y_i a_{ij} \leq c_j$
$x_j \geq 0$	$y_i \geq 0$

Theorem

If x and y are feasible primal and dual solutions, then any solution to the primal has a value no less than any feasible solution to dual.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof

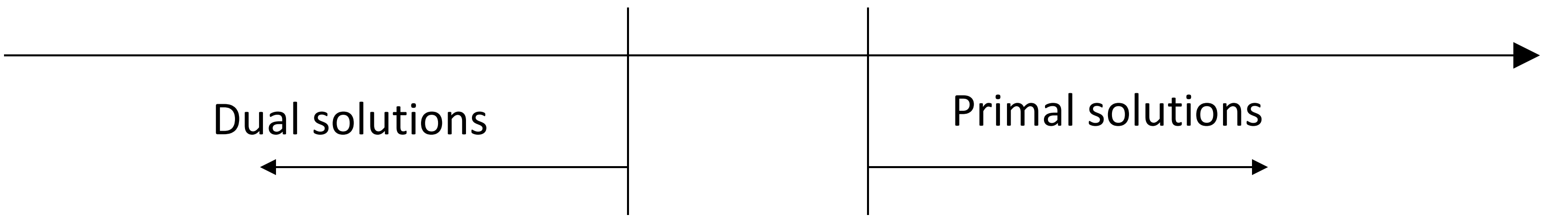
$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \Big(\underline{\sum_{i=1}^{m} a_{ij} y_i} \Big) x_j$$

$$= \sum_{i=1}^{m} \Big(\underline{\sum_{j=1}^{n} a_{ij} x_j} \Big) y_i \geq \sum_{i=1}^{m} b_i y_i$$

Primal Dual Programs

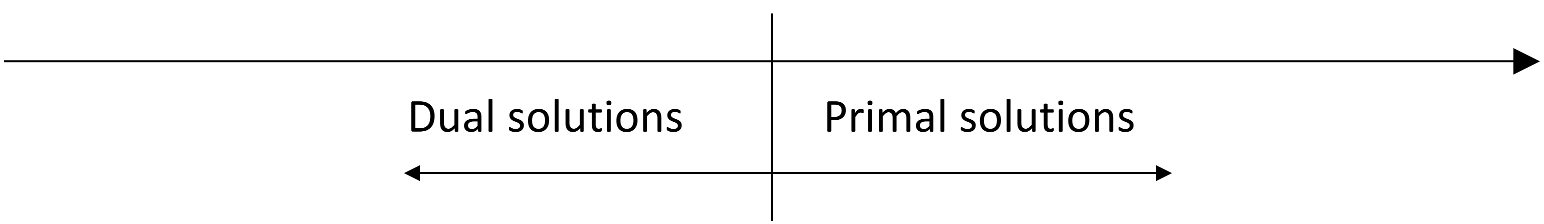
Primal Program

Dual Program



Von Neumann [1947]

Primal optimal = Dual optimal



Strong Duality – Prove that if primal solution = dual solution, then the solution is optimal for both

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j$$

$$\min \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} y_i a_{ij} = c_j$$

$$y_i \geq 0$$

PROVE:

$$\max \sum_{j=1}^{n} c_j x_j = \min \sum_{i=1}^{m} b_i y_i$$

Farka’s Lemma

- Exactly one of the following is solvable:

$$Ax \leq 0$$
$$c^{\mathrm{T}}x > 0$$

and:

$$A^{\mathrm{T}}y = c$$
$$y \geq 0$$

where:

– x and c are n-vectors

– y is an m-vector

– A is $m \times n$ matrix

Fundamental Theorem on Linear Inequalities

Let $a_1, a_2, \ldots, a_m, b$ be vectors in n-dimensional space. Then either one of the following happens:

(1) b is a nonnegative linear combination of linearly independent vectors from $a_1, \ldots, a_m$.

(2) There exists a hyperplane $\{x | cx = 0\}$, containing $t-1$ linearly independent vectors from $a_1, a_2, \ldots, a_m$, such that $cb < 0$ and $ca_1, \ldots, ca_m \geq 0$, where $t = rank\{a_1, \ldots, a_m, b\}$.

Proof of Fundamental Theorem

(i) Write $b = \lambda_{i_1} a_{i_1} + \ldots + \lambda_{i_n} a_{i_n}$. If $\lambda_{i_1}, \ldots, \lambda_{i_n} \geq 0$, we are in case 1.

(ii) Otherwise choose the smallest h among $i_1, \ldots, i_n$ with $\lambda_h < 0$. Let $\{x | cx = 0\}$ be the hyperplane spanned by $D \setminus \{a_h\}$ so that $cb = \lambda_h < 0$.

(iii) If $ca_1, \ldots, ca_m \geq 0$, then we are in case 2.

(iv) Otherwise choose the smallest s such that $ca_s < 0$. Then replace D by $(D \setminus \{a_h\}) \cup \{a_s\}$, and repeat.

Strong Duality

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j$$

$$\min \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} y_i a_{ij} = c_j$$

$$y_i \geq 0$$

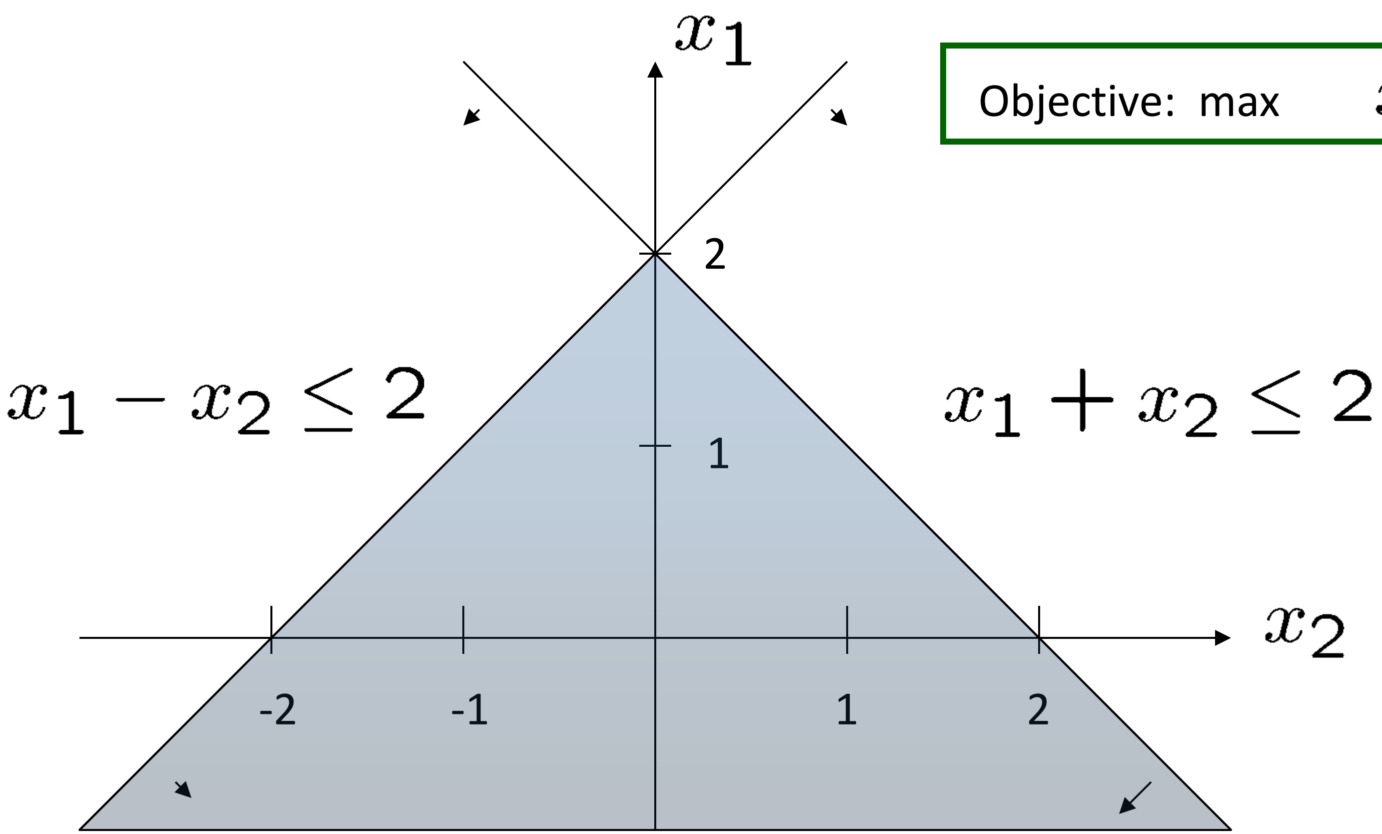
PROVE:

$$\max \sum_{j=1}^{n} c_j x_j = \min \sum_{i=1}^{m} b_i y_i$$

In other words, the optimal value for the primal is the optimal value for the dual.

Example

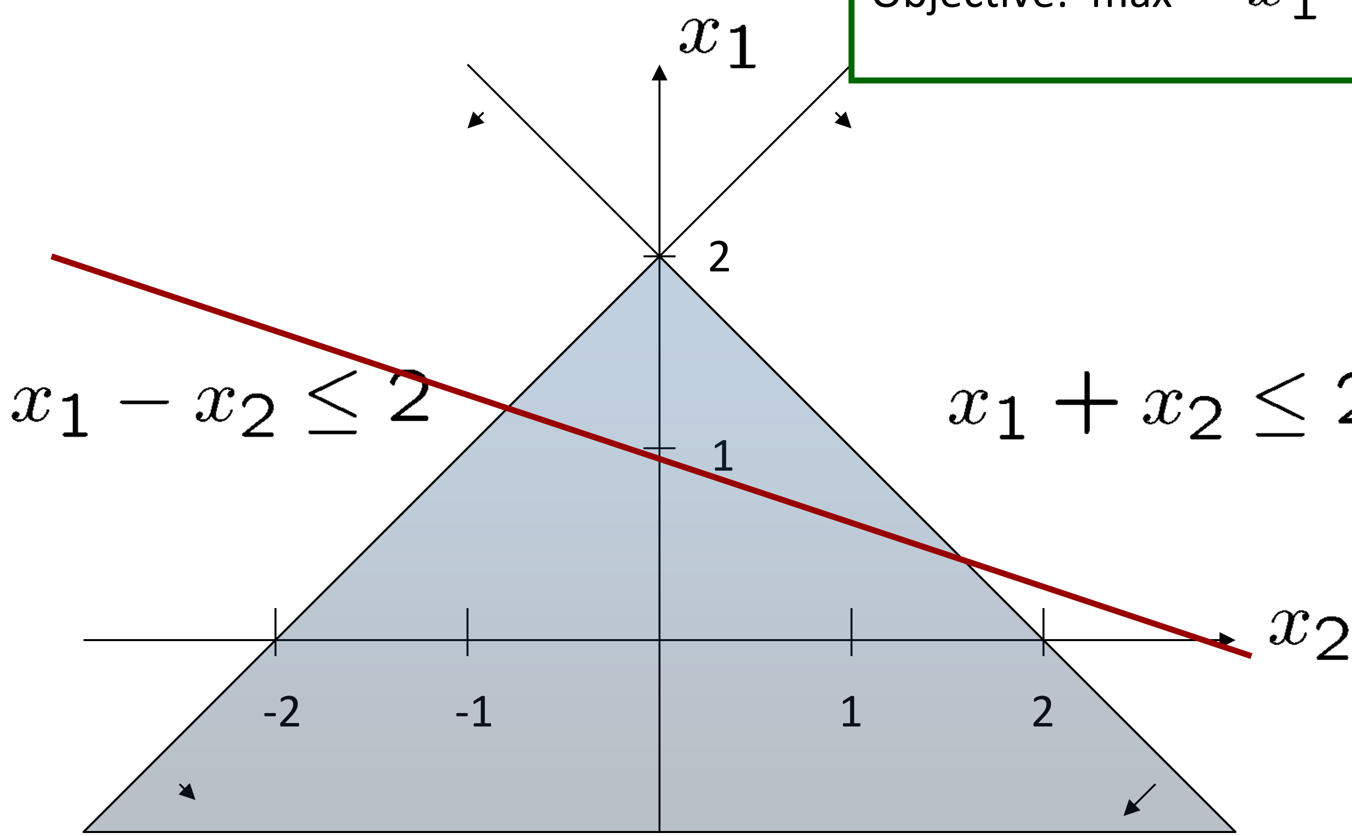
Objective: max x_1



$$x_1 = \frac{1}{2}(x_1 - x_2) + \frac{1}{2}(x_1 + x_2) \qquad 2 = \frac{1}{2}(2 + 2)$$

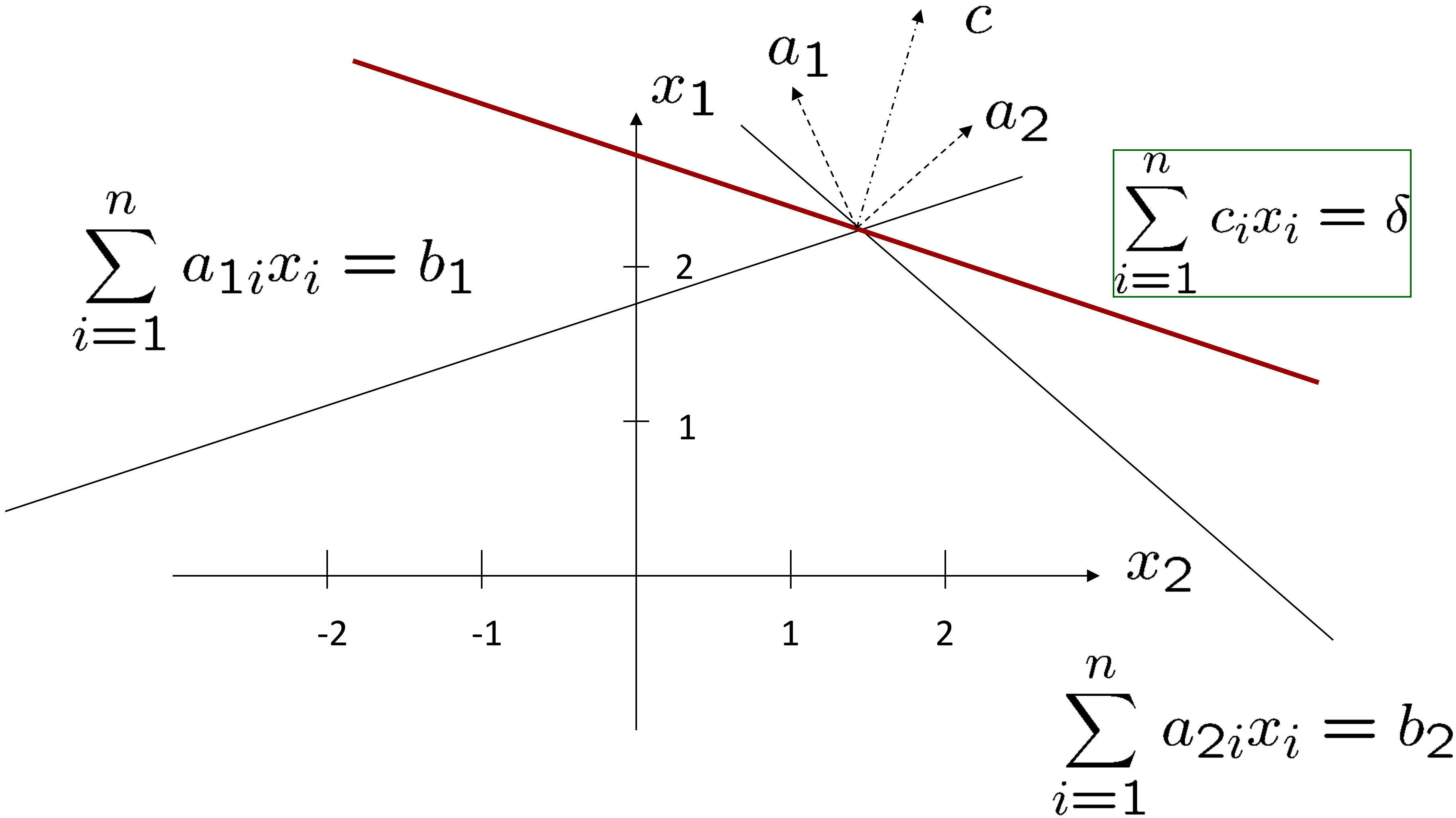
Example

Objective: max $x_1 + \frac{1}{3}x_2$

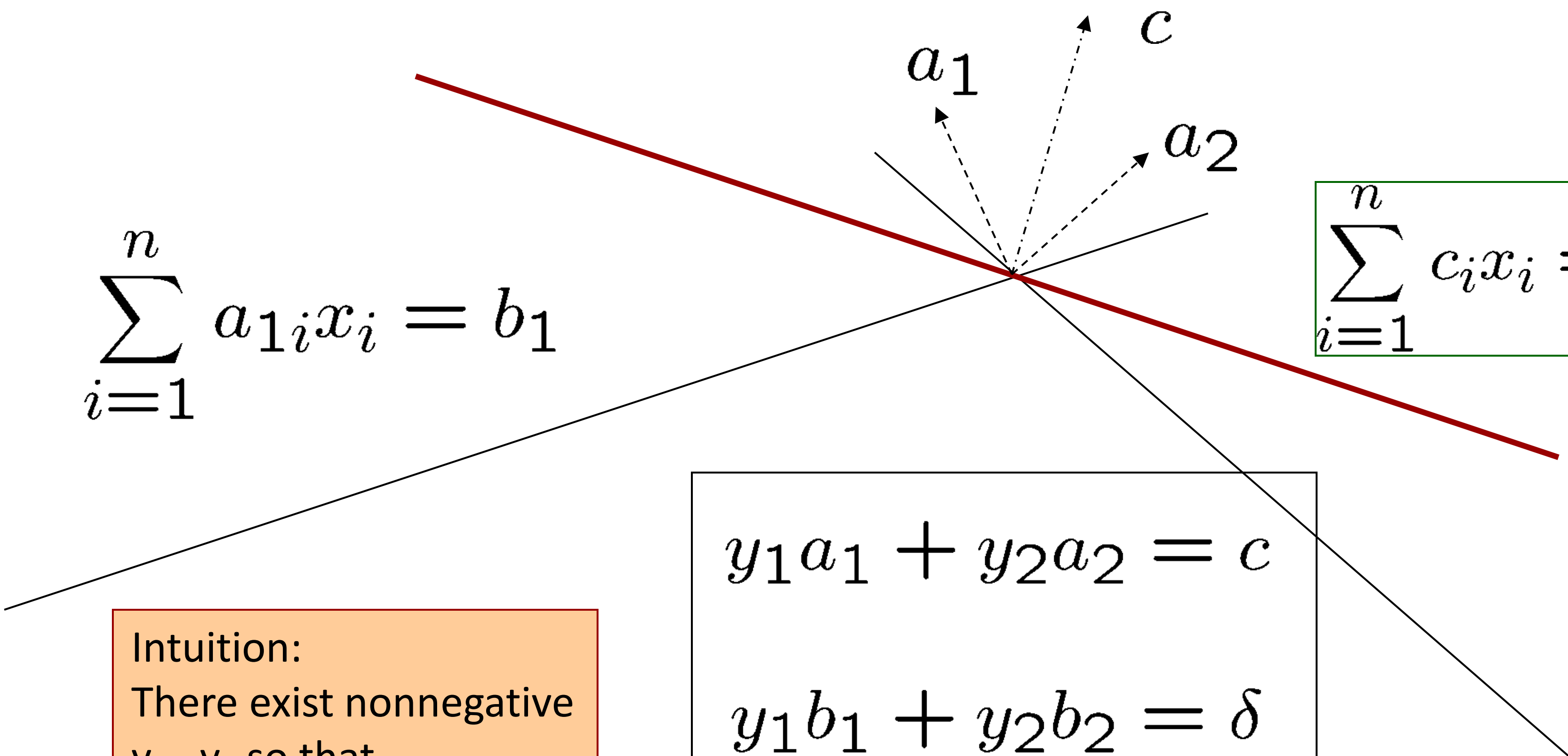


$$x_1 + \frac{1}{3}x_2 = \frac{1}{3}(x_1 - x_2) + \frac{2}{3}(x_1 + x_2) \quad 2 = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot 2$$

Geometric Intuition



Geometric Intuition



$$\sum_{i=1}^{n} a_{1i} x_i = b_1$$

$$\sum_{i=1}^{n} c_i x_i = \delta$$

$$y_1 a_1 + y_2 a_2 = c$$

$$y_1 b_1 + y_2 b_2 = \delta$$

$$\sum_{i=1}^{n} a_{2i} x_i = b_2$$

Intuition:
There exist nonnegative y_1 , y_2 so that

The vector c can be generated by a_1, a_2.

Y = (y_1, y_2) is the dual optimal solution!

Strong Duality

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j$$

$$\min \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} y_i a_{ij} = c_j$$

$$y_i \geq 0$$

$$y_1 a_1 + y_2 a_2 = c$$

$$y_1 b_1 + y_2 b_2 = \delta$$

Primal optimal value

Intuition: There exist y_1 , y_2 so that

Y = (y_1, y_2) is the dual optimal solution!

Here's another analogy: 2 Player Game

Column player

			
	0	-1	1
	1	0	-1
	-1	1	0

Row player

Strategy:
A probability distribution

Row player tries to maximize the payoff, column player tries to minimize

2 Player Game

Strategy:
A probability distribution

Column player

Row player

	A(i,j)	

Is it fair??

You have to decide your strategy first.

Von Neumann Minimax Theorem

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx = \min_{x \in \Delta^n} \max_{y \in \Delta^m} yAx$$

Strategy set

Which player decides first doesn't matter!

Key Observation

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx$$

If the row player fixes his strategy,

then we can assume that y chooses a **pure** strategy

$$\min_{x \in \Delta^n} yAx$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0$$

Vertex solution
is of the form
(0,0,…,1,…0),
i.e. a pure strategy

Key Observation

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx = \max_{y \in \Delta^m} \min_i (yA)_i$$

similarly

$$\min_{x \in \Delta^n} \max_{y \in \Delta^m} yAx = \min_{x \in \Delta^n} \max_j (Ax)_j$$

Primal-Dual Programs

$\max_{y \in \Delta^m} \min_i (yA)_i$	$\min_{x \in \Delta^n} \max_j (Ax)_j$
$\max \quad t$	$\min \quad w$
(x_j) $\sum_{i=1}^{m} y_i a_{ij} \geq t$	$\sum_{j=1}^{n} a_{ij} x_j \leq w$
(w) $\sum_{i=1}^{m} y_i = 1$	$\sum_{j=1}^{n} x_j = 1$
$y_i \geq 0$	$x_j \geq 0$

duality

Reading Assignments

- Reading assignment for next class:
  - Chapter 9.3 (Selection in Linear Time)
  - Chapter 34 (NP Completeness)