Today:
– NP-Completeness (con’t.)

COSC 581, Algorithms

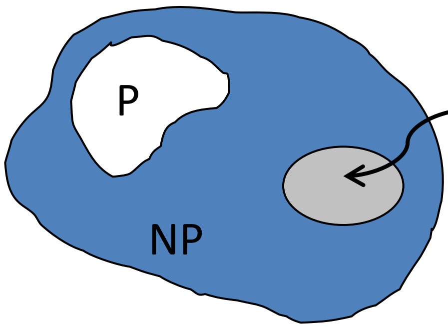
April 17, 2014

Many of these slides are adapted from several online sources

Reading Assignments

- Today's class:
  - Chapter 34 (con't.)

Recall: NP-Completeness



- A problem B is **NP-complete** if:

  (1) B $\in$ **NP**

  (2) A $\leq_p$ B for all A $\in$ **NP**

- If B satisfies only property (2) we say that B is **NP-hard**
- No polynomial time algorithm has been discovered for an **NP-Complete** problem
- No one has ever proven that no polynomial time algorithm can exist for any **NP-Complete** problem

- Significance: If one NP-Complete problem can be solved in poly-time, then all NP problems can be solved in poly-time

Recall: We always cast NP-Complete problems as decision problems

- **Decision problems**
  - Given an input and a question regarding a problem, determine if the answer is yes or no
- **Optimization problems**
  - Find a solution with the “best” value

- Interesting question:
  - Let’s presume that someone (amazingly) proves that P=NP.
  - But, all the NP-complete (NPC) problems are expressed as decision problems.
  - So, if P=NP, how can we make use of the poly-time algorithm that solves an NPC decision problem to also solve the optimization version of the same problem in poly-time?

Example: Using Poly-time alg. for decision problem to solve optimization problem in poly-time

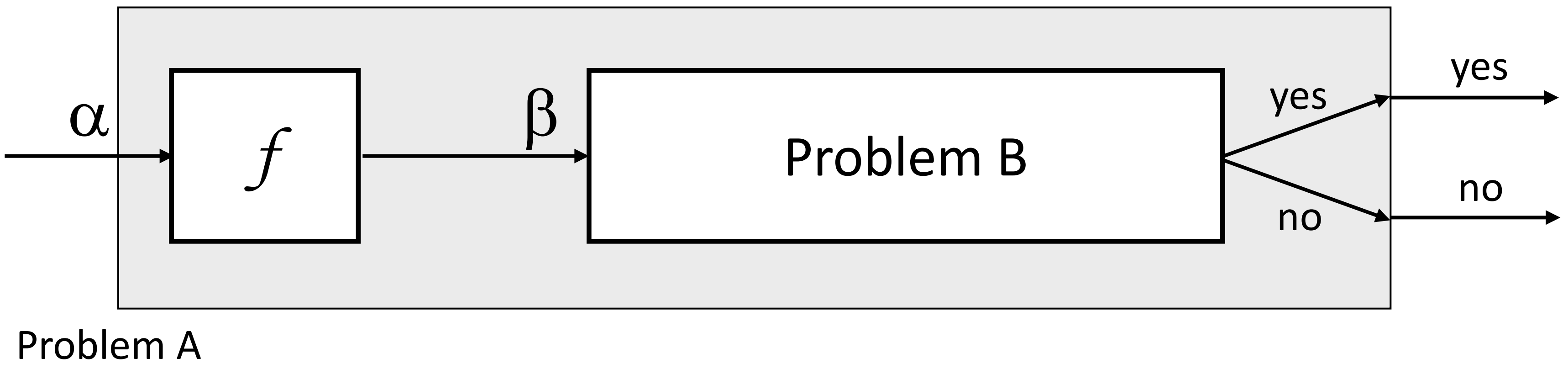
Example: Show that if P = NP, then there is a polynomial time algorithm that, given a Boolean formula ϕ, actually produces a satisfying assignment for ϕ (assuming ϕ is satisfiable).

Recall: Polynomial Reductions

- Reduction is a way of saying that one problem is **no harder** than another.
- We say that problem A is no harder than problem B,

  (i.e., we write **"A $\leq_p$ B"**)

  if we can solve A using the algorithm that solves B.
- **Idea:** transform the inputs of A to inputs of B in poly time



Recall: Polynomial Reductions

- Given two problems A, B, we say that A is polynomially **reducible** to B ($A \leq_p B$) if:

1. There exists a function f that converts the input of A to inputs of B in polynomial time
2. $A(\alpha) = YES \Leftrightarrow B(f(\alpha)) = YES$

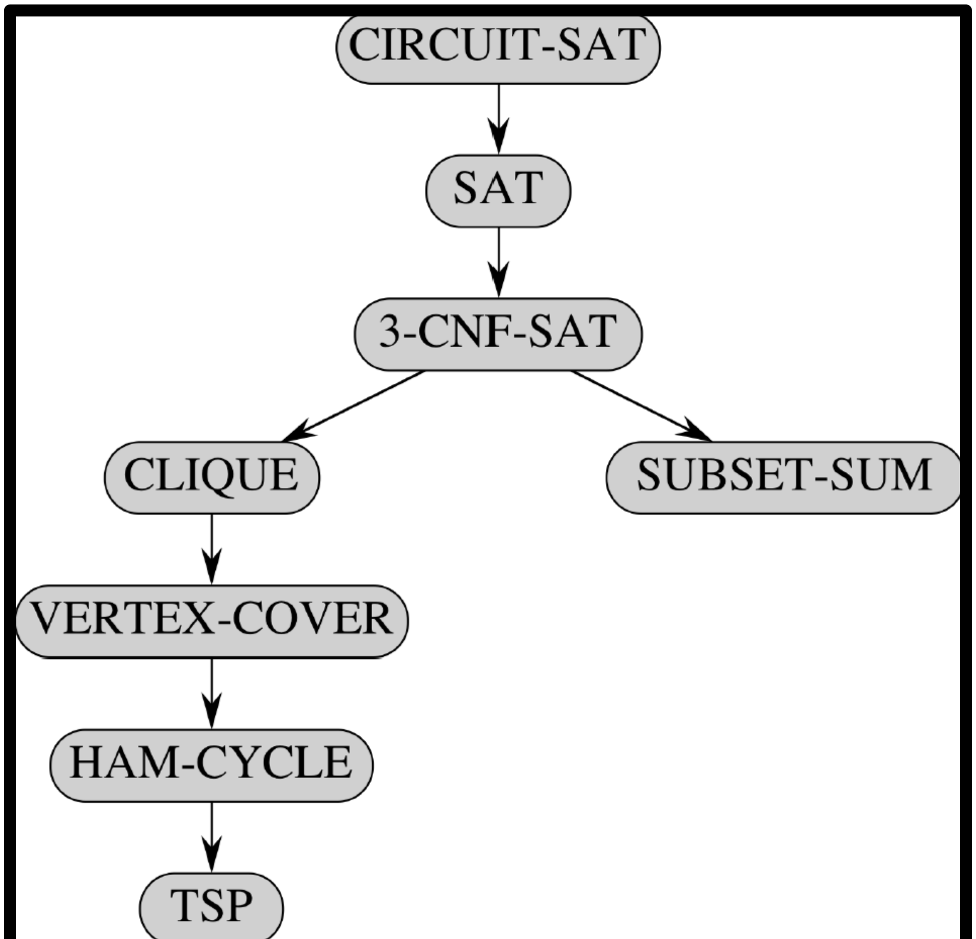
Proving NP-Completeness In Practice

1) Prove that the new problem $B \in NP$
2) Show that **one known** NP-Complete problem, A, can be transformed to B in polynomial time (i.e., $A \leq_p B$)

- Conclude that B is NP-Complete

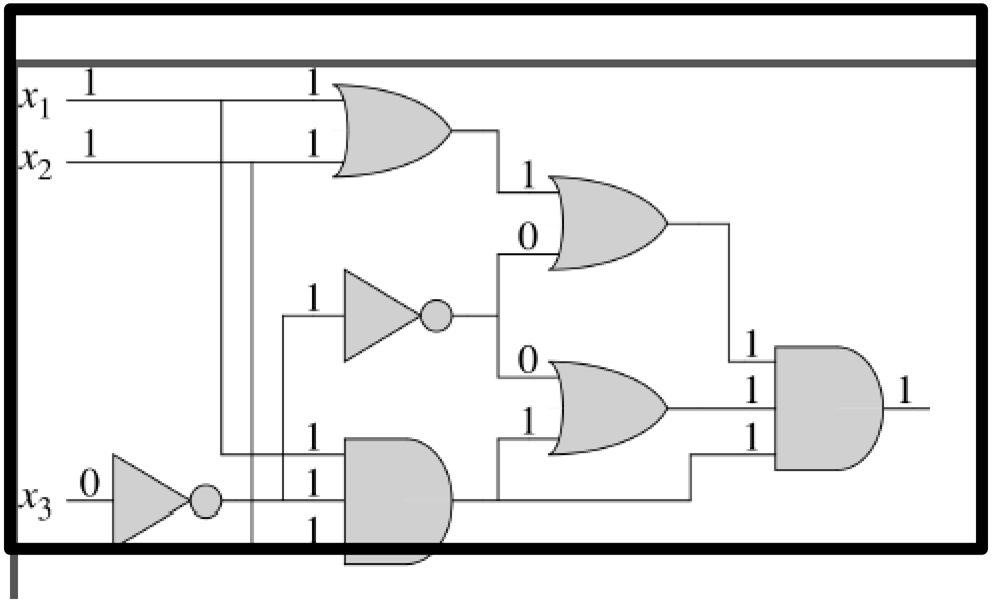
Once one problem (SAT) shown to be NP-Complete, can show many others…

Example reductions (From CLRS, Ch. 34):



NP-Complete Problem: Circuit-SAT

- *Circuit-SAT problem:* Given a boolean combinational circuit, C, determine if there is a satisfying assignment to inputs such that the circuit's output is 1.
- CLRS proves this is NP-Complete (more next week)



Example circuit with satisfying assignment

NP-Complete Problem:
Satisfiability (SAT)

- ***Satisfiability problem***: Given a logical expression ϕ, find an assignment of values (F, T) to variables x_i that causes ϕ to evaluate to T:

$$\phi = x_1 \vee \neg x_2 \wedge x_3 \vee \neg x_4$$

- SAT was the historically first problem shown to be NP-complete

- Here, we'll presume CIRCUIT-SAT is known NP-Complete (per CLRS), and prove SAT is NP-Complete by reduction

Prove SAT is NP-complete

- Step 1: SAT $\in$ NP
  - Argue that, given a certificate, you can verify that the certificate provides a solution in polynomial time
- Step 2: Show that some known NP-Complete problem is reducible in poly-time to SAT (i.e., A $\leq_p$ SAT)
  - What known NP-Complete problem do we choose?

Show Circuit-SAT $\leq_p$ SAT

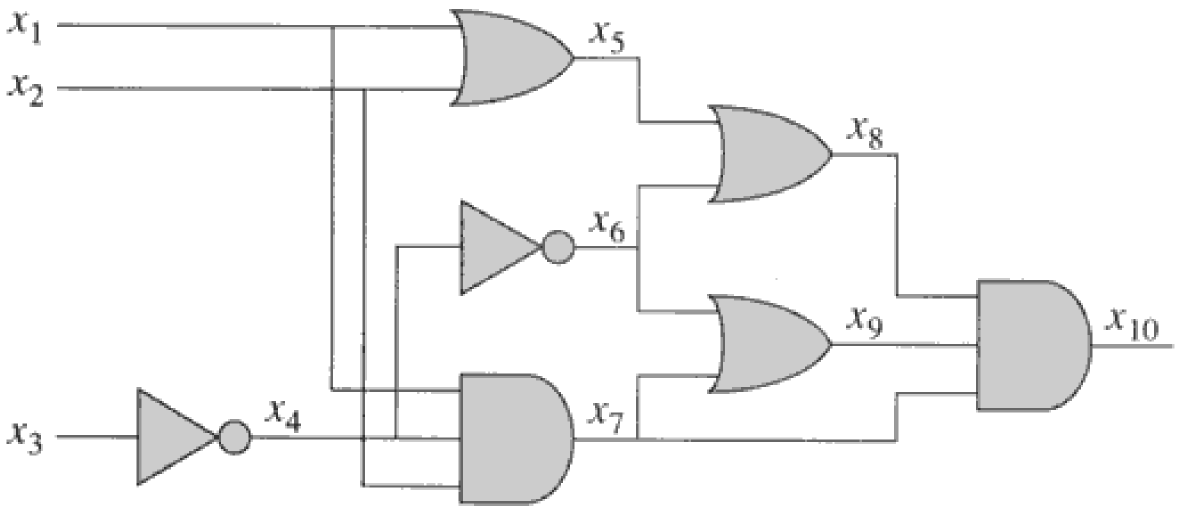
- What do we have to do?
  1) Given an instance <C> of Circuit-SAT, define poly-time function *f* that converts <C> to instance <ϕ> of SAT
  2) Argue that *f* is poly-time
  3) Argue that *f* is correct (i.e., <C> of Circuit-SAT is satisfiable iff < ϕ > of SAT is satisfiable)

- Here's a proposed poly-time reduction, *f*:
  - For every wire x_i of C, define a variable x_i in the formula.
  - Every gate can be expressed as:

    $x_i \leftrightarrow$ (*boolean operations consistent with gate)*
  - The final formula ϕ is the conjunction (AND) of the circuit output variable and conjunction of all clauses describing the operation of each gate.

Example of reduction of Circuit-SAT to SAT

Here's an input instance <C> if Circuit-SAT:



Here's the result of *f*(C), which gives instance < ϕ > of SAT:

$$
\begin{aligned}
\phi = x_{10} &\wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)) \\
&\wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\
&\wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\
&\wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\
&\wedge (x_6 \leftrightarrow \neg x_4)) \\
&\wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\
&\wedge (x_4 \leftrightarrow \neg x_3)
\end{aligned}
$$

Next, prove properties of f

- Argue that f is poly-time:
  - Obvious -- Clearly the reduction can be done in poly time
- Argue that f is correct:
  - C is satisfiable if and only if ϕ is satisfiable:
    - $\Rightarrow$ If C is satisfiable, then there is a satisfying assignment. This means that each wire of C has a well-defined value and the output of C is 1. Thus, the assignment of wire values to variables in ϕ makes each clause in ϕ evaluate to 1. So ϕ is 1 when C is satisfiable.
    - $\Leftarrow$ The reverse proof should also be done (i.e., if ϕ evaluates to 1, then C must be satisfiable); proof mirrors the argument above.
- Since (1) SAT $\in$ NP, and (2) Circuit-SAT $\leq_p$ SAT, we conclude that SAT is NP-Complete.

Important note about reductions…

- Note that we never make use of the *solution* to a problem in creating reduction function *f*

- In the proof of correctness, we mention the solution of one problem helping us to get the solution of the other (if such a solution were known), based on our construction

- But, this solution is not used for the construction defined by *f*. Why?
  - We don't know the solution, because finding it is an NP-complete problem.
  - Thus, our reduction function could not be polynomial-time if it required solving an NP-complete problem to create the construction.
  - The reduction function *f* must therefore work for any possible instance of the known NP-complete problem, but without knowledge of the solution.

Another NP-Complete Problem: CNF Satisfiability

- CNF is a special case of SAT
- ϕ is in "Conjuctive Normal Form" (CNF)
  - "AND" of expressions (i.e., clauses)
  - Each clause contains only "OR"s of the variables and their complements

E.g.: $\phi = (x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$

clauses

Another NP-Complete Problem: 3-CNF Satisfiability

3-CNF is a subcase of CNF problem:

- A *literal* in a boolean formula is an occurrence of a variable or its negation.
- CNF (Conjunctive Nornal Form) is a boolean formula expressed as AND of clauses, each of which is the OR of one or more literals.
- 3-CNF is a CNF in which each clause has exactly 3 distinct literals (a literal and its negation are distinct)

- Example:

$$\Phi = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

- Let us prove that **3-CNF** is NP-Complete

Prove 3-CNF-SAT is NP-complete

- Step 1: 3-CNF-SAT $\in$ NP
  - Argue that, given a certificate, you can verify that the certificate provides a solution in polynomial time

- Step 2: Show that some known NP-Complete problem is reducible in poly-time to 3-CNF-SAT (i.e., A $\leq_p$ 3-CNF-SAT)
  - What known NP-Complete problem do we choose?

SAT $\leq_p$ 3-CNF-SAT

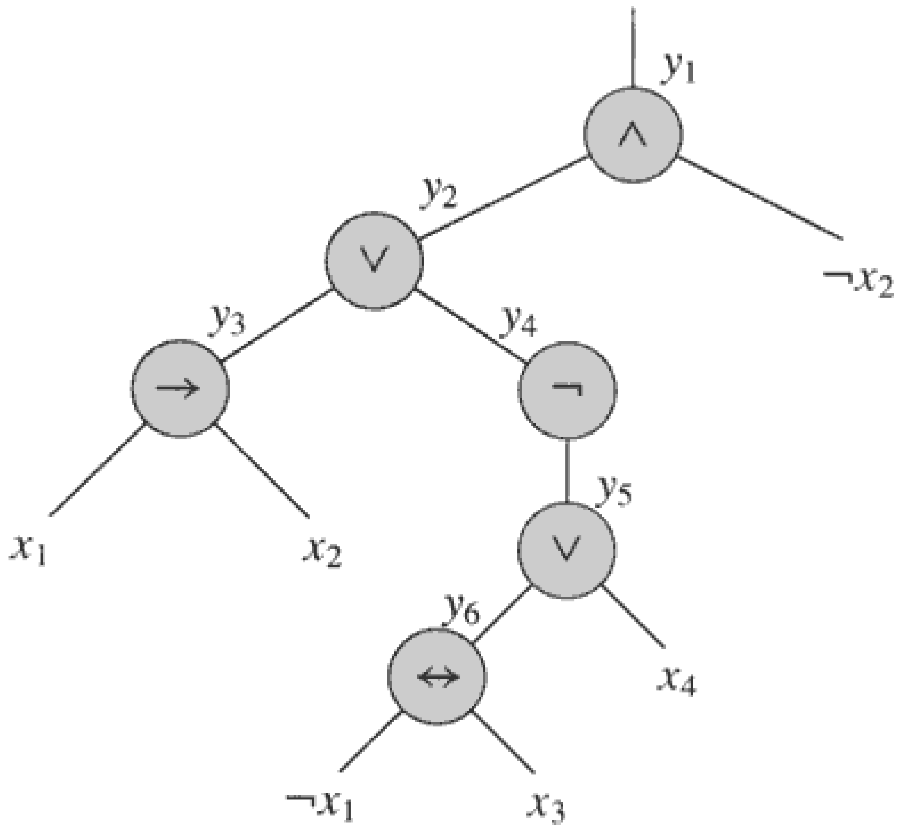
- What do we have to do?
  1) Given an instance < ϕ > of SAT, define poly-time function *f* that converts < ϕ > to instance < ϕ''' > of SAT
  2) Argue that *f* is poly-time
  3) Argue that *f* is correct (i.e., < ϕ > of SAT is satisfiable iff < ϕ''' > of 3-CNF-SAT is satisfiable)

SAT $\leq_p$ 3-CNF-SAT

- Proposed definition of *f:*
  - Suppose ϕ is any boolean formula, Construct a binary parse tree, with literals as leaves and connectives as internal nodes.
  - Introduce a variable y_i for the output of each internal nodes.
  - Rewrite the formula to ϕ' as the AND of the root variable and a conjunction of clauses describing the operation of each node.
  - The result is that in ϕ', each clause has at most three literals.
  - Change each clause into conjunctive normal form as follows:
    - Construct a truth table
    - Write the disjunctive normal form for all truth-table items evaluating to 0
    - Using DeMorgans law to change to CNF.
  - The resulting ϕ'' is in CNF but each clause has 3 or fewer literals.
  - Change 1 or 2-literal clauses into a 3-literal clause ϕ''' as follows:
    - If a clause has one literal *l*, change it to $(l \vee p \vee q) \wedge (l \vee p \vee \neg q) \wedge (l \vee \neg p \vee q) \wedge (l \vee \neg p \vee \neg q)$.
    - If a clause has two literals $(l_1 \vee l_2)$, change it to $(l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$.

Example: Binary parse tree for:

$$\phi=((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$



$$\begin{aligned}\phi' = y_1 &\wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2))\\ &\wedge (y_2 \leftrightarrow (y_3 \vee y_4))\\ &\wedge (y_4 \leftrightarrow \neg y_5)\\ &\wedge (y_3 \leftrightarrow (x_1 \rightarrow x_2))\\ &\wedge (y_5 \leftrightarrow (y_6 \vee x_4))\\ &\wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))\end{aligned}$$

Figure 34.11 The tree corresponding to the formula $\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$.

Example of Converting a 3-literal clause into CNF format

y_1	y_2	x_2	$(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

Disjunctive Normal Form:

$$\phi_i' = (y_1 \wedge y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge \neg x_2) \vee (\neg y_1 \wedge y_2 \wedge \neg x_2)$$

Conjunctive Normal Form:

$$\phi_i'' = (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)$$

Figure 34.12 The truth table for the clause $(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$.

3-CNF-SAT is NP-complete

Now, prove correctness of *f:*

- First, prove reduction is poly time:
  - From ϕ to ϕ', we introduce at most 1 variable and 1 clause per connective in ϕ.
  - From ϕ' to ϕ'', we introduce at most 8 clauses for each clause in ϕ'.
  - From ϕ'' to final 3-CNF, we introduce at most 4 clauses for each clause in ϕ''.
- Then, prove reduction is correct – i.e., ϕ and resulting 3-CNF formula are equivalent:
  - From ϕ to ϕ', keep equivalence by construction.
  - From ϕ' to ϕ'', keep equivalence by construction.
  - From ϕ'' to final 3-CNF, keep equivalence by construction.

Since: (1) 3-CNF-SAT $\in$ NP, and (2) SAT $\leq_p$ 3-CNF-SAT, we conclude that 3-CNF-SAT is NP-Complete.

Another NP-Complete Problem: Clique

Clique Problem:

- Given: undirected graph G = (V, E)
- **Clique:** a subset of vertices in V' $\subseteq$ V, each pair of which is connected by an edge in E, i.e., a clique is a complete subgraph of G.
- **Size of a clique:** number of vertices it contains

Optimization problem:

- Find a clique of maximum size

Decision problem:

- Does G have a clique of size k?
- Input instance = <G,k>

Prove Clique is NP-complete

- Step 1: Clique $\in$ NP
  - Argue that, given a certificate, you can verify that the certificate provides a solution in polynomial time
- Step 2: Show that some known NP-Complete problem is reducible in poly-time to Clique (i.e., $A \leq_p$ Clique)
  - What known NP-Complete problem do we choose?

3-CNF-SAT $\leq_p$ Clique

- What do we have to do?
  1) Given an instance < ϕ > of 3-CNF-SAT, define poly-time function *f* that converts < ϕ > to instance < G, k > of Clique
  2) Argue that *f* is poly-time
  3) Argue that *f* is correct (i.e., < ϕ > of 3-CNF-SAT is satisfiable iff G has a Clique of size k)

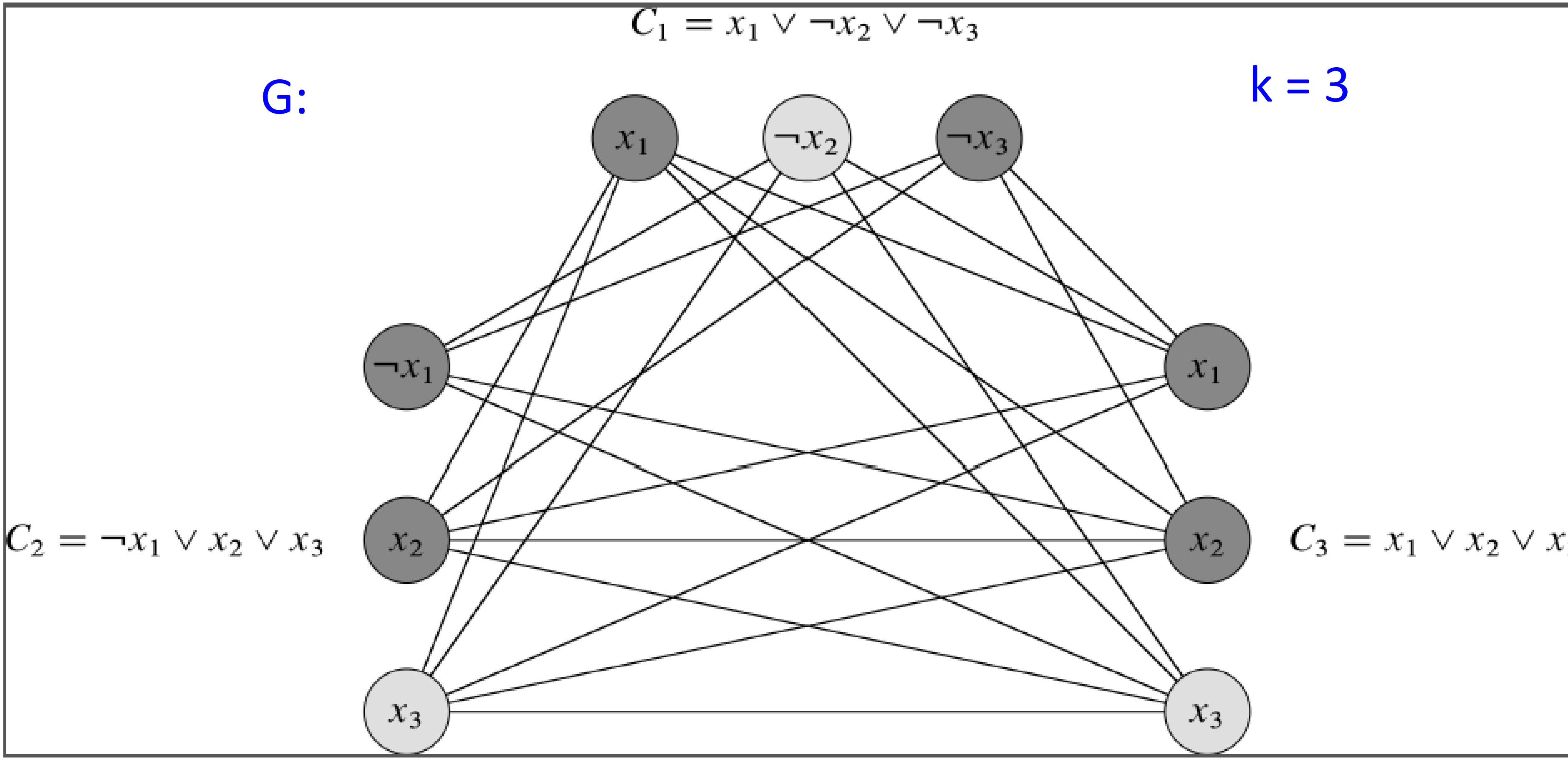
3-CNF-SAT $\leq_p$ Clique

- Reduction function *f* from 3-CNF-SAT to Clique:
  - Suppose $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_k$ is a boolean formula in 3-CNF form with k clauses.
  - We construct a graph G=(V,E) as follows:
    - For each clause $C_r = (l_1^r \vee l_2^r \vee l_3^r)$, place a triple of v_1^r, v_2^r, v_3^r into V
    - Place an edge between two vertices v_i^r and v_j^s when:
      - $r \neq s$, that is v_i^r and v_j^s are in different triples, and
      - Their corresponding literals are consistent, i.e, l_i^r is not negation of l_j^s.
  - Our resulting instance of Clique is <G, k>

- Then we argue that ϕ is satisfiable if and only if G has a clique of size k.

Example reduction from < ϕ > to <G, k>:

$\phi=(x_1\vee\neg x_2\vee\neg x_3)\wedge(\neg x_1\vee x_2\vee x_3)\wedge(x_1\vee x_2\vee x_3)$



Clique is NP-Complete

Now, prove correctness of *f:*

- First, prove reduction is poly time:
  - Should be apparent – only create 3 variables per clause; do this k times

- Then, prove reduction is correct – i.e., ϕ is satisfiable if and only if G has a clique of size *k*:
  - $\Rightarrow$ If ϕ is satisfiable, then there exists a satisfying assignment that makes at least one literal in each clause evaluate to True. Pick one of these literals from each clause. Then consider the subgraph V' consisting of the corresponding vertex of each such literal. For each pair, $v_i^r, v_j^s \in V'$, where $r \neq s$, since l_i^r, l_j^s both evaluate to 1, and l_i^r is not negation of l_j^s, then there must be an edge between v_i^r and v_j^s. So V' is a clique of size *k*.
  - $\Leftarrow$ If G has a clique V' of size *k*, then V' contains exactly one vertex from each triple. Assign all the literals corresponding to the vertices in V' to True, and other literals to either True or False (i.e., they don't matter). Then each clause will evaluate to True. So ϕ is satisfiable.

Since: (1) Clique $\in$ NP, and (2) 3-CNF-SAT $\leq_p$ Clique, we conclude that Clique is NP-Complete.

Another NP-Complete Problem: Vertex Cover

Vertex Cover Problem:

- Undirected graph $G = (V, E)$
- **Vertex cover:** a subset $V' \subseteq V$ such that each edge in the graph is covered by some vertex in V' (i.e., if $(u, v) \in E$, then $u \in V'$ or $v \in V'$ or both.)
- **Size of a VC:** number of vertices it contains

Optimization problem:

- Find a VC of maximum size

Decision problem:

- Does G have a VC of size k?
- Instance of VC = <G, k>

Prove Vertex-Cover is NP-complete

- Step 1: Vertex-Cover $\in$ NP
  - Argue that, given a certificate, you can verify that the certificate provides a solution in polynomial time

- Step 2: Show that some known NP-Complete problem is reducible in poly-time to Vertex-Cover (i.e., A $\leq_p$ Vertex-Cover)
  - What known NP-Complete problem do we choose?

Clique $\leq_p$ Vertex-Cover

- What do we have to do?
    1) Given an instance < G, k > of Clique, define poly-time function *f* that converts < G, k > to instance < G’, k’ > of Vertex-Cover
    2) Argue that *f* is poly-time
    3) Argue that *f* is correct (i.e., G has a clique of size k iff G’ has a vertex cover of size k)

Clique $\leq_p$ Vertex Cover

- Reduction *f*, from Clique to Vertex Cover:
  – Convert G(V, E) to complement graph G'(V,E'):
    - The edges E' of G' contain only those edges *not* in E
  – Output vertex cover instance <G', |V|- k>

- Then we argue that G has a clique of size *k* iff G' has a vertex cover of size |V| - *k*

Proof of correctness of *f*

Now, prove correctness of *f:*

- First, prove reduction is poly time; straight-forward

Next, prove reduction is correct – i.e., G has a clique of size *k* iff G’ has a vertex cover of size |V| - *k*

- $\Rightarrow$ If G has a clique of size *k*, G’ has a vertex cover of size |V| - *k*
  - Let V’ be the *k*-clique
  - Then V - V’ is a vertex cover in G’
    - Let (*u*,*v*) be any edge in G’. Then *u* and *v* cannot both be in V’ (*Why?*)
    - Thus at least one of *u* or *v* is in V-V’ (*why?*), so
      edge (*u*, *v*) is covered by V-V’
    - Since this is true for *any* edge in G’, V-V’ is a vertex cover.

Vertex-Cover is NP-Complete (con't)

- $\Leftarrow$ If G' has a vertex cover V' $\subseteq$ V, with |V'| = |V| - *k*, then G has a clique of size *k*
  - For all *u*,*v* $\in$ V, if (*u*,*v*) $\in$ G' then *u* $\in$ V' or *v* $\in$ V' or both (*why?*)
  - Contrapositive: if *u* $\notin$ V' and *v* $\notin$ V', then (*u*,*v*) $\in$ E
  - In other words, all vertices in V-V' are connected by an edge, thus V-V' is a clique
  - Since |V| - |V'| = *k*, the size of the clique is *k*
- Thus we conclude that G has a clique of size *k* iff G' has a vertex cover of size |V| - *k*

Since: (1) Vertex-Cover $\in$ NP, and (2) Clique $\leq_p$ Vertex-Cover, we conclude that Vertex-Cover is NP-Complete.

Reading Assignments

- Next class:
  - Chapter 34.3
    - Looking more deeply at reductions