Chapter 15 (continued)

Probabilistic Reasoning Over Time

Inference Tasks

Filtering: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$

belief state—input to the decision process of a rational agent

Prediction: $\mathbf{P}(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$ for $k > 0$

evaluation of possible action sequences;
like filtering without the evidence

Smoothing: $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$

better estimate of past states, essential for learning

Most likely explanation: $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$

speech recognition, decoding with a noisy channel

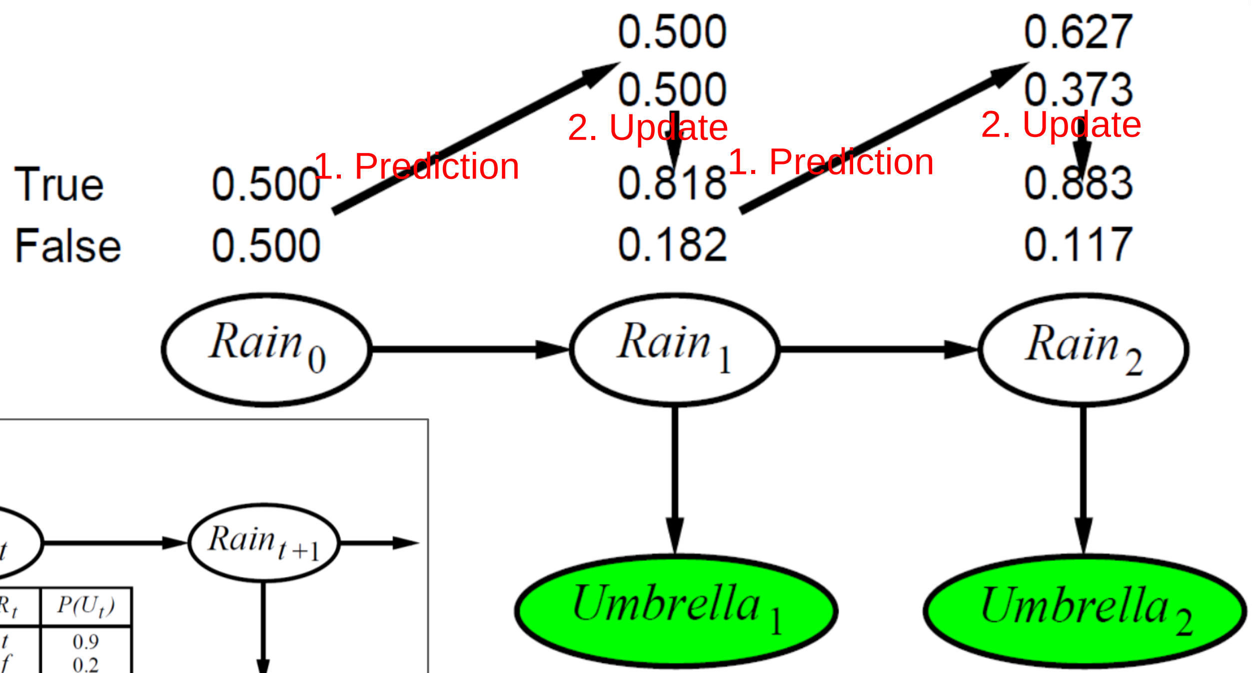
Example: Filtering

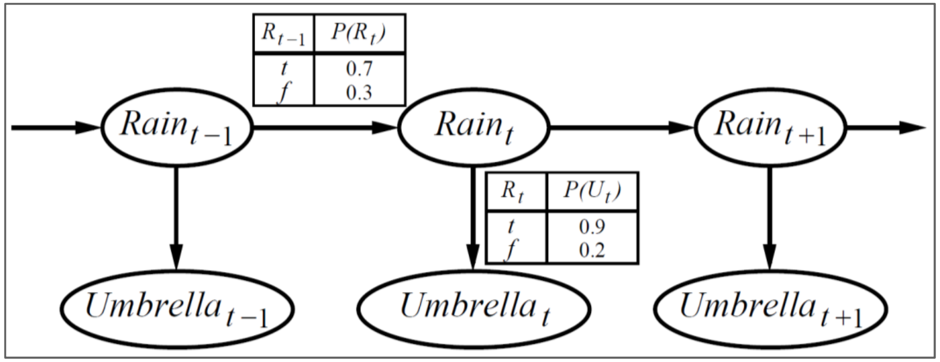
$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \Sigma_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})$$

2. Update 1. Prediction

- View as massage passing

$$\mathbf{f}_{1:t+1} = \text{FORWARD}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1}) \text{ where } \mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$$





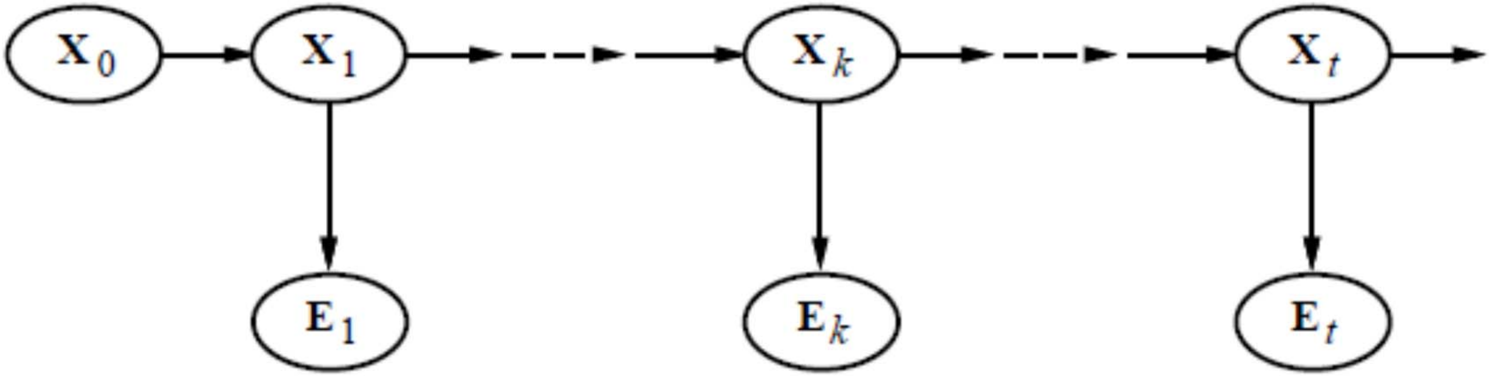
Prediction

- Prediction can be viewed as filtering without new evidence
- Predication can be recursively computed by:

$$
\begin{aligned}
& P(\boldsymbol{X}_{t+1}|\boldsymbol{e}_{1:t}) \\
= & \sum_{\boldsymbol{x}_t} P(\boldsymbol{X}_{t+1}, \boldsymbol{x}_t|\boldsymbol{e}_{1:t}) \\
= & \sum_{\boldsymbol{x}_t} P(\boldsymbol{X}_{t+1}|\boldsymbol{x}_t, \boldsymbol{e}_{1:t}) P(\boldsymbol{x}_t|\boldsymbol{e}_{1:t}) \\
= & \sum_{\boldsymbol{x}_t} P(\boldsymbol{X}_{t+1}|\boldsymbol{x}_t) \boxed{P(\boldsymbol{x}_t|\boldsymbol{e}_{1:t})}
\end{aligned}
$$

Filtering: $\mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$

Smoothing



- Objective: compute the distribution over past states given evidence up to the present: $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$
- Method: divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$, $\mathbf{e}_{k+1:t}$

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\
&= \alpha \mathbf{f}_{1:k} \mathbf{b}_{k+1:t}
\end{aligned}$$

Forward-backward algorithm

Smoothing (continued)

- Smoothing: $\mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:t}) = \alpha\, \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) = \alpha\, \mathbf{f}_{1:k}\, \mathbf{b}_{k+1:t}$
- Backward message: $\mathbf{b}_{k+1:t} = \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k)$

$$
\begin{aligned}
\boxed{\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k)} &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k, \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} \mid \mathbf{x}_{k+1}) \boxed{P(\mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1})} \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k)
\end{aligned}
$$

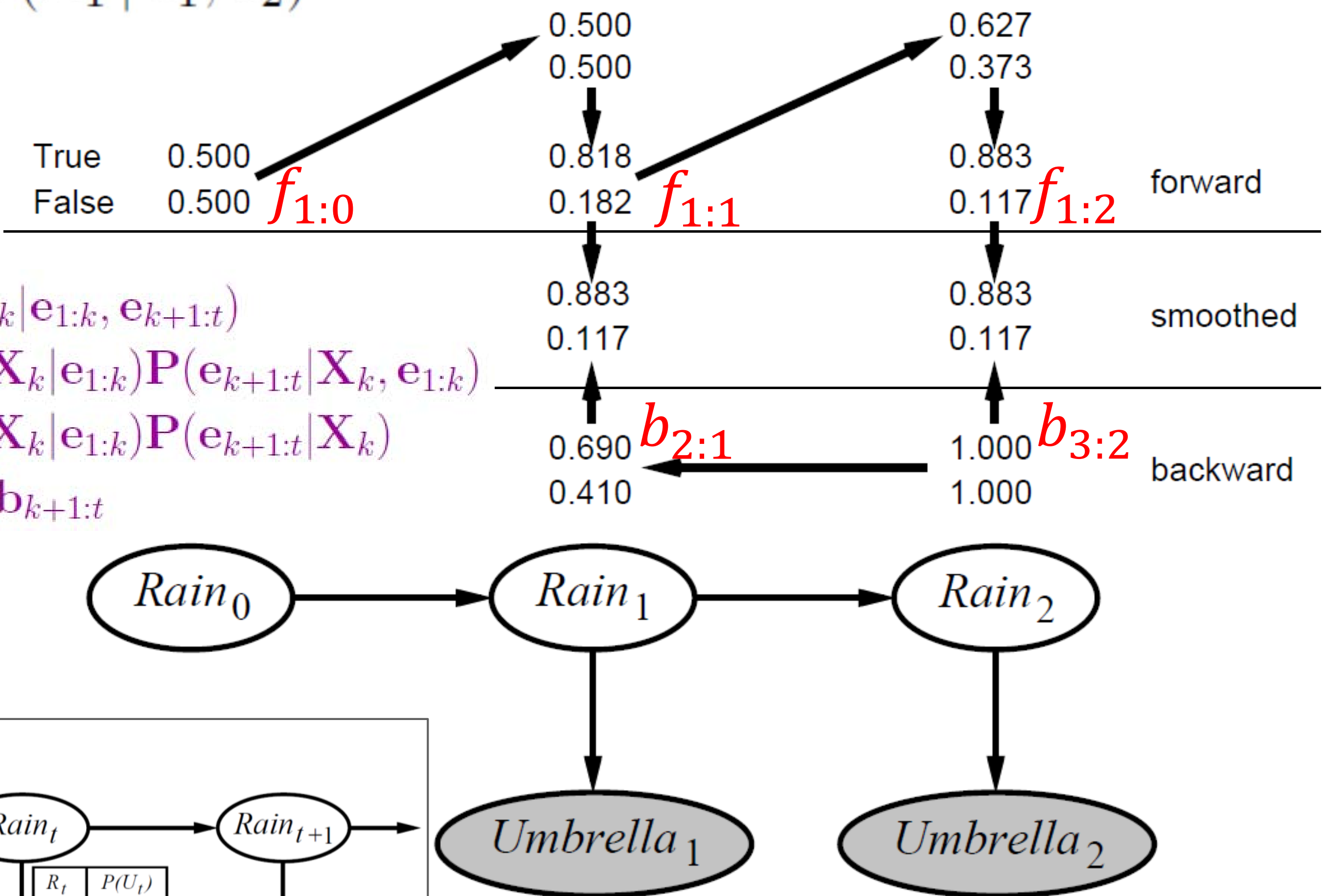
Recursion

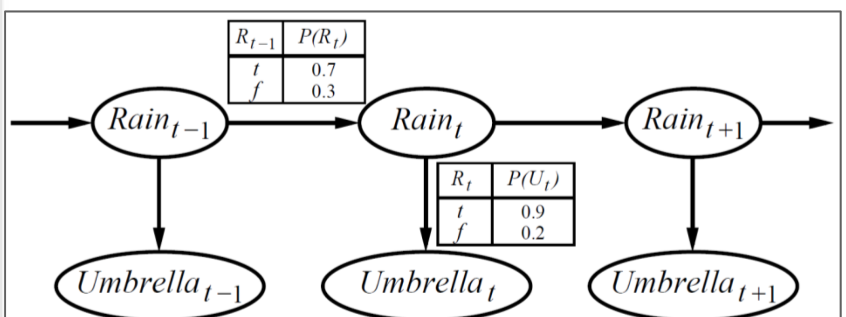
$$\boxed{\mathbf{b}_{k+1:t} = \textsc{Backward}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1})}$$

Example: Smoothing

- Compute $\mathbf{P}(R_1 \mid u_1, u_2)$

$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\
&= \alpha \mathbf{f}_{1:k}\mathbf{b}_{k+1:t}
\end{aligned}
$$





R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} \mid \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k)$$

Most Likely Explanation

- Most likely sequence $\neq$ sequence of most likely states!!!!

$$\max_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_t} P(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_t|\boldsymbol{e}_{1:t}) \qquad \max_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_t} \sum_{i=1}^{t} P(\boldsymbol{x}_i|\boldsymbol{e}_{1:t})$$

Most likely path to each $\mathbf{x}_{t+1}$
$=$ most likely path to **some** $\mathbf{x}_t$ plus one more step

$$\max_{\mathbf{x}_1\ldots\mathbf{x}_t} \mathbf{P}(\mathbf{x}_1,\ldots,\mathbf{x}_t,\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$$
$$= \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x}_t}\left(\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) \max_{\mathbf{x}_1\ldots\mathbf{x}_{t-1}} P(\mathbf{x}_1,\ldots,\mathbf{x}_{t-1},\mathbf{x}_t|\mathbf{e}_{1:t})\right)$$

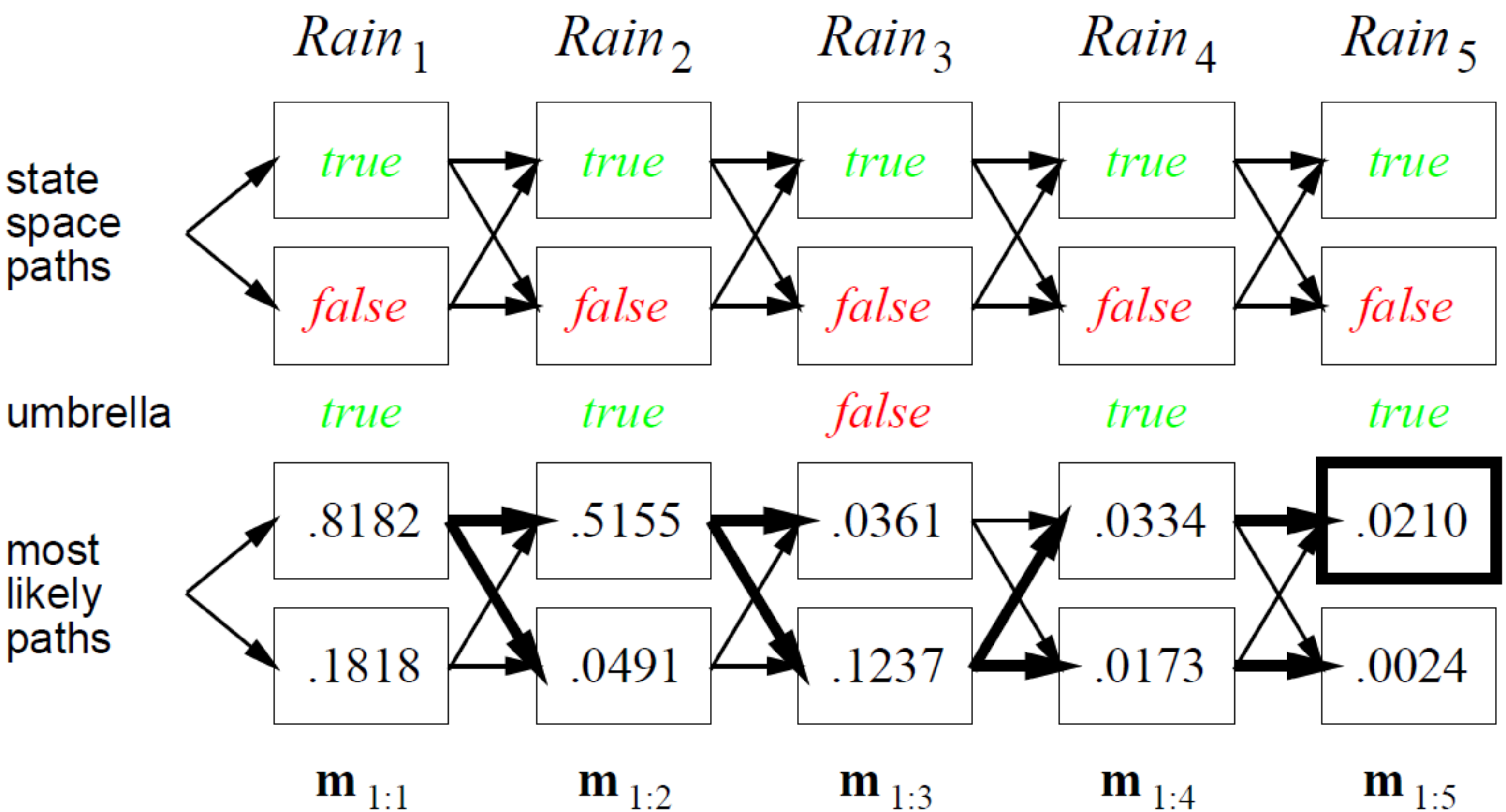
Identical to filtering, except $\mathbf{f}_{1:t}$ replaced by

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1\ldots\mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1,\ldots,\mathbf{x}_{t-1},\mathbf{X}_t|\mathbf{e}_{1:t}),$$

I.e., $\mathbf{m}_{1:t}(i)$ gives the probability of the most likely path to state i.
Update has sum replaced by max, giving the Viterbi algorithm:

$$\mathbf{m}_{1:t+1} = \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x}_t}(\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)\mathbf{m}_{1:t})$$

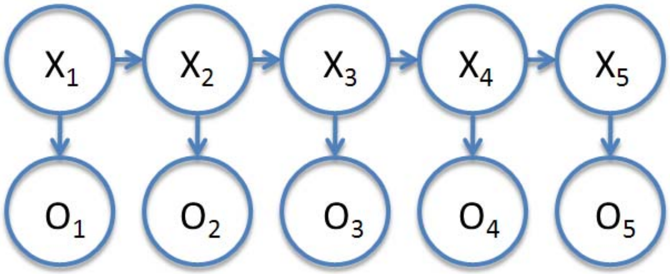
Most Likely Explanation: Example

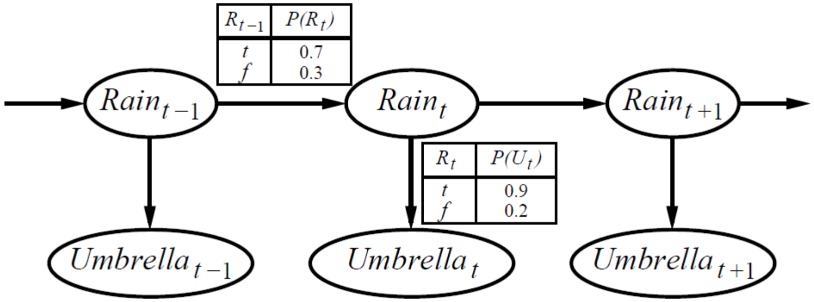


$$\max_{\mathbf{x}_1 \ldots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$$

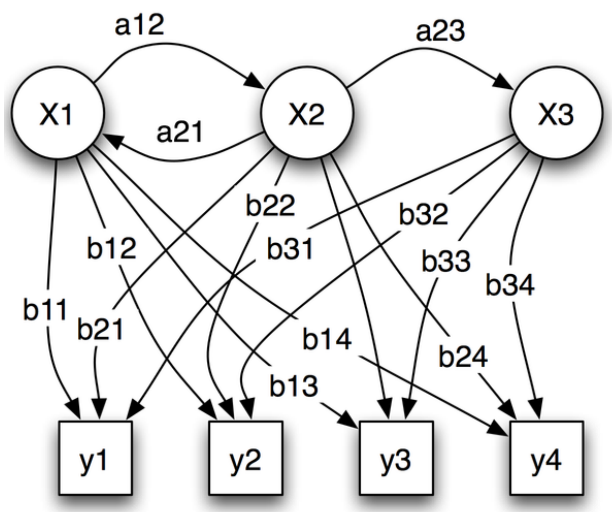
$$= \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t}) \right)$$

Hidden Markov Models





Bayesian network representation



State machine representation

$\mathbf{X}_t$ is a single, discrete variable (usually $\mathbf{E}_t$ is too)
Domain of X_t is $\{1, \ldots, S\}$

Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$, e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$

Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
e.g., with $U_1 = true$, $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$

Forward and backward messages as column vectors:

$$\begin{aligned} \mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^{\top} \mathbf{f}_{1:t} \\ \mathbf{b}_{k+1:t} &= \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t} \end{aligned}$$

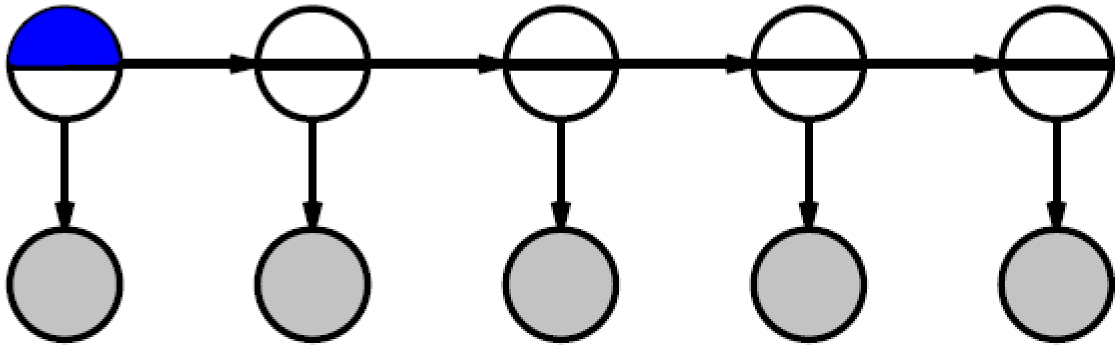
Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$

Hidden Markov Models (continued)

Can avoid storing all forward messages in smoothing by running forward algorithm backwards:

$$
\begin{aligned}
\mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t} \\
\mathbf{O}_{t+1}^{-1} \mathbf{f}_{1:t+1} &= \alpha \mathbf{T}^\top \mathbf{f}_{1:t} \\
\alpha' (\mathbf{T}^\top)^{-1} \mathbf{O}_{t+1}^{-1} \mathbf{f}_{1:t+1} &= \mathbf{f}_{1:t}
\end{aligned}
$$

Algorithm: forward pass computes $\mathbf{f}_t$, backward pass does $\mathbf{f}_i$, $\mathbf{b}_i$



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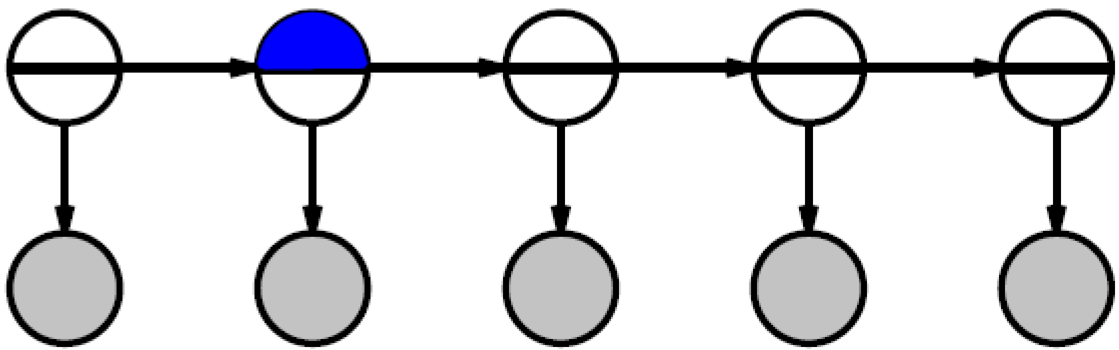

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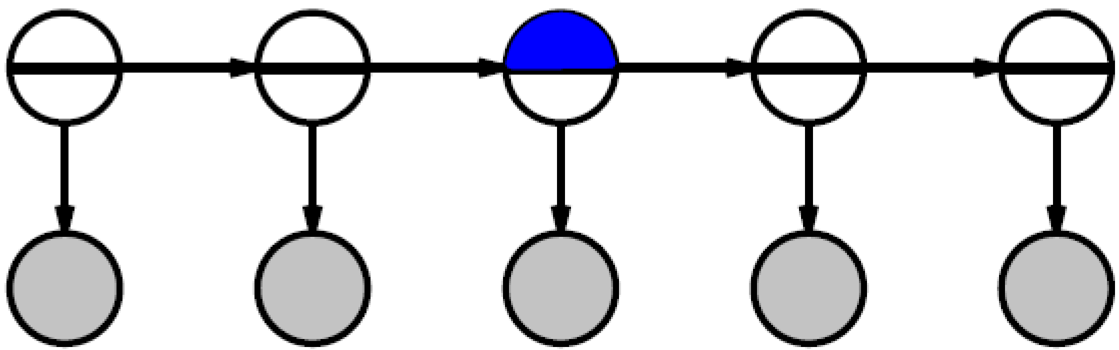

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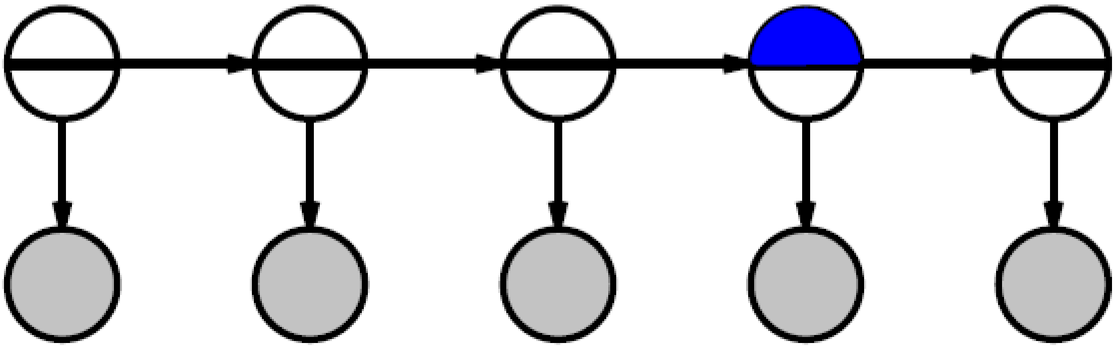

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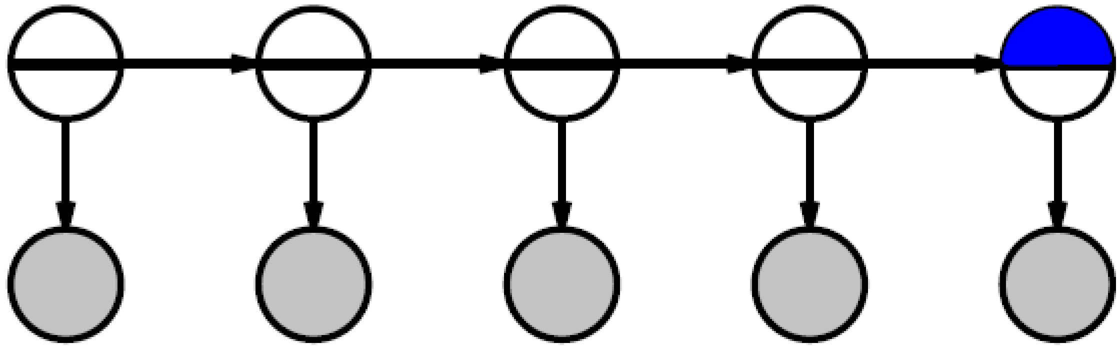

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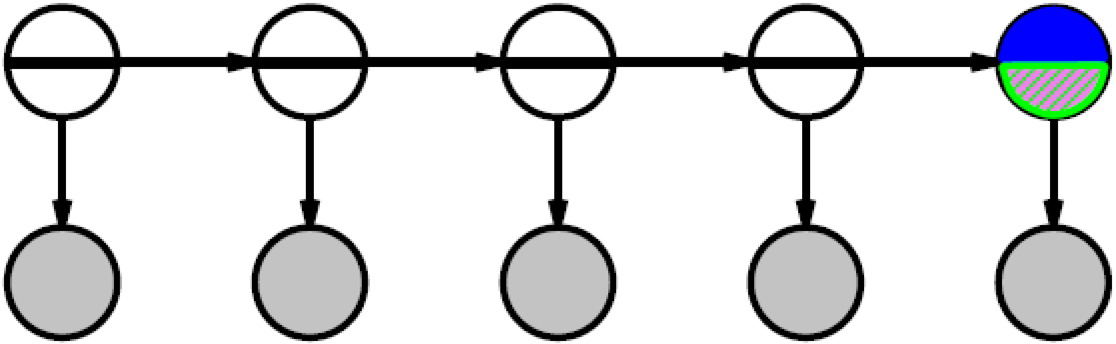

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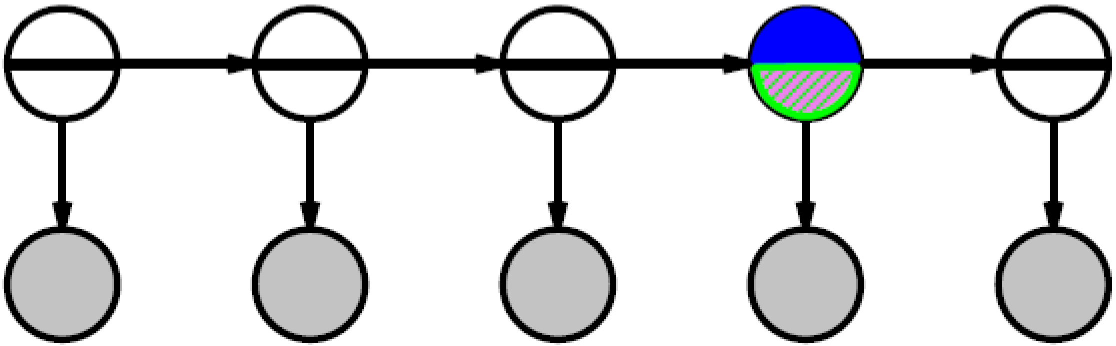

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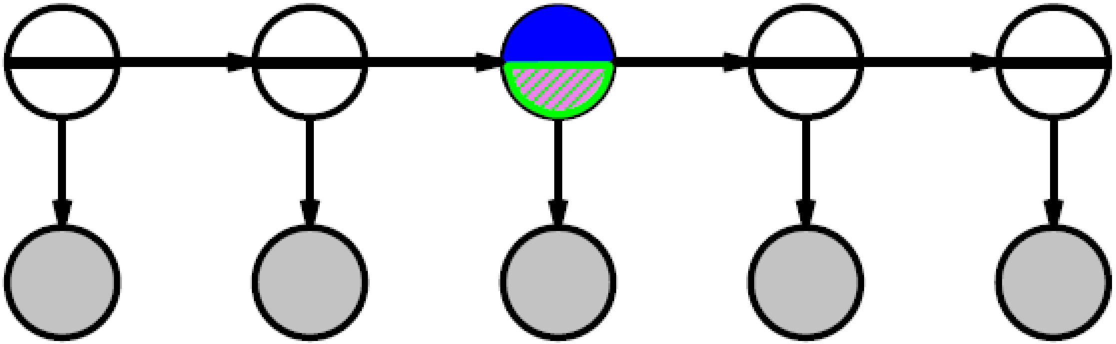

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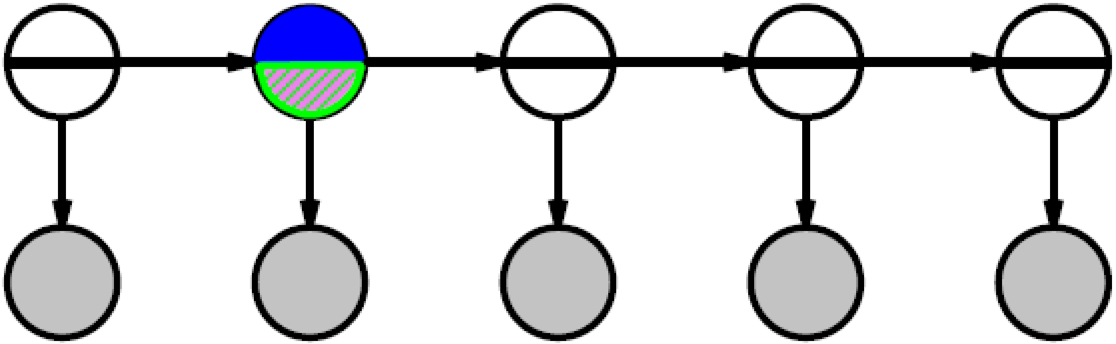

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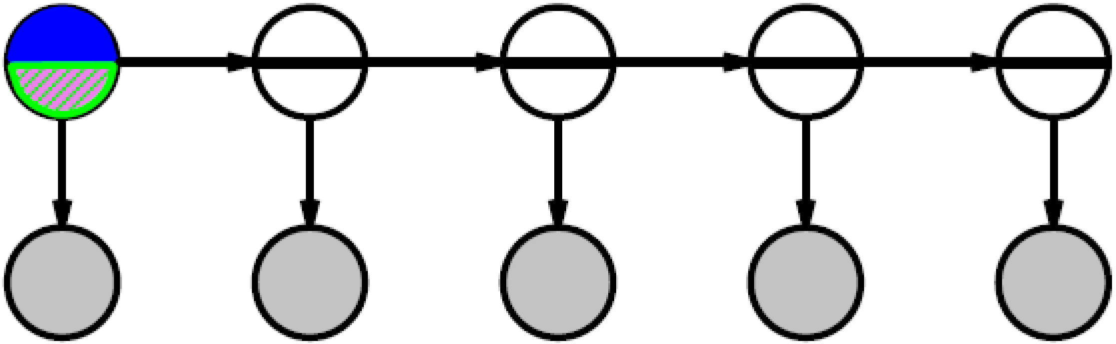
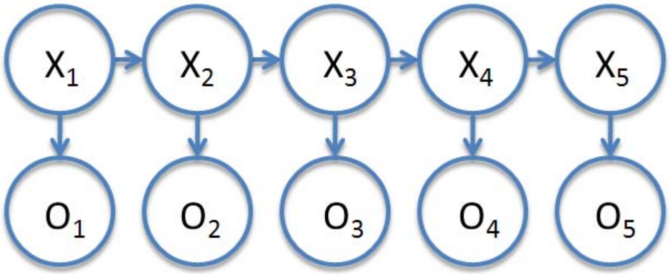
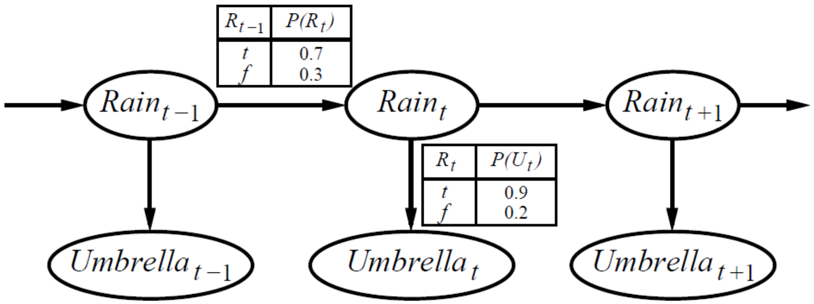
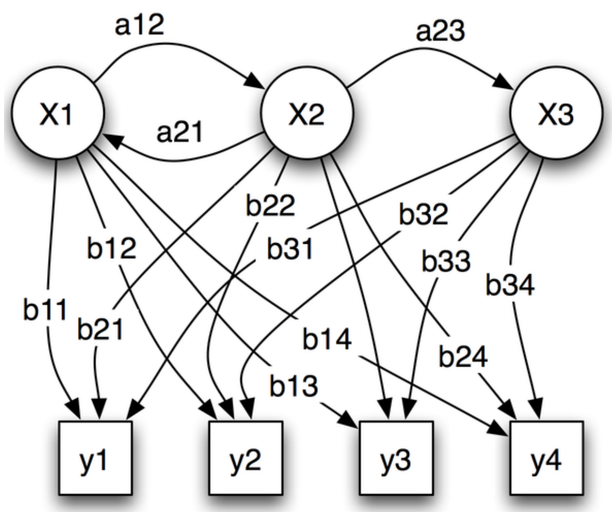


Hidden Markov Models (continued)





Bayesian network representation



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