Exercise 4 - The Continuous-Time and Discrete-Time Fourier Transform

1. Calculate the FT of \( x(t) = e^{-at}u(t) \) for \( a > 0 \).
   Answer: \( X(j\omega) = \frac{1}{a+j\omega} \) for \( a > 0 \), which is complex.

2. Calculate the FT of \( x(t) = \delta(t) \).
   Answer: \( X(j\omega) = 1 \).

3. Calculate the FT of \( x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \)
   Answer: \( X(j\omega) = \frac{2\sin T_1 \omega}{\omega} \), which is real.

4. Revisit the problem we did during the discussion of time-domain convolution. Derive the impulse response to
   \[ \frac{dy(t)}{dt} + 2y(t) = x(t) \]
   where we kind of used the eigenfunction property of the LTI system. Here we solve the same problem in the frequency domain.
   Answer: \( H(j\omega) = \frac{1}{j\omega + 2} \)

5. Consider a stable LTI system characterized by the following LCDE
   \[ \frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \]
   Find the impulse response. If the input signal is \( x(t) = e^{-t}u(t) \), what is the output signal?
   Answer: \( h(t) = \frac{1}{2} e^{-t}u(t) + \frac{1}{2} e^{-3t}u(t) \), \( y(t) = \left( \frac{1}{4} e^{-t} + \frac{1}{2} t e^{-t} - \frac{1}{4} e^{-3t} \right) u(t) \)

6. Find the DTFT for signal \( x[n] = a^n u[n] \) where \( |a| < 1 \).
   Answer: \( X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \). Should be able to plot its magnitude and phase. Note the periodicity of \( X(e^{j\omega}) \).
7. Find the frequency response and impulse response of the LTI system represented by the following LCDE

\[ y[n] - \frac{3}{4} y[n - 1] + \frac{1}{8} y[n - 2] = 2x[n] \]

Calculate the output of the system when the input signal is \( x[n] = (\frac{1}{4})^n u[n] \).

Answer: The frequency response is

\[ H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \]

and the impulse response is

\[ h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n] \]

The output to \( x[n] \) is

\[ Y(e^{j\omega}) = -\frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})^2} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}} \]

and

\[ y[n] = -4(\frac{1}{4})^n u[n] - 2(n + 1)(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n] \]