Review 4 - Fourier Series Representation of Periodic Signals (ECE 315 S17)

1. Complex exponentials as basis functions to decompose a signal

(a) All signals are “mixture” that can be decomposed into linear combination of some basis signals. There are two classes of basis signals,
   i. delayed impulses $\implies$ time-domain system analysis or convolution
   ii. complex exponentials $\implies$ frequency-domain system analysis

(b) Eigenfunction vs. eigenvalue: A signal for which the system output is a constant times the input is referred to as an eigenfunction of the system, and the amplitude factor is referred to as the system’s eigenvalue

(c) Complex exponentials ($e^{st}$ or $z^n$) are eigenfunctions of LTI systems. $H(s)$ or $H(z)$ is the eigenvalue associated with the eigenfunction for a specific $s$ or $z$, where $s$ and $z$ are complex variables. That is,

\[
\begin{align*}
\text{CT: } e^{st} & \longrightarrow H(s)e^{st} \\
\text{DT: } z^n & \longrightarrow H(z)z^n
\end{align*}
\]

where

\[
H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau} d\tau
\]

\[
H(z) = \sum_{k=\infty}^{+\infty} h[k]z^{-k}
\]

and so for LTI systems, if we can write the input signal as weighted sum of the complex exponentials, we can figure out the output.

\[
\begin{align*}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{s_k t} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k H(s_k)e^{s_k t} \\
x[n] &= \sum_{k=-\infty}^{+\infty} a_k z_k^n \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} a_k H(z_k)z_k^n
\end{align*}
\]

2. Fourier series representation of continuous-time periodic signals.

(a) A periodic function written in the format of complex exponential: $e^{j\omega_0 t}$
   (as compared to the complex exponential for a generic signal, $e^{st}$), i.e.,
   $s = j\omega_0$
(b) The set of harmonically related complex exponentials, $e^{jk\omega_0 t}$

(c) The Fourier series representation of continuous-time periodic signal with period $T$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T} t}$$

Need to be able to figure out the zeroth harmonic component, first harmonic component, second harmonic component, etc., as well as the frequency of each component.

(d) The alternative trigonometric form for the Fourier series representation of “real” periodic signals:

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

where $a_k = A_k e^{j\theta_k}$.

(e) The synthesis and analysis equations.

The “analysis” equation

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

where $\{a_k\}$ refers to as the Fourier series coefficients or spectral coefficients. And the “synthesis” equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T} t}$$

3. Convergence of CTFS

(a) partial sum: $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$, $e_N(t) = x(t) - x_N(t)$

(b) $x(t)$ is square integrable: if $\int_T |x(t)|^2 dt < \infty$, then $\int_T |e_N(t)|^2 dt \to 0$ as $N \to \infty$.

(c) The Dirichlet condition: $x(t)$ is absolute integrable and well behaved (i.e., finite number of minima/maxima, finite number of discontinuities). If $\int_T |x(t)| dt < \infty$, then $e_N(t) \to 0$ as $N \to \infty$ except at discontinuities.
(d) Gibb’s phenomenon: as \( N \) increases, the ripples in the partial sums become compressed toward the discontinuity, but for any finite value of \( N \), the peak amplitude of the ripples remains constant.

4. FS representation of DT periodic signals. The synthesis equation:

\[
x[n] = \sum_{k=-N}^{N} a_k e^{j\omega_0 n}
\]

The analysis equation:

\[
a_k = \frac{1}{N} \sum_{n=-N}^{N} x[n]e^{-j\omega_0 n}
\]

Finite sum over one period. No convergence issue.

5. Properties of Fourier series

(a) Should be very familiar with the properties and know how to use the property table

(b) Given the FS pairs of some popular signal, need to know how to use these to find FS pairs of more complex signals that can be obtained by transformations of the popular signals.