1. Continuous-time Fourier transform (CTFT) of aperiodic signals. Aperiodic signals can be interpreted as periodic signals with the period $T \to \infty$, then the frequency $\omega_0 \to 0$, so the harmonics are infinitely close to each other. As a result, the summation in CTFS becomes the integral

(a) The definition:
- The inverse Fourier Transform (the synthesis equation):
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]
- The Fourier Transform (the analysis equation):
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

(b) Relationship between the FS coefficients, $a_k$, and the Fourier transform $X(j\omega)$:
- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.
  \[ a_k = \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0} \]
- The FT of a periodic signal with FS coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the $k$th harmonic frequency $k\omega_0$ is $2\pi$ times the $k$th Fourier series coefficient $a_k$.
  \[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \]

That way, $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ would be exactly $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
(c) CTFT convergence. (similar to the discussion of convergence of FS)

- Let $\hat{x}(t)$ be the estimate of $x(t)$ calculated from the synthesis equation, $\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$, and $e(t) = x(t) - \hat{x}(t)$
- $x(t)$ is square integrable: if $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$, then $\int_{-\infty}^{\infty} |e(t)|^2 dt \rightarrow 0$. That is, the energy of the error is zero.
- The Dirichlet condition: $x(t)$ is absolute integrable and well-behaved (i.e., finite number of minima/maxima, finite number of finite discontinuities). If $\int_{-\infty}^{\infty} |x(t)| dt < \infty$, then $e(t) \rightarrow 0$ except at discontinuities. That is, $\hat{x}(t)$ approaches $x(t)$ at every moment of $t$ except at discontinuities.
- Gibb’s phenomenon still exists at discontinuities.

(d) Properties of FT and FT pairs (see handout in class): Need to know how to use the tables of properties and transform pairs to solve problems.

(e) System characterization by linear constant-coefficient differential equations (LCDE) - finding the frequency response.

- In the family of LTI systems, a subset of which can be represented using LCDE.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where $N$ is the order of the output and $M$ is the order of the input. We often assume $N \geq M$.

- Use LCDE to find the frequency response $H(j\omega)$.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

2. Discrete-time Fourier transform (DTFT) of aperiodic signals

(a) The definition:

- The inverse Fourier Transform (the synthesis equation):

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- The Fourier Transform (the analysis equation):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$
(b) Relationship between the DTFS coefficients, $a_k$, and the DTFT, $X(e^{j\omega})$.

- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.

$$a_k = \frac{1}{N} X(e^{j\omega})|_{\omega = k\omega_0}$$

- The FT of a periodic signal with FS coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the $k$th harmonic frequency $k\omega_0$ is $2\pi$ times the $k$th Fourier series coefficient $a_k$.

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

(c) DTFT vs. CTFT

- DTFT is periodic in $\omega$ with $2\pi$ as period, which is why $x[n]$ uses finite interval of integration in the synthesis equation

- Notation wise, DTFT uses $X(e^{j\omega})$ while CTFT uses $X(j\omega)$.

(d) DTFT convergence.

- Absolute summable: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

- Square summable (signal has finite energy): $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

(e) Properties of DTFT and DTFT pairs: Need to know how to use the tables of properties and transform pairs to solve problems.

(f) System characterization by linear constant-coefficient difference equation (LCDE) - finding the frequency response.

- Generalized representation:

$$\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]$$

- Use LCDE to find the frequency response $H(e^{j\omega})$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

(g) Duality.
<table>
<thead>
<tr>
<th>Frequency domain (analysis equations)</th>
<th>Dualities</th>
<th>Time domain (synthesis equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CTFT</strong></td>
<td>$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$</td>
<td>$\iff$</td>
</tr>
<tr>
<td>aperiodic in frequency</td>
<td></td>
<td>aperiodic in time</td>
</tr>
<tr>
<td>continuous in frequency</td>
<td></td>
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</tr>
<tr>
<td><strong>DTFT</strong></td>
<td>$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$</td>
<td></td>
</tr>
<tr>
<td>periodic in frequency (with period $2\pi$)</td>
<td></td>
<td>aperiodic in time</td>
</tr>
<tr>
<td>continuous in frequency</td>
<td></td>
<td>discrete in time</td>
</tr>
<tr>
<td><strong>CTFS</strong></td>
<td>$a_k = \frac{1}{T_0} \int_{T_0} x(t)e^{-jk\omega_0 t}dt$</td>
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