Continuous-time Fourier transform (CTFT) of aperiodic signals. (Aperiodic signals can be interpreted as periodic signals with the period \( T \to \infty \), then the frequency \( \omega_0 \to 0 \), so the harmonics are infinitely close to each other. As a result, the summation in CTFS becomes the integral.

(a) The definition:
- The inverse Fourier Transform (the synthesis equation):
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]
- The Fourier Transform (the analysis equation):
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

(b) Relationship between the FS coefficients, \( a_k \), and the Fourier transform \( X(j\omega) \):
- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.
  \[ a_k = \frac{1}{T} X(j\omega)|_{\omega = k\omega_0} \]
- The FT of a periodic signal with FS coefficients \( \{a_k\} \) can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the \( k \)th harmonic frequency \( k\omega_0 \) is \( 2\pi \) times the \( k \)th Fourier series coefficient \( a_k \).
  \[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \]

That way, \( x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \) would be exactly \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k} \).
(c) CTFT convergence. (similar to the discussion of convergence of FS)

- Let \( \hat{x}(t) \) be the estimate of \( x(t) \) calculated from the synthesis equation,
  \[
  \hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega,
  \]
  and \( e(t) = x(t) - \hat{x}(t) \).
- \( x(t) \) is square integrable: if \( \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \), then \( \int_{-\infty}^{\infty} |e(t)|^2 dt \to 0 \).
  That is, the energy of the error is zero.
- The Dirichlet condition: \( x(t) \) is absolute integrable and well behaved (i.e., finite number of minima/maxima, finite number of finite discontinuities). If \( \int_{-\infty}^{\infty} |x(t)| dt < \infty \), then \( e(t) \to 0 \) except at discontinuities. That is, \( \hat{x}(t) \) approaches \( x(t) \) at every moment of \( t \) except at discontinuities.
- Gibb’s phenomenon still exists at discontinuities.

(d) Properties of FT and FT pairs (see handout in class): Need to know how to use the tables of properties and transform pairs to solve problems.

(e) System characterization by linear constant-coefficient differential equations (LCDE) - finding the frequency response.

- In the family of LTI systems, a subset of which can be represented using LCDE.
  \[
  \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}
  \]
  where \( N \) is the order of the output and \( M \) is the order of the input. We often assume \( N \geq M \).
- Use LCDE to find the frequency response \( H(j\omega) \).
  \[
  H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}
  \]

2. Discrete-time Fourier transform (DTFT) of aperiodic signals

(a) The definition:

- The inverse Fourier Transform (the synthesis equation):
  \[
  x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega
  \]
- The Fourier Transform (the analysis equation):
  \[
  X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}
  \]
(b) Relationship between the DTFS coefficients, \( a_k \), and the DTFT, \( X(e^{j\omega}) \).

- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.

\[
an_k = \frac{1}{N} X(e^{j\omega})|_{\omega = k\omega_0}
\]

- The FT of a periodic signal with FS coefficients \( \{a_k\} \) can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the \( k \)th harmonic frequency \( k\omega_0 \) is \( 2\pi \) times the \( k \)th Fourier series coefficient \( a_k \).

\[
X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
\]

(c) DTFT vs. CTFT

- DTFT is periodic in \( \omega \) with \( 2\pi \) as period, which is why \( x[n] \) uses finite interval of integration in the synthesis equation
- Notation wise, DTFT uses \( X(e^{j\omega}) \) while CTFT uses \( X(j\omega) \).

(d) DTFT convergence.

- Absolute summable: \( \sum_{n=-\infty}^{\infty} |x[n]| < \infty \)
- Square summable (signal has finite energy): \( \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \)

(e) Properties of DTFT and DTFT pairs: Need to know how to use the tables of properties and transform pairs to solve problems.

(f) System characterization by linear constant-coefficient difference equation (LCDE) - finding the frequency response.

- Generalized representation:

\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]
\]

- Use LCDE to find the frequency response \( H(e^{j\omega}) \)

\[
H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}
\]

(g) Duality.
### Frequency domain (analysis equations)

<table>
<thead>
<tr>
<th>Domain</th>
<th>Equation</th>
<th>Duality</th>
<th>Time domain (synthesis equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CTFT</td>
<td>$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$</td>
<td>$\iff$</td>
<td>$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$</td>
</tr>
<tr>
<td></td>
<td>aperiodic in frequency</td>
<td></td>
<td>aperiodic in time</td>
</tr>
<tr>
<td></td>
<td>continuous in frequency</td>
<td></td>
<td>continuous in time</td>
</tr>
<tr>
<td>DTFT</td>
<td>$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$</td>
<td>$\iff$</td>
<td>$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega}) e^{j\omega n} d\omega$</td>
</tr>
<tr>
<td></td>
<td>periodic in frequency (with period $2\pi$)</td>
<td></td>
<td>aperiodic in time</td>
</tr>
<tr>
<td></td>
<td>continuous in frequency</td>
<td></td>
<td>discrete in time</td>
</tr>
<tr>
<td>CTFS</td>
<td>$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 t} dt$</td>
<td>$\iff$</td>
<td>$x(t) = \sum_{n=-\infty}^{+\infty} a_k e^{j\omega_0 t}$</td>
</tr>
<tr>
<td></td>
<td>aperiodic in frequency</td>
<td></td>
<td>periodic in time</td>
</tr>
<tr>
<td></td>
<td>discrete in frequency</td>
<td></td>
<td>continuous in time</td>
</tr>
<tr>
<td>DTFS</td>
<td>$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 n}$</td>
<td>$\iff$</td>
<td>$x[n] = \sum_{k=0}^{N-1} a_k e^{j\omega_0 n}$</td>
</tr>
<tr>
<td></td>
<td>periodic in frequency</td>
<td></td>
<td>periodic in time</td>
</tr>
<tr>
<td></td>
<td>discrete in frequency</td>
<td></td>
<td>discrete in time</td>
</tr>
</tbody>
</table>