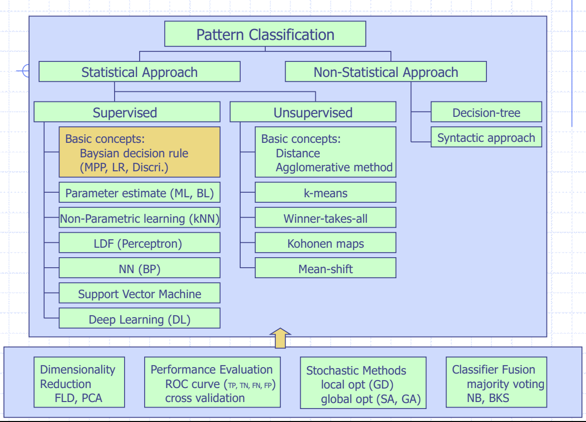
ECE 471/571 – Lecture 2

Bayesian Decision Theory



Bayes' formula (Bayes' rule)

From domain knowledge

a-priori probability (prior probability)

Conditional probability density (pdf – probability density function) (likelihood)

$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j) P(\omega_j)}{p(x)}$$

$$p(x) = \sum_{j=1}^{c} p(x \mid \omega_j) P(\omega_j)$$

a-posteriori probability (posterior probability)

normalization constant (evidence)

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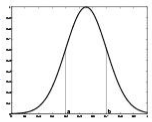
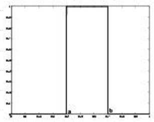
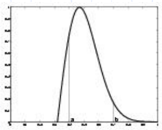
pdf examples

- Gaussian distribution
  - Bell curve
  - Normal distribution
- Uniform distribution
- Rayleigh distribution







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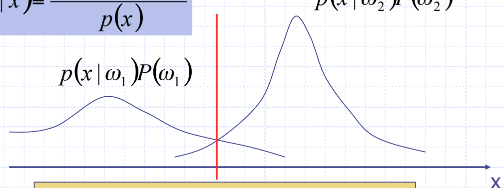
Bayes decision rule

$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j)P(\omega_j)}{p(x)}$$



$p(x \mid \omega_1)P(\omega_1)$

$p(x \mid \omega_2)P(\omega_2)$

Maximum *a-posteriori* Probability (MAP):
For a given x, if $P(\omega_1 \mid x) > P(\omega_2 \mid x)$, then x belongs to class 1, otherwise, 2.

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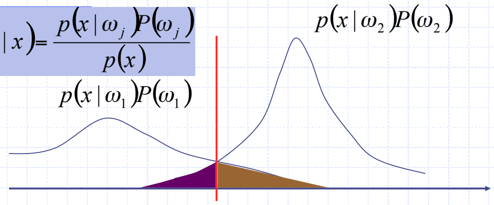
Conditional probability of error

$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j)P(\omega_j)}{p(x)}$$

$p(x \mid \omega_1)P(\omega_1)$

$p(x \mid \omega_2)P(\omega_2)$



$$P(error \mid x) = \begin{cases} P(\omega_1 \mid x) & \text{if we decide } \omega_2 \\ P(\omega_2 \mid x) & \text{if we decide } \omega_1 \end{cases} = \min[P(\omega_1 \mid x), P(\omega_2 \mid x)]$$

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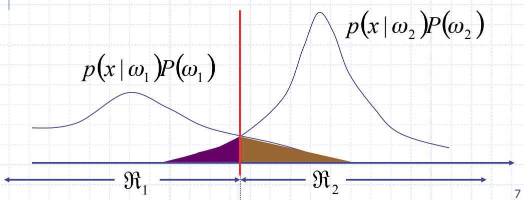
Decision regions

- The effect of any decision rule is to partition the feature space into c decision regions $\Re_1, \Re_2, \cdots, \Re_c$



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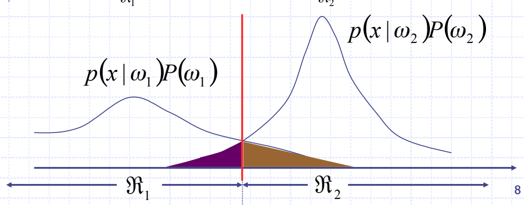
$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j) P(\omega_j)}{p(x)}$$

Overall probability of error

Or unconditional risk, unconditional probability of error

$$P(error) = \int_{-\infty}^{\infty} P(error, x) dx = \int_{-\infty}^{\infty} P(error \mid x) p(x) dx$$

$$P(error) = \int_{\Re_1} P(\omega_2 \mid x) p(x) dx + \int_{\Re_2} P(\omega_1 \mid x) p(x) dx$$



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The conditional risk

- Given x, the conditional risk of taking action α_i is:

λ_{ij}

$$R(\alpha_i \mid x) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$

- λ_{ij} is the loss when decide x belongs to class i while it should be j

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Likelihood ratio - two category classification

$$R(\alpha_1 | x) = \lambda_{11} P(\omega_1 | x) + \lambda_{12} P(\omega_2 | x)$$
$$R(\alpha_2 | x) = \lambda_{21} P(\omega_1 | x) + \lambda_{22} P(\omega_2 | x)$$
If $R(\alpha_1 | x) < R(\alpha_2 | x)$, then decide ω_1
$$(\lambda_{21} - \lambda_{11}) P(\omega_1 | x) > (\lambda_{12} - \lambda_{22}) P(\omega_2 | x)$$

$$\frac{p(x | \omega_1)}{p(x | \omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

Likelihood ratio

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Zero-One loss

$$\lambda(\alpha_i | \omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$

$$R(\alpha_i | x) = \sum_{j \neq i} P(\omega_j | x) = 1 - P(\omega_i | x)$$

$$\frac{p(x | \omega_1)}{p(x | \omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)} = \frac{P(\omega_2)}{P(\omega_1)}$$

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Recap

- Bayes decision rule → maximum *a-posteriori* probability
- Conditional risk
- Likelihood ratio
- Decision regions → How to calculate the overall probability of error

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Recap

$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j)P(\omega_j)}{p(x)} \qquad R(\alpha_i \mid x) = \sum_{j=1}^{c} \lambda(\alpha_i \mid \omega_j)P(\omega_j \mid x)$$

Maximum *a-posteriori* Probability

For a given x, if $P(\omega_1 \mid x) > P(\omega_2 \mid x)$, then x belongs to class 1, otherwise, 2.

Likelihood ratio

If $R(\alpha_1 \mid x) < R(\alpha_2 \mid x)$, then decide ω_1, that is

$$\frac{p(x \mid \omega_1)}{p(x \mid \omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

Overall probability of error

$$P(error) = \int_{\Re_1} P(\omega_2 \mid x)p(x)dx + \int_{\Re_2} P(\omega_1 \mid x)p(x)dx$$