ECE 471/571 – Lecture 2

Bayesian Decision Theory

Bayes’ formula (Bayes’ rule)

\[ p(\omega_j | x) = \frac{p(x | \omega_j) p(\omega_j)}{p(x)} = \frac{\sum_{\omega_j} p(x | \omega_j) p(\omega_j)}{p(x)} \]

- Conditional probability density (pdf – probability density function)
- a-posteriori probability (posterior probability)
- a-priori probability (prior probability)
- normalization constant (evidence)
- From domain knowledge

Dimensionality Reduction: PCA, LDA
Performance Evaluation: ROC curve (TP, TN, FP, FN)
Classifier Fusion: Majority Voting, ML, SQS
pdf examples

- Gaussian distribution
  - Bell curve
  - Normal distribution
- Uniform distribution
- Rayleigh distribution

Bayes decision rule

\[
\begin{align*}
P(\omega_1 \mid x) &= \frac{p(x \mid \omega_1) P(\omega_1)}{p(x)} \\
P(\omega_2 \mid x) &= \frac{p(x \mid \omega_2) P(\omega_2)}{p(x)}
\end{align*}
\]

Maximum a-posteriori Probability (MAP):
For a given \( x \), if \( P(\omega_1 \mid x) > P(\omega_2 \mid x) \)
then \( x \) belongs to class 1, otherwise, 2.

Conditional probability of error

\[
P(error \mid x) = \begin{cases} 
P(\omega_1 \mid x) & \text{if we decide } \omega_1 \\
\min\{P(\omega_1 \mid x), P(\omega_2 \mid x)\} & \text{if we decide } \omega_2
\end{cases}
\]
The effect of any decision rule is to partition the feature space into $c$ decision regions $R_1, R_2, \ldots, R_c$.

**Overall probability of error**

Or unconditional risk, unconditional probability of error

$$P(error) = \int P(error, x) dx = \int P(error | x) p(x) dx$$

$$P(error) = \int_{R_1} P(\omega_1 | x) p(x) dx + \int_{R_2} P(\omega_2 | x) p(x) dx$$

**The conditional risk**

Given $x$, the conditional risk of taking action $\alpha_i$ is:

$$R(\alpha_i | x) = \sum_{j} \lambda_{ij} \cdot p(\omega_j | x)$$

$\lambda_{ij}$ is the loss when decide $x$ belongs to class $i$ while it should be $j$. 

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Likelihood ratio - two category classification

\[ R(\xi_1 | x) = \frac{\lambda_{11} P(\omega_1 | x)}{\lambda_{12} P(\omega_2 | x)} \]

\[ R(\xi_2 | x) = \frac{\lambda_{21} P(\omega_1 | x)}{\lambda_{22} P(\omega_2 | x)} \]

If \( R(\xi_1 | x) < R(\xi_2 | x) \), then decide \( \omega_1 \)

\( (\lambda_{21} - \lambda_{11}) P(\omega_1 | x) > (\lambda_{22} - \lambda_{12}) P(\omega_2 | x) \)

\[ \frac{p(x | \omega_1)}{p(x | \omega_2)} > \frac{\lambda_{22} - \lambda_{12}}{\lambda_{21} - \lambda_{11}} \]

Zero-One loss

\[ \lambda(\xi_i | \omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \]

\[ R(\xi_i | x) = \sum_{j} P(\omega_j | x) = 1 - P(\omega_i | x) \]

\[ \frac{p(x | \omega_1)}{p(x | \omega_2)} > \frac{\lambda_{22} - \lambda_{12}}{\lambda_{21} - \lambda_{11}} \]

Recap

- Bayes decision rule → maximum \textit{a-posteriori} probability
- Conditional risk
- Likelihood ratio
- Decision regions → How to calculate the overall probability of error
### Recap

For a given $x$, if $P(\omega_1 \mid x) > P(\omega_2 \mid x)$ then $x$ belongs to class 1, otherwise, 2.

#### Maximum a-posteriori Probability

$$P(\omega_j \mid x) = \frac{p(x \mid \omega_j) \cdot P(\omega_j)}{p(x)}$$

$$R(\omega_j \mid x) = \sum_{i \neq j} \lambda(x, \omega_i) P(\omega_i \mid x)$$

#### Likelihood ratio

If $R(\omega_1 \mid x) < R(\omega_2 \mid x)$ then decide $\omega_i$, that is

$$\frac{p(x \mid \omega_j)}{p(x \mid \omega_i)} = \frac{\lambda_{ji}}{\lambda_{ij}}$$

$$P(\omega_j \mid x) = \frac{p(\omega_j \mid x) p(x) dx}{\int_{x_1}^x} + \int_{x_1}^x P(\omega_j \mid x) p(x) dx$$

#### Overall probability of error