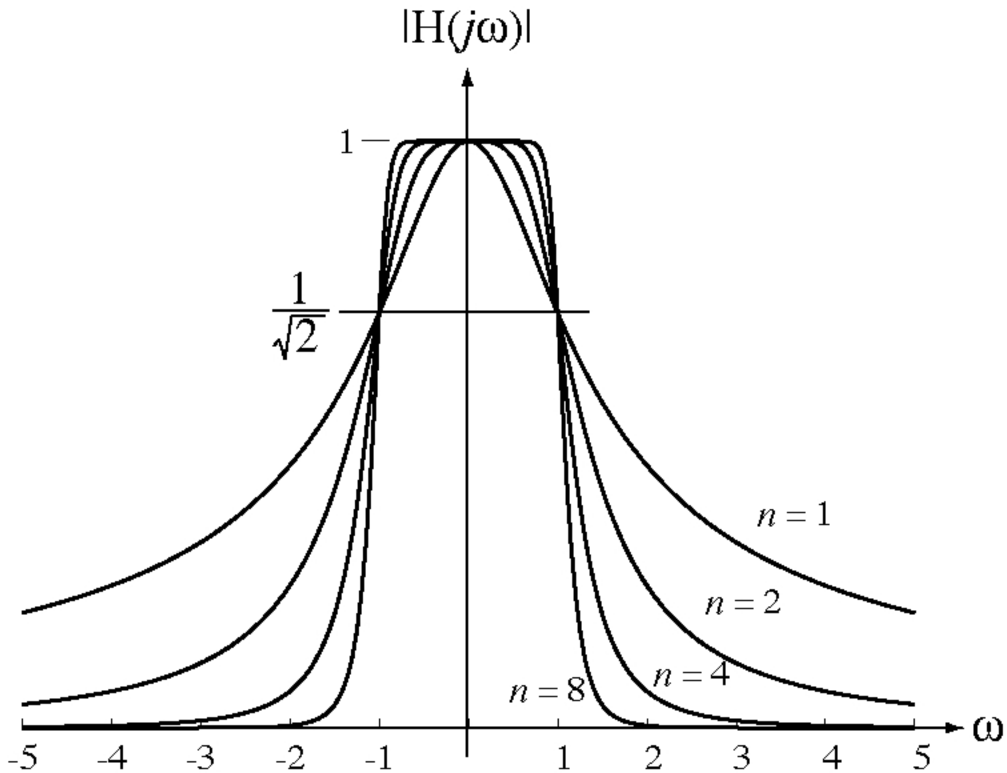
Filter Analysis and Design

Butterworth Filters

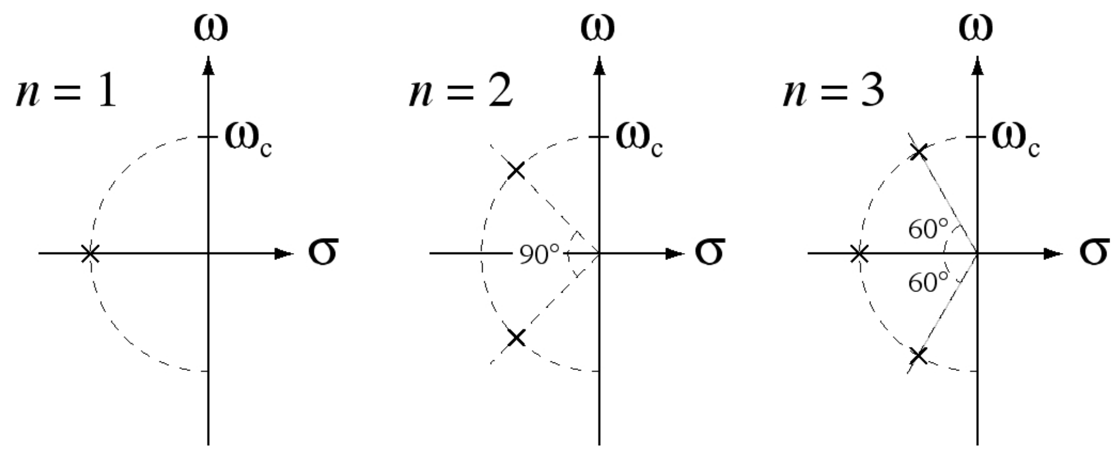
Butterworth filters have a transfer function whose squared magnitude has the form $\left|\mathrm{H}_a\left(j\omega\right)\right|^2 = \dfrac{1}{1+\left(\omega/\omega_c\right)^{2n}}$.



Butterworth Filters

The poles of a lowpass Butterworth filter lie on a semicircle of radius ω_c in the open left half-plane. The number of poles is n and the angular spacing between the poles is always π / n. For n odd there is one pole on the negative real axis and all the others occur in complex conjugate pairs. For even n all the poles occur in complex conjugate pairs. The form of the transfer function is

$$\mathrm{H}_a(s) = \frac{1}{(1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_n)} = \prod_{k=1}^{n} \frac{1}{1 - s/p_k} = \prod_{k=1}^{n} \frac{p_k}{p_k - s}$$



Frequency Transformation

A normalized lowpass filter can be transformed into an un-normalized lowpass filter or into a highpass, bandpass or bandstop filter by **frequency transformation**.

$$\text{Lowpass to lowpass} \Rightarrow s \to s / \omega_c$$

$$\text{Lowpass to highpass} \Rightarrow s \to \omega_c / s$$

$$\text{Lowpass to bandpass} \Rightarrow s \to \frac{s^2 + \omega_L \omega_H}{s(\omega_H - \omega_L)}$$

$$\text{Lowpass to bandstop} \Rightarrow s \to \frac{s(\omega_H - \omega_L)}{s^2 + \omega_L \omega_H}$$

(ω_c is the cutoff frequency in rad/s of a lowpass or highpass filter and ω_L and ω_H are the lower and upper cutoff frequencies in rad/s of a bandpass or bandstop filter.)

Frequency Transformation

Example

Design a first-order bandstop Butterworth filter with a cutoff frequencies of 1 kHz and 2 kHz.

$$\mathrm{H}_{norm}(s) = \frac{1}{s+1} \Rightarrow \mathrm{H}_{BS}(s) = \frac{1}{\dfrac{s(\omega_{\mathrm{H}} - \omega_L)}{s^2 + \omega_L \omega_{\mathrm{H}}} + 1} = \frac{s^2 + \omega_L \omega_{\mathrm{H}}}{s^2 + s(\omega_{\mathrm{H}} - \omega_L) + \omega_L \omega_{\mathrm{H}}}$$

$$\omega_L = 2\pi \times 1000 = 2000\pi \quad , \quad \omega_H = 4000\pi$$

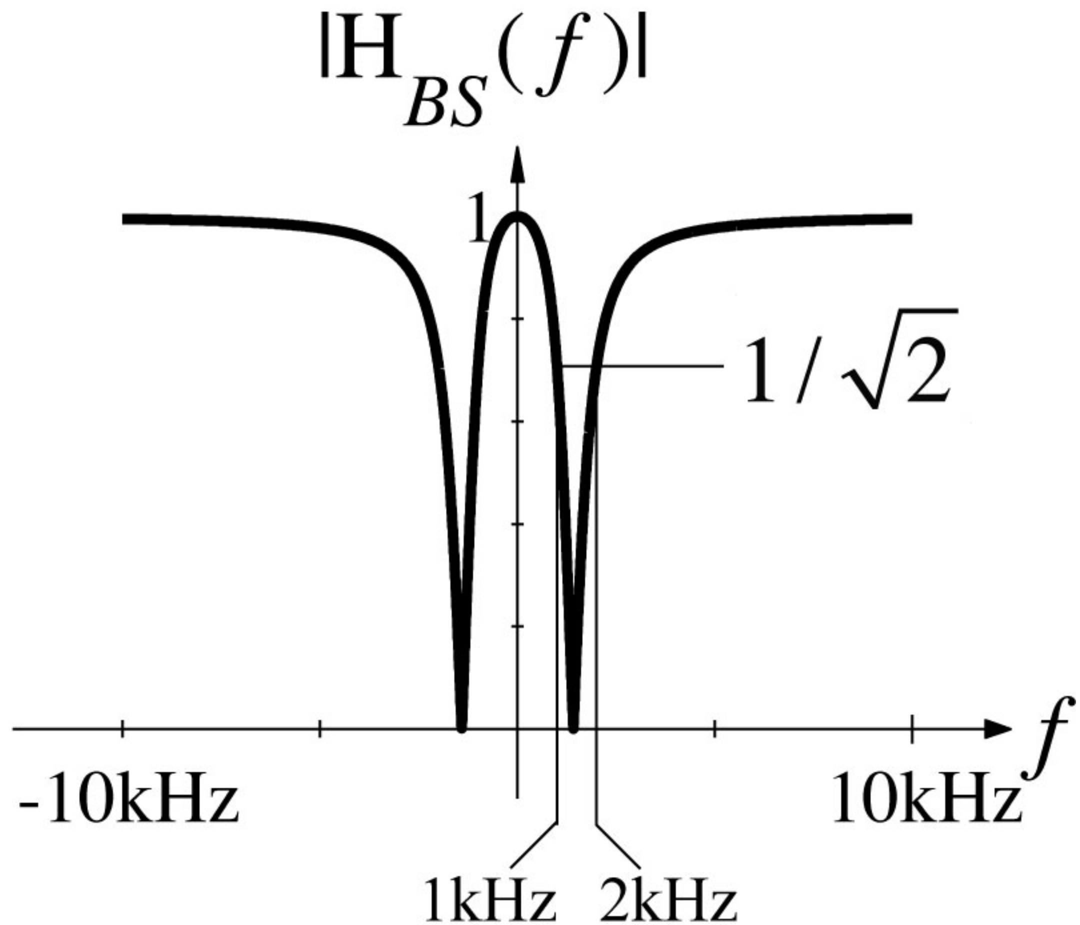
$$\mathrm{H}_{BS}(s) = \frac{s^2 + 8\pi^2 \times 10^6}{s^2 + 2000\pi s + 8\pi^2 \times 10^6} \cong \frac{s^2 + 7.896 \times 10^7}{s^2 + 6283 s + 7.896 \times 10^7}$$

Zeros at $s = \pm j8886$ which correspond to $\omega = \pm 8886$ rad/s and $f = 1414$ Hz

Poles at $s = -3142 \pm j8886$.

Frequency Transformation

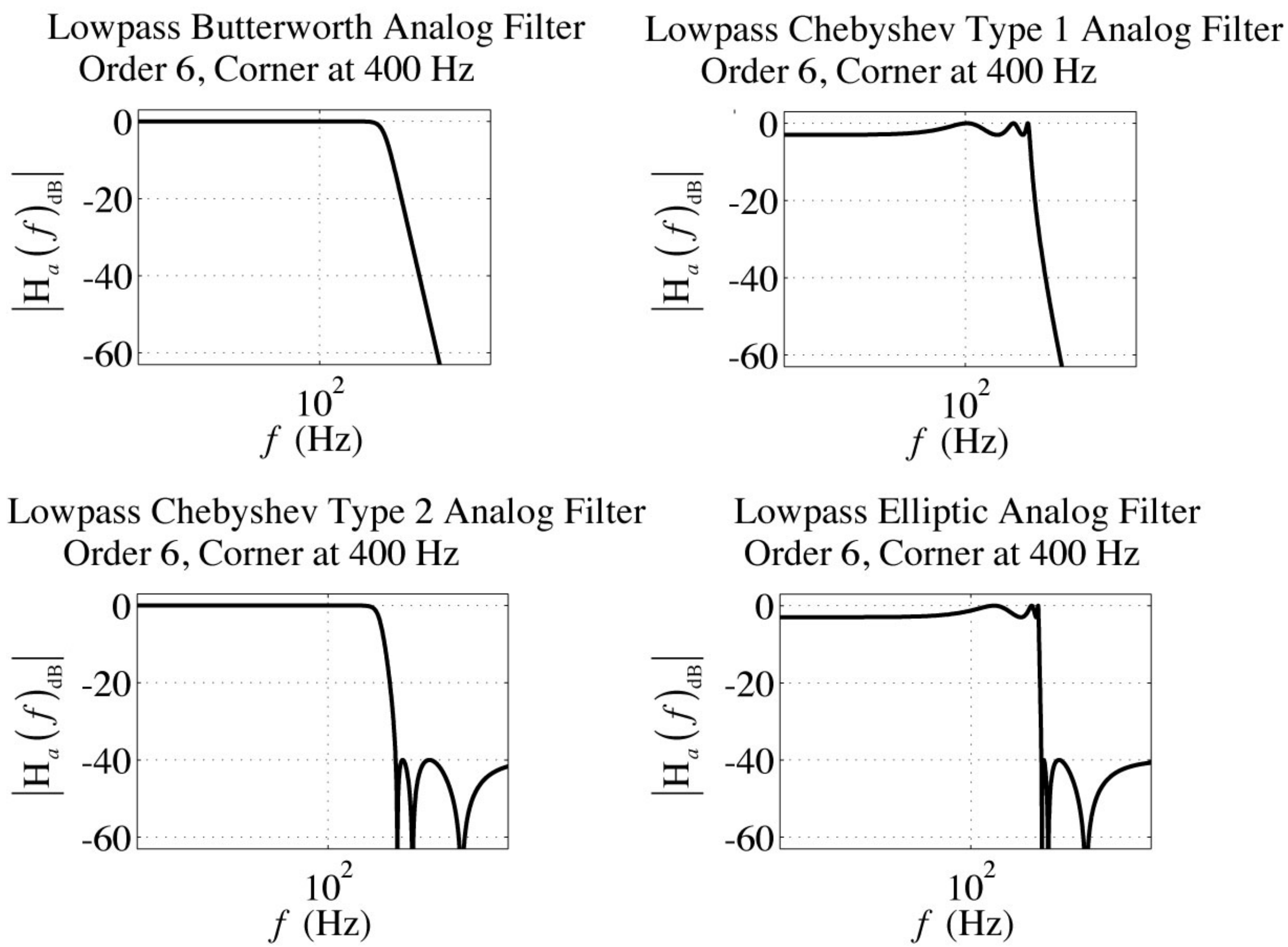
Example



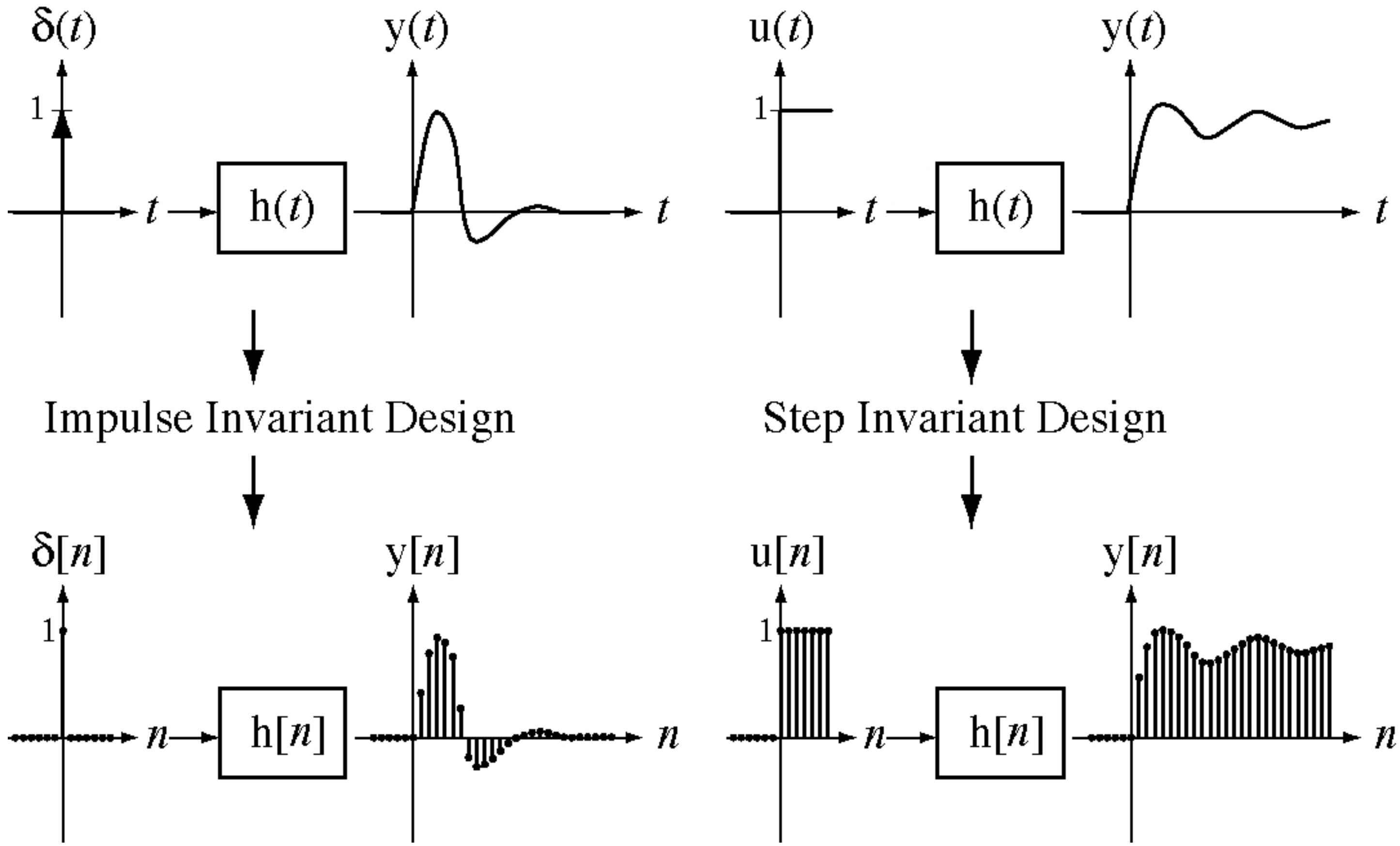
Chebyshev, Elliptic and Bessel Filters

The Butterworth filter is "maximally flat" in its passband. This means that its frequency response in the passband is monotonic and the slope approaches zero at the maximum response. The **Chebyshev** filter has **ripple** in either its passband or stopband depending on which type of Chebyshev filter it is. Ripple is a variation of the frequency response between two limits. Allowing this ripple in the passband or stopband lets the transition from pass to stop band be faster than for a Butterworth filter of the same order. The **Elliptic** (or **Cauer**) filter has ripple in both the passband and stopband and also has an even faster transition between passband and stopband than the Chebyshev filters.

Chebyshev, Elliptic and Bessel Filters



Impulse and Step Invariant Design



Impulse Invariant Design

Let $\mathrm{h}_\delta(t) = \mathrm{h}_a(t)\delta_{T_s}(t)$. Then

$$\mathrm{H}_\delta(j\omega) = f_s \sum_{k=-\infty}^{\infty} \mathrm{H}_a\left(j(\omega - k\omega_s)\right)$$

where $\mathrm{h}_a(t) \xleftrightarrow{\mathscr{L}} \mathrm{H}_a(s)$.

Also let $\mathrm{h}_d[n] = \mathrm{h}_a(nT_s)$. Then, using $\omega/f_s = \Omega \Rightarrow \omega = f_s\Omega$

$$\mathrm{H}_d\left(e^{j\Omega}\right) = f_s \sum_{k=-\infty}^{\infty} \mathrm{H}_a\left(jf_s(\Omega - 2\pi k)\right)$$

Impulse Invariant Design

In the expression

$$\mathrm{H}_d\left(e^{j\Omega}\right) = f_s \sum_{k=-\infty}^{\infty} \mathrm{H}_a\left(jf_s\left(\Omega - 2\pi k\right)\right)$$

it is clear that the digital filter's frequency response consists of multiple scaled aliases of the analog filter's frequency response and, to the extent that the aliases overlap, the two frequency responses must differ.

Impulse Invariant Design

As an example of impulse invariant design let

$$\mathrm{H}_a(s) = \frac{A\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}.$$

Then $\mathrm{h}_a(t) = \sqrt{2}A\omega_c e^{-\omega_c t/\sqrt{2}} \sin\left(\omega_c t/\sqrt{2}\right)\mathrm{u}(t)$. Sample f_s times per second to form $\mathrm{h}_d[n] = \sqrt{2}A\omega_c e^{-\omega_c n T_s/\sqrt{2}} \sin\left(\omega_c n T_s/\sqrt{2}\right)\mathrm{u}[n]$.

Then

$$\mathrm{H}_d(z) = \sqrt{2}A\omega_c \frac{z e^{-\omega_c T_s/\sqrt{2}} \sin\left(\omega_c T_s/\sqrt{2}\right)}{z^2 - 2e^{-\omega_c T_s/\sqrt{2}} \cos\left(\omega_c T_s/\sqrt{2}\right) z + e^{-2\omega_c T_s/\sqrt{2}}}$$

and

$$\mathrm{H}_d\left(e^{j\Omega}\right) = \sqrt{2}A\omega_c \frac{e^{j\Omega} e^{-\omega_c T_s/\sqrt{2}} \sin\left(\omega_c T_s/\sqrt{2}\right)}{e^{j2\Omega} - 2e^{-\omega_c T_s/\sqrt{2}} \cos\left(\omega_c T_s/\sqrt{2}\right) e^{j\Omega} + e^{-2\omega_c T_s/\sqrt{2}}}$$

Impulse Invariant Design

The digital filter's frequency response is

$$\mathrm{H}_d\left(e^{j\Omega}\right)=\sqrt{2}A\omega_c\frac{e^{j\Omega}e^{-\omega_c T_s/\sqrt{2}}\sin\left(\omega_c T_s/\sqrt{2}\right)}{e^{j2\Omega}-2e^{-\omega_c T_s/\sqrt{2}}\cos\left(\omega_c T_s/\sqrt{2}\right)e^{j\Omega}+e^{-2\omega_c T_s/\sqrt{2}}}$$

and, from a previous slide, $\mathrm{H}_d\left(e^{j\Omega}\right)=f_s\sum_{k=-\infty}^{\infty}\mathrm{H}_a\left(jf_s\left(\Omega-2\pi k\right)\right)$.

Therefore

$$\mathrm{H}_d\left(e^{j\Omega}\right)=f_s\sum_{k=-\infty}^{\infty}\frac{A\omega_c^2}{\left[jf_s\left(\Omega-2\pi k\right)\right]^2+j\sqrt{2}\omega_c f_s\left(\Omega-2\pi k\right)+\omega_c^2}$$

$$=\sqrt{2}A\omega_c\frac{e^{j\Omega}e^{-\omega_c T_s/\sqrt{2}}\sin\left(\omega_c T_s/\sqrt{2}\right)}{e^{j2\Omega}-2e^{-\omega_c T_s/\sqrt{2}}\cos\left(\omega_c T_s/\sqrt{2}\right)e^{j\Omega}+e^{-2\omega_c T_s/\sqrt{2}}}$$

Impulse Invariant Design

Let $A = 10$ and ω_c=100 and f_s=200. Then

$$\mathrm{H}_d\left(e^{j\Omega}\right) = 2000\sum_{k=-\infty}^{\infty}\frac{1}{\left[j2\left(\Omega-2\pi k\right)\right]^2 + j2\sqrt{2}\left(\Omega-2\pi k\right)+1}$$

and

$$\mathrm{H}_d\left(e^{j\Omega}\right) = 1000\sqrt{2}\,\frac{e^{j\Omega}e^{-1/2\sqrt{2}}\sin\left(1/2\sqrt{2}\right)}{e^{j2\Omega} - 2e^{-1/2\sqrt{2}}\cos\left(1/2\sqrt{2}\right)e^{j\Omega} + e^{-1/\sqrt{2}}}$$

At $\Omega = 0$,

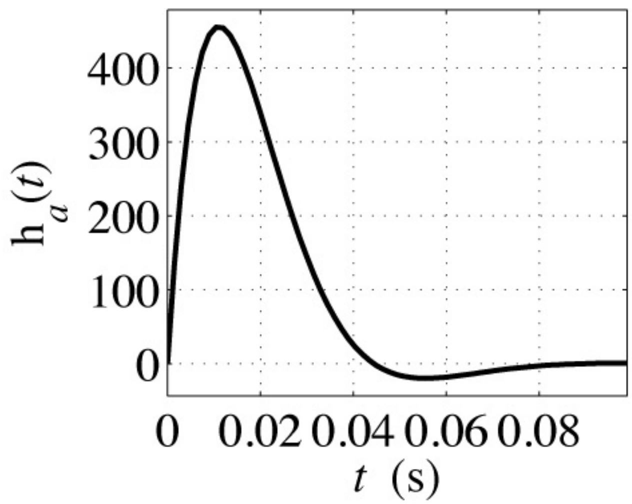
$$\mathrm{H}_d\left(1\right) = 2000\sum_{k=-\infty}^{\infty}\frac{1}{1-16\pi^2k^2 - j4\sqrt{2}\pi k} = 1958.5$$

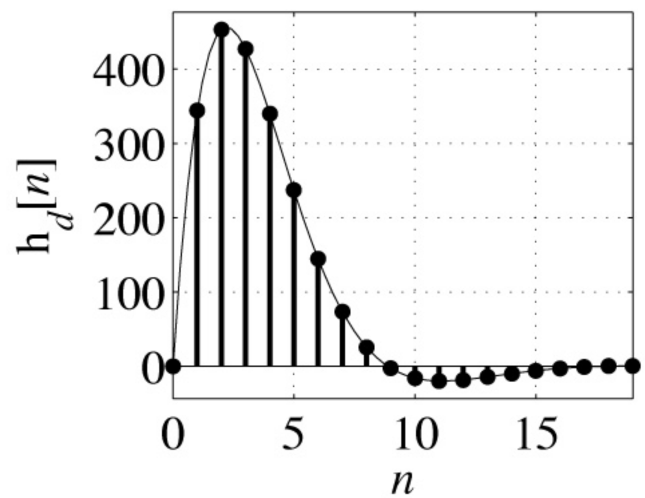
and

$$\mathrm{H}_d\left(1\right) = \frac{343.825}{1-1.31751+0.49306} = 1958.5$$

Impulse Invariant Design

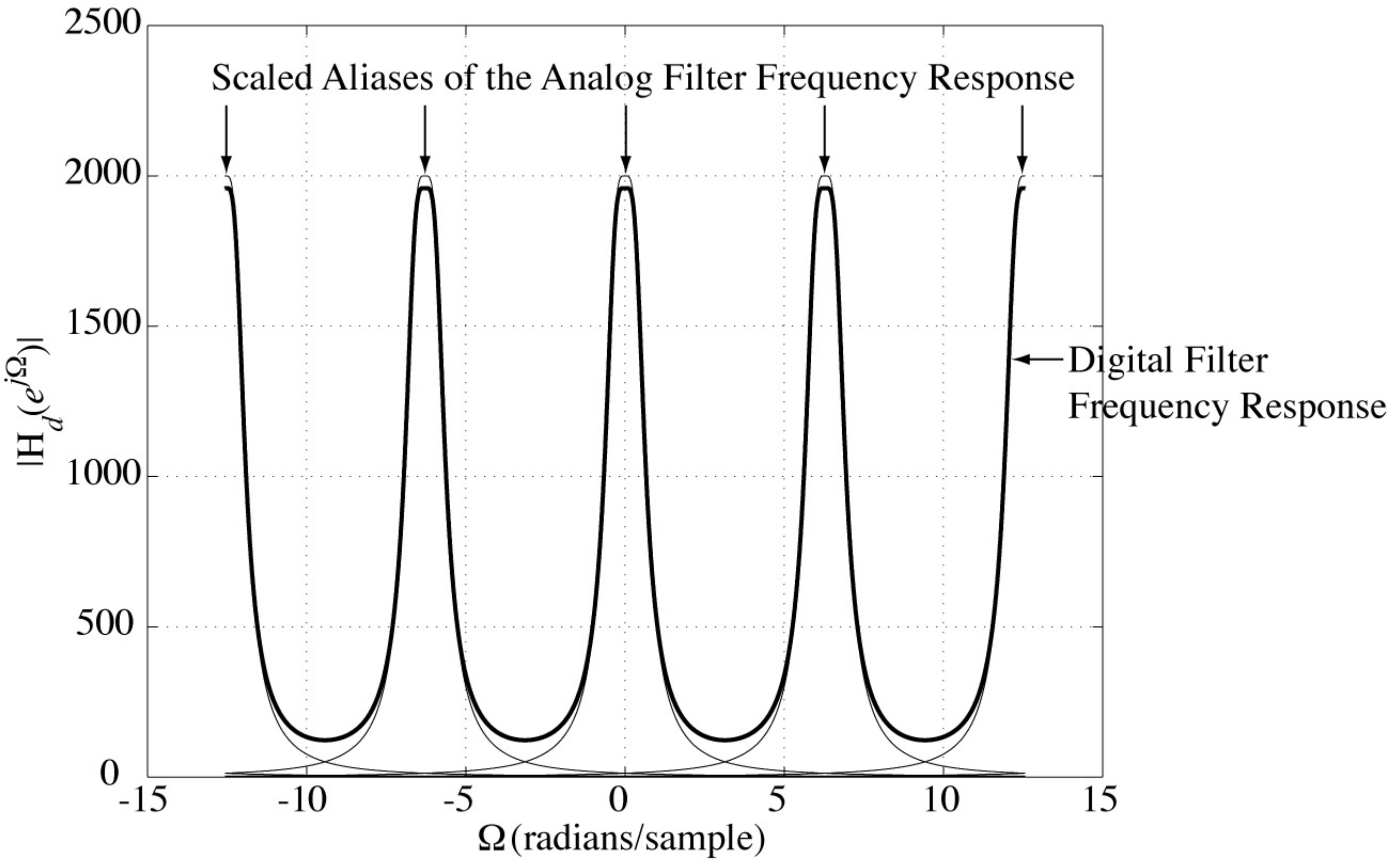
By design the digital filter's impulse response consists of samples of the analog filter's impulse response.





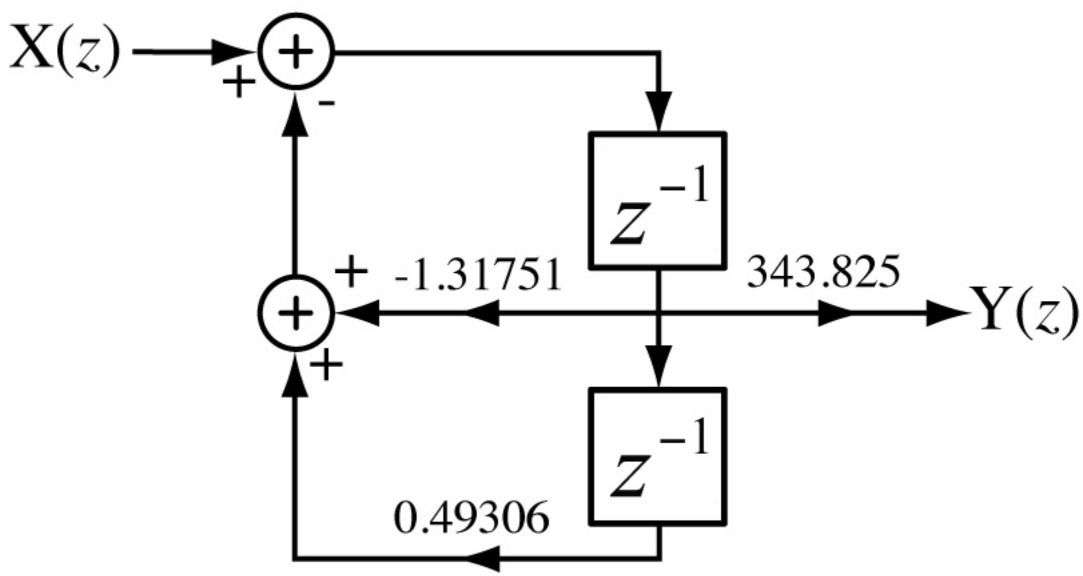
Impulse Invariant Design

The frequency response of the digital filter differs from the frequency response of the analog filter due to aliasing.



Impulse Invariant Design

Direct Form II Realization of the Digital Filter Designed by the Impulse Invariant Technique



Impulse Invariant Design

Let $\mathrm{H}_a(s) = \dfrac{A\omega_c}{s+\omega_c} \Rightarrow \mathrm{H}_a(j2\pi f) = \dfrac{A\omega_c}{j2\pi f + \omega_c} \Rightarrow \mathrm{H}_a(j\omega) = \dfrac{A\omega_c}{j\omega + \omega_c}$.

Then $\mathrm{h}_a(t) = A\omega_c e^{-\omega_c t}\,\mathrm{u}(t)$. Sample at rate f_s to form $\mathrm{h}_d[n] = A\omega_c e^{-\omega_c n T_s}\,\mathrm{u}[n]$

and $\mathrm{H}_d(z) = A\omega_c \dfrac{z}{z - e^{-\omega_c T_s}} \Rightarrow \mathrm{H}_d(e^{j\Omega}) = A\omega_c \dfrac{e^{j\Omega}}{e^{j\Omega} - e^{-\omega_c T_s}}$

$$\mathrm{H}_d(e^{j\Omega}) = f_s \sum_{k=-\infty}^{\infty} \frac{A\omega_c}{jf_s(\Omega - 2\pi k) + \omega_c} = A\omega_c \frac{e^{j\Omega}}{e^{j\Omega} - e^{-\omega_c T_s}}$$

Let $A = 10$, $\omega_c = 50$ and $f_s = 100$ and check equality at $\Omega = 0$.

$$f_s \sum_{k=-\infty}^{\infty} \frac{A\omega_c}{jf_s(\Omega - 2\pi k) + \omega_c} = \sum_{k=-\infty}^{\infty} \frac{50000}{-j200\pi k + 50} = 1020.7$$

$$A\omega_c \frac{e^{j\Omega}}{e^{j\Omega} - e^{-\omega_c T_s}} = 500 \frac{1}{1 - e^{-1/2}} = 1270.7$$

Impulse Invariant Design

Why are the two results

$$f_s \sum_{k=-\infty}^{\infty} \frac{A\omega_c}{jf_s\left(\Omega - 2\pi k\right) + \omega_c} = \sum_{k=-\infty}^{\infty} \frac{50000}{-j200\pi k + 50} = 1020.7$$

and

$$A\omega_c \frac{e^{j\Omega}}{e^{j\Omega} - e^{-\omega_c T_s}} = 500 \frac{1}{1 - e^{-1/2}} = 1270.7$$

different by almost 25%? The difference comes from sampling $\mathrm{h}_a\left(t\right)$ to form $\mathrm{h}_d\left[n\right] = A\omega_c e^{-\omega_c n T_s}\,\mathrm{u}\left[n\right]$. $\mathrm{h}_a\left(t\right)$ has a discontinuity at $t = 0$, a sample time. If we let the first sample be $A\omega_c / 2$ instead of $A\omega_c$ the two expressions for $\mathrm{H}_d\left(e^{j\Omega}\right)$ are now exactly equal. $A\omega_c / 2$ is the average of the two limits at the point of discontinuity. This is consistent with Fourier transform theory which says that the Fourier representation at a discontinuity is the average of the two limits.

Impulse Invariant Design

Let $\mathrm{H}_a(s) = \dfrac{9.87\times10^4 s^2}{s^4 + 444.3s^3 + 2.467\times10^6 s^2 + 5.262\times10^8 s + 1.403\times10^{12}}$

and let $f_s = 1000$.

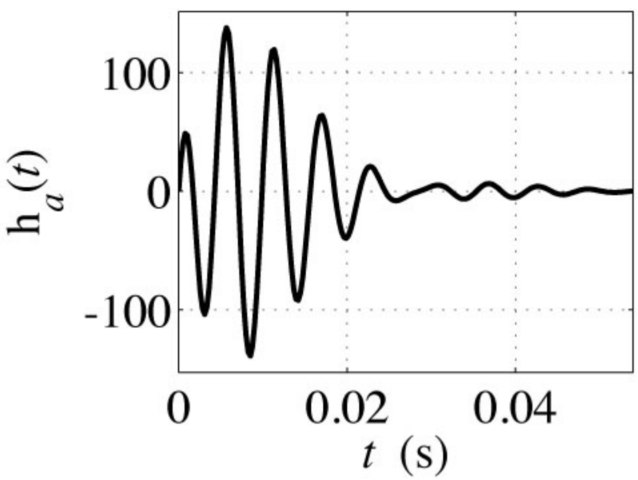
$$\mathrm{h}_a(t) = \left[246.07e^{-122.41t}\cos(1199.4t - 1.48) + 200.5e^{-99.74t}\cos(977.27t + 1.683)\right]\mathrm{u}(t)$$

$$\mathrm{h}_d[n] = \left[246.07e^{-0.12241n}\cos(1.1994n - 1.48) + 200.5e^{-0.09974n}\cos(0.97727n + 1.683)\right]\mathrm{u}[n]$$

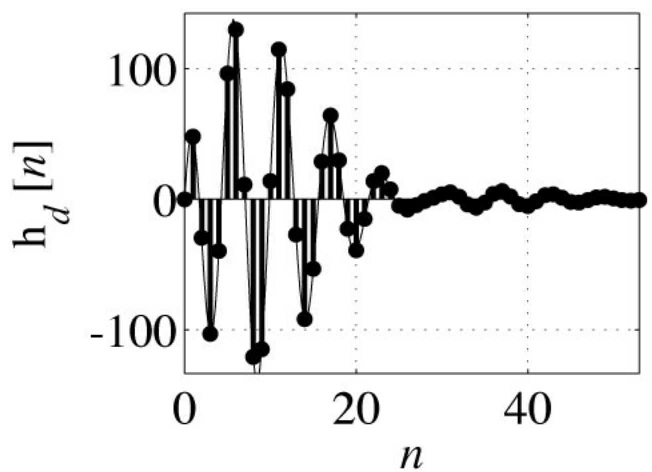
$$\mathrm{H}_d(z) = \frac{48.4z^3 - 107.7z^2 + 51.46z}{z^4 - 1.655z^3 + 2.252z^2 - 1.319z + 0.6413}$$

Impulse Invariant Design

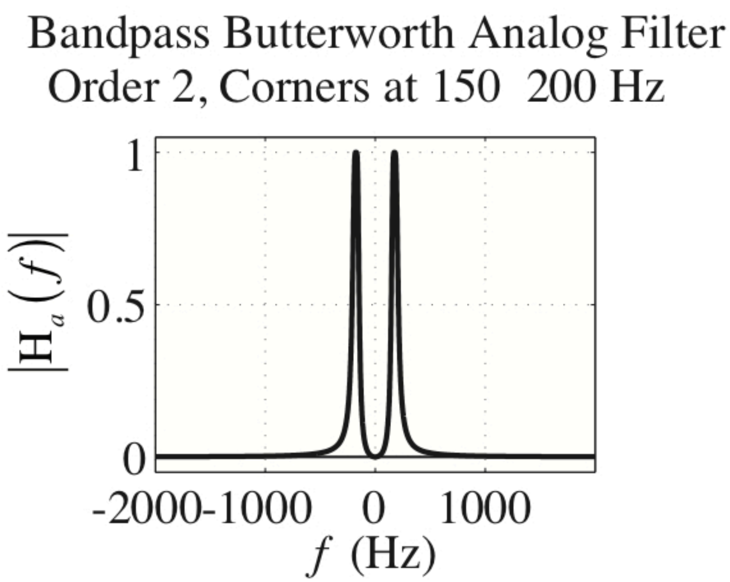
Analog Filter Impulse Response

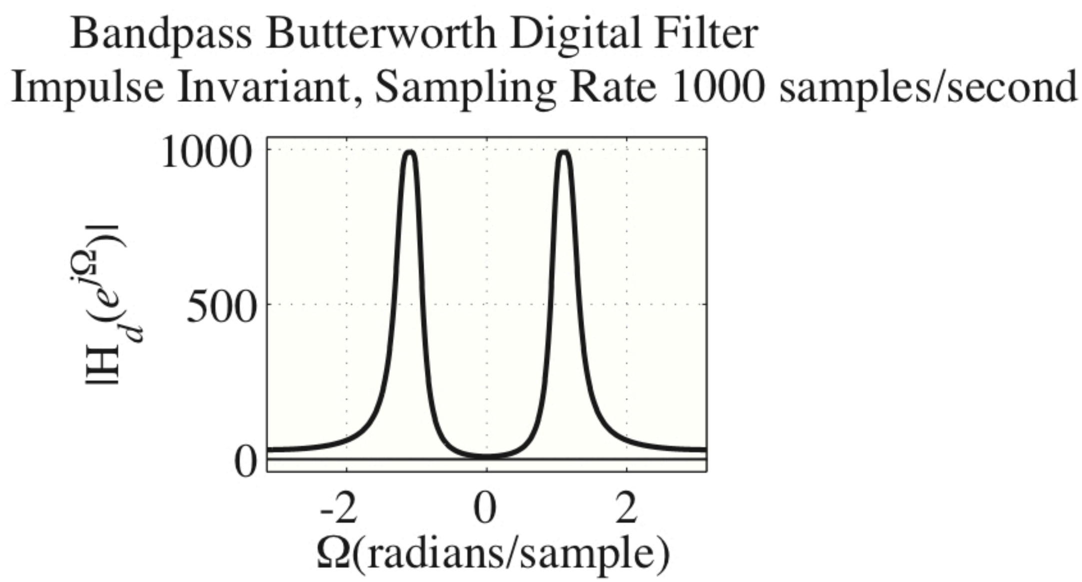


Digital Filter Impulse Response - Impulse Invariant

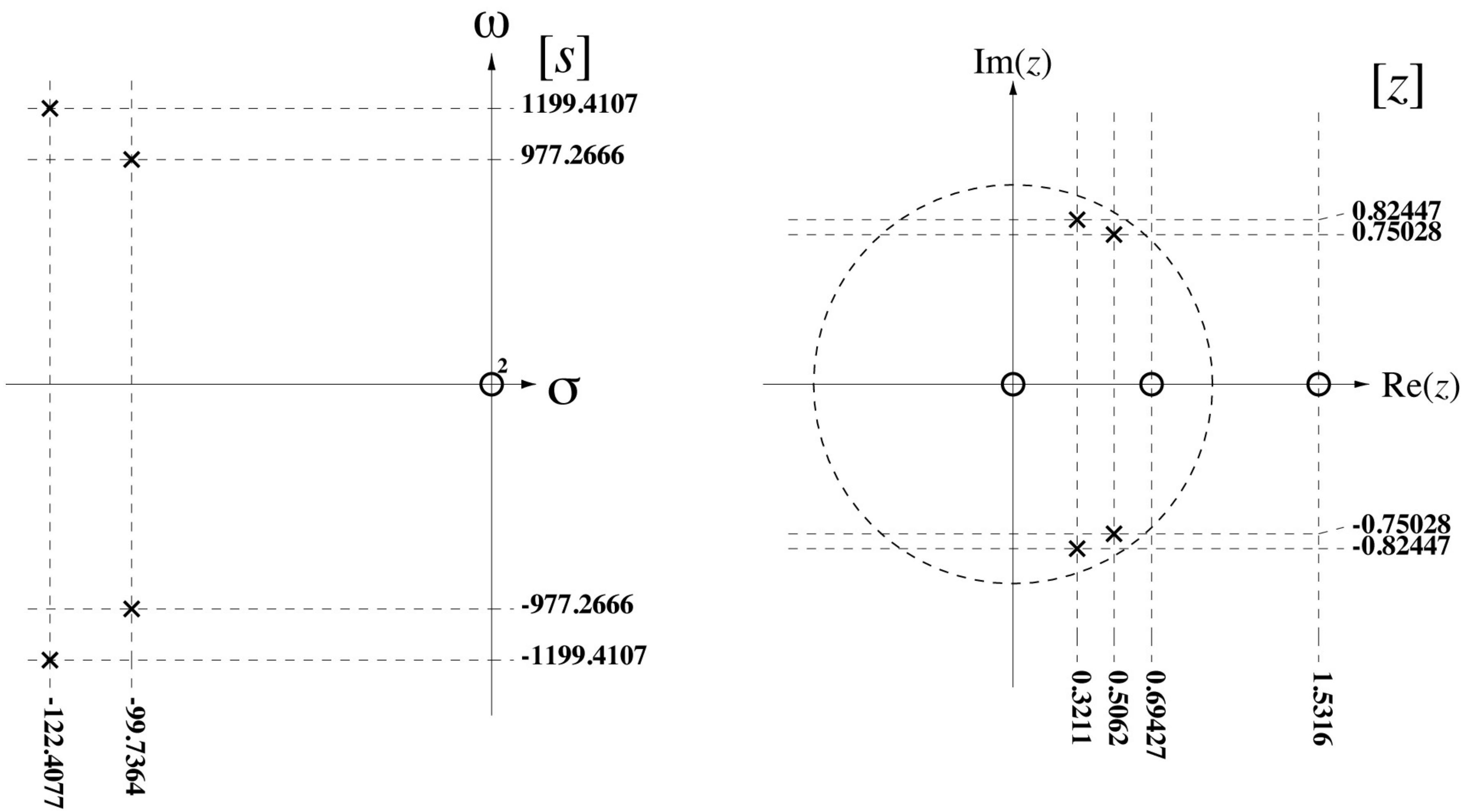


Impulse Invariant Design





Impulse Invariant Design



Step Invariant Design

In step invariant design the analog unit step response is sampled to form the digital unit sequence response.

$$h_{-1a}(t) = \mathcal{L}^{-1}\left(\frac{H_a(s)}{s}\right) \Rightarrow h_{-1d}[n] = h_{-1,a}(nT_s)$$

$$\mathcal{Z}\left(h_{-1d}[n]\right) = \frac{z}{z-1} H_d(z) \Rightarrow H_d(z) = \frac{z-1}{z} \mathcal{Z}\left(h_{-1d}[n]\right)$$

Let $H_a(s) = \dfrac{9.87 \times 10^4 s^2}{s^4 + 444.3s^3 + 2.467 \times 10^6 s^2 + 5.262 \times 10^8 s + 1.403 \times 10^{12}}$

and let $f_s = 1000$.

Step Invariant Design

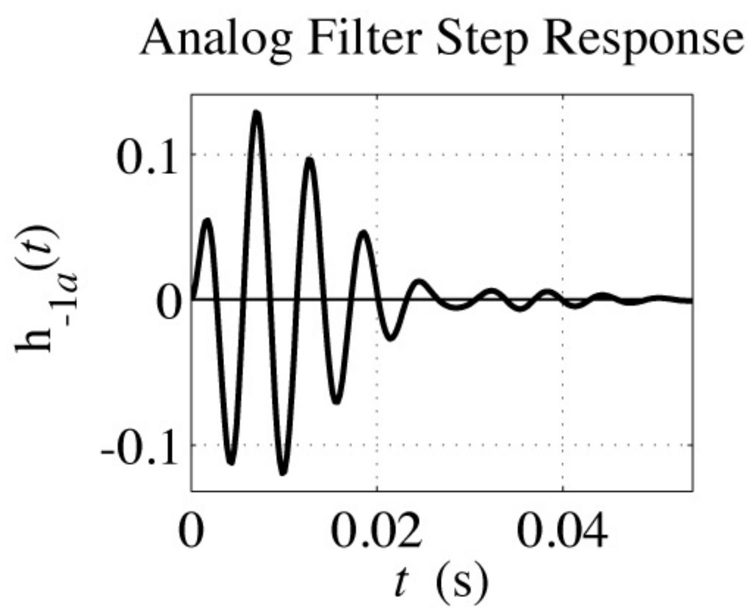
$$\mathrm{h}_{-1a}(t)=\begin{bmatrix}0.2041e^{-122.408t}\cos(1199.4t+3.1312)\\+0.2041e^{-99.74t}\cos(977.27t+0.01042)\end{bmatrix}\mathrm{u}(t)$$

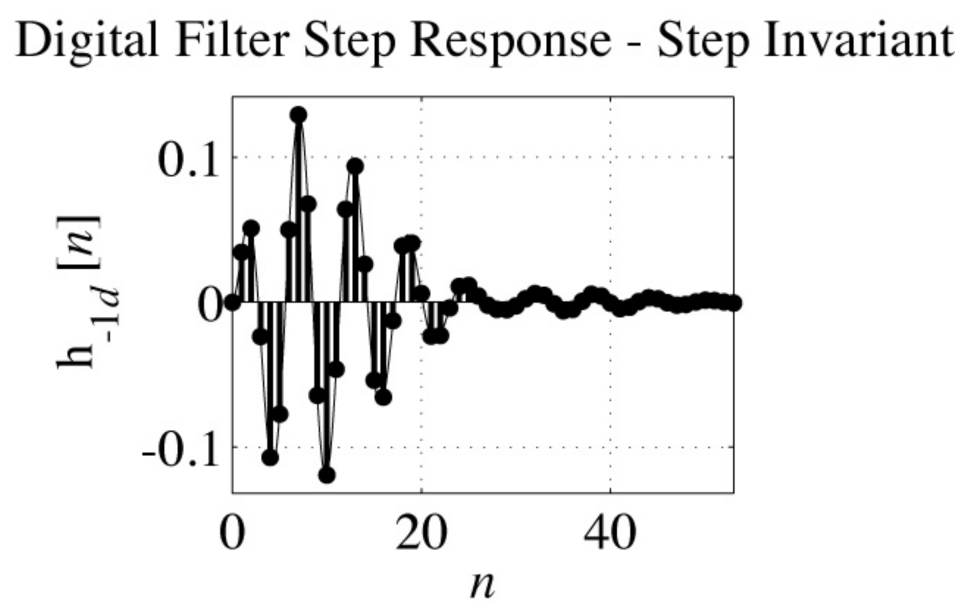
$$\mathrm{h}_{-1d}[n]=\begin{bmatrix}0.2041(0.8847)^n\cos(1.1994n+3.1312)\\+0.2041(0.9051)^n\cos(0.97727n+0.0102)\end{bmatrix}\mathrm{u}[n]$$

$$\mathrm{H}_d(z)=\frac{0.03443z^3-0.03905z^2-0.02527z+0.02988}{z^4-1.655z^3+2.252z^2-1.319z+0.6413}$$

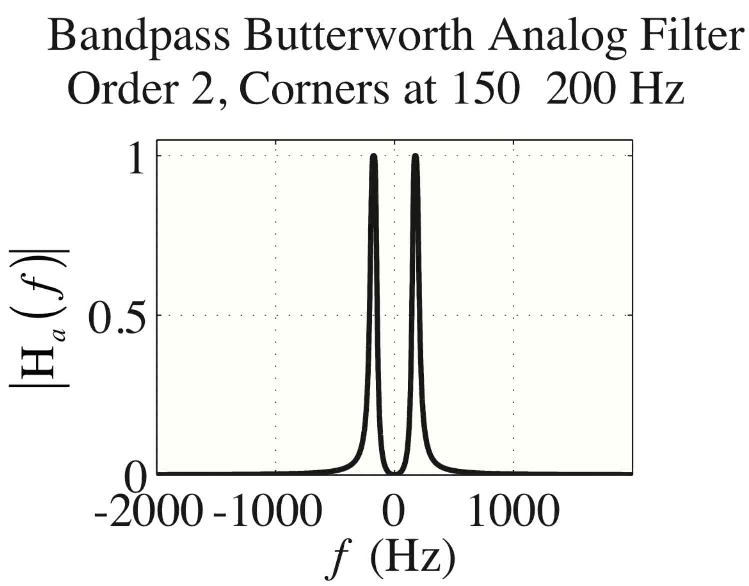
$$\begin{aligned}\mathrm{y}[n]=&\ 0.03443\,\mathrm{x}[n-1]-0.03905\,\mathrm{x}[n-2]-0.02527\,\mathrm{x}[n-2]\\&+0.02988\,\mathrm{x}[n-4]+1.655\mathrm{y}[n-1]-2.252\,\mathrm{y}[n-2]\\&+1.319\,\mathrm{y}[n-3]-0.6413\mathrm{y}[n-4]\end{aligned}$$

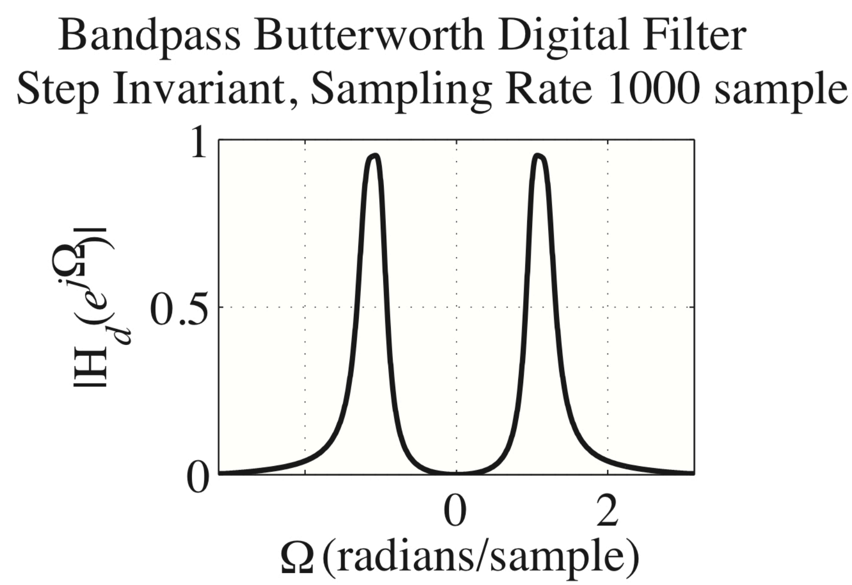
Step Invariant Design



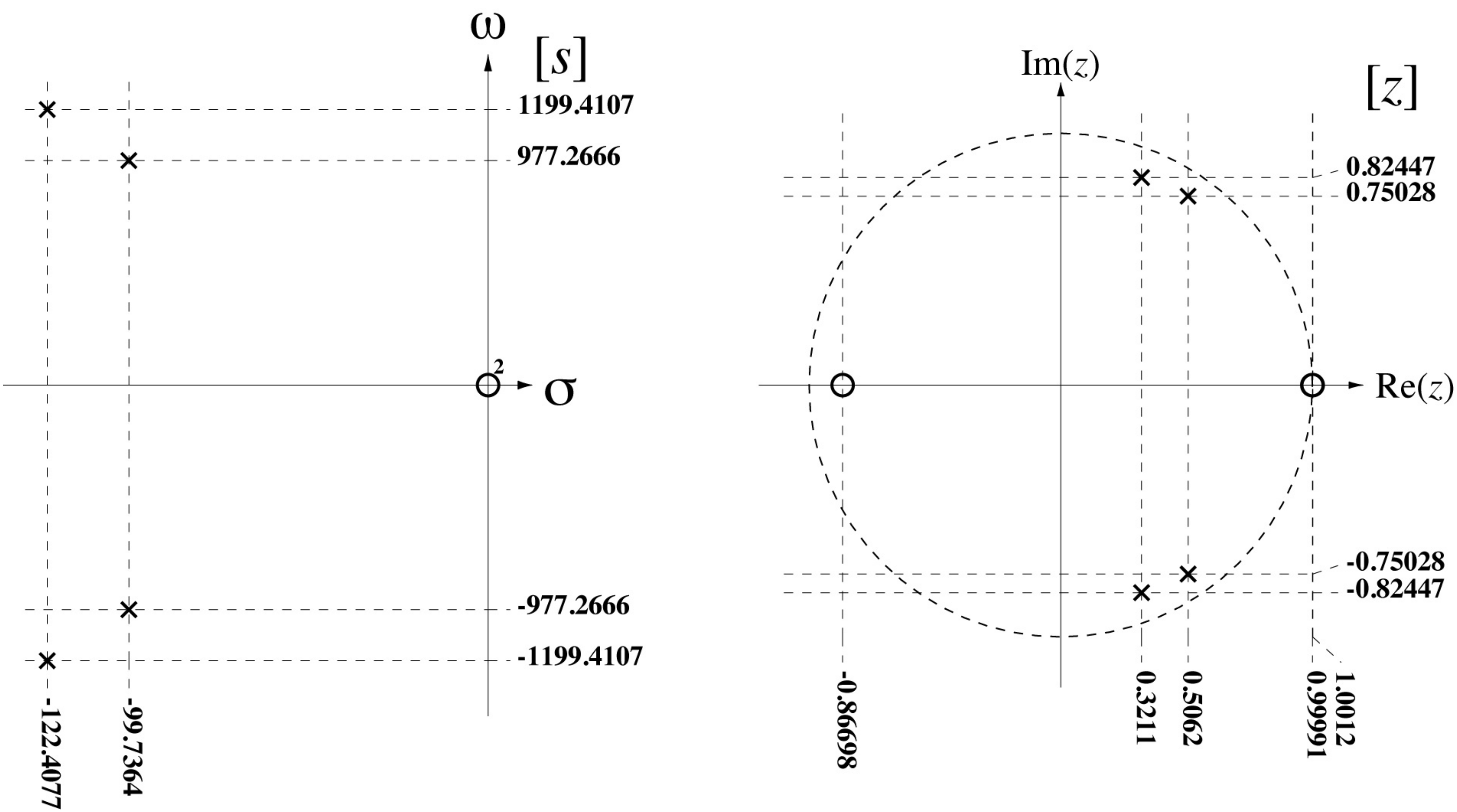


Step Invariant Design





Step Invariant Design



Finite-Difference Design

Finite-difference design is based on approximating continuous-time derivatives by finite differences. The transfer function $H_a(s) = \frac{1}{s+a}$ implies the differential equation $\frac{d}{dt}\left(y_a(t)\right) + a\,y_a(t) = x_a(t)$. The derivative can be approximated by $\frac{d}{dt}\left(y_a(t)\right) \cong \frac{y_d[n+1] - y_d[n]}{T_s}$. Then

$\frac{y_d[n+1] - y_d[n]}{T_s} + a\,y_d[n] = x_d[n]$ and the differential equation in continuous time has become a difference equation in discrete time. Then we can form the discrete-time transfer function $H_d(z) = \frac{Y_d(z)}{X_d(z)} = \frac{T_s}{z - (1 - aT_s)}$.

Finite-Difference Design

A derivative can be approximated by a forward difference

$$\frac{d}{dt}\left(\mathrm{y}_a(t)\right) \cong \frac{\mathrm{y}_d[n+1]-\mathrm{y}_d[n]}{T_s} \xleftrightarrow{\mathcal{Z}} \frac{z-1}{T_s}\mathrm{Y}_d(z)$$

a backward difference

$$\frac{d}{dt}\left(\mathrm{y}(t)\right) \cong \frac{\mathrm{y}_d[n]-\mathrm{y}_d[n-1]}{T_s} \xleftrightarrow{\mathcal{Z}} \frac{1-z^{-1}}{T_s}\mathrm{Y}_d(z) = \frac{z-1}{zT_s}\mathrm{Y}_d(z)$$

or a central difference

$$\frac{d}{dt}\left(\mathrm{y}_a(t)\right) \cong \frac{\mathrm{y}_d[n+1]-\mathrm{y}_d[n-1]}{2T_s} \xleftrightarrow{\mathcal{Z}} \frac{z-z^{-1}}{2T_s}\mathrm{Y}_d(z) = \frac{z^2-1}{2zT_s}\mathrm{Y}_d(z).$$

Since every multiplication by s represents a differentiation in time, finite difference design can be done by simply replacing s with a z-transform expression for a difference that approximates a derivative.

Finite-Difference Design

Using a forward difference,

$$\mathrm{H}_d(z) = \mathrm{H}_a(s) = \left[\frac{1}{s+a}\right]_{s\to\frac{z-1}{T_s}} = \frac{1}{\frac{z-1}{T_s}+a} = T_s\frac{1}{z-(1-aT_s)}$$

Using a backward difference,

$$\mathrm{H}_d(z) = \mathrm{H}_a(s) = \left[\frac{1}{s+a}\right]_{s\to\frac{z-1}{zT_s}} = \frac{1}{\frac{z-1}{zT_s}+a} = \frac{T_s}{1+aT_s}\frac{z}{z-\frac{1}{1+aT_s}}$$

Using a central difference,

$$\mathrm{H}_d(z) = \mathrm{H}_a(s) = \left[\frac{1}{s+a}\right]_{s\to\frac{z^2-1}{2zT_s}} = \frac{1}{\frac{z^2-1}{2zT_s}+a} = 2T_s\frac{z}{z^2+2aT_s z-1}$$

Finite-Difference Design

Using the finite-difference technique it is possible to approximate a stable continuous-time filter by an unstable discrete-time filter.

In $\mathrm{H}_a(s) = \dfrac{1}{s+a}$ the pole is at $s = -a$. In the digital filter

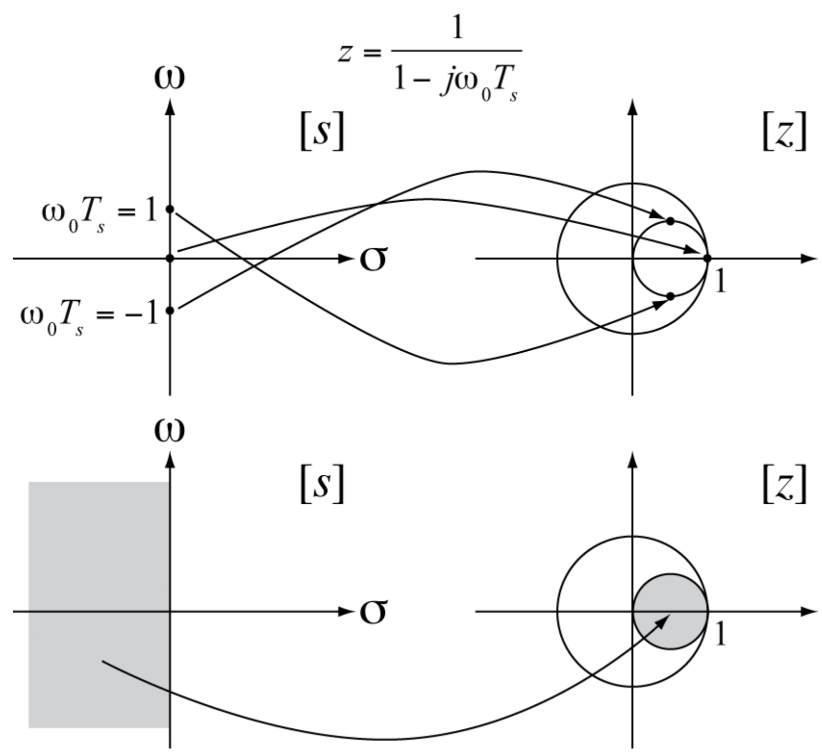
$\mathrm{H}_d(z) = T_s \dfrac{1}{z-(1-aT_s)}$ the pole is at $z = (1-aT_s)$. If $\mathrm{H}_a(s)$ is stable, $a > 0$ and $1-aT_s$ is on the real axis of the z plane. If $aT_s \geq 2$ the digital filter is unstable. One way to stabilize the design is to use a smaller value for T_s, which implies a higher sampling rate. This design was based on using a forward difference.

Finite-Difference Design

If we use a backward difference instead of a forward difference we get the digital filter

$$\mathrm{H}_d(z) = \mathrm{H}_a(s) = \left[\frac{1}{s+a}\right]_{s \to \frac{z-1}{zT_s}}$$

$$= \frac{T_s}{1+aT_s}\frac{z}{z-1/(1+aT_s)}.$$



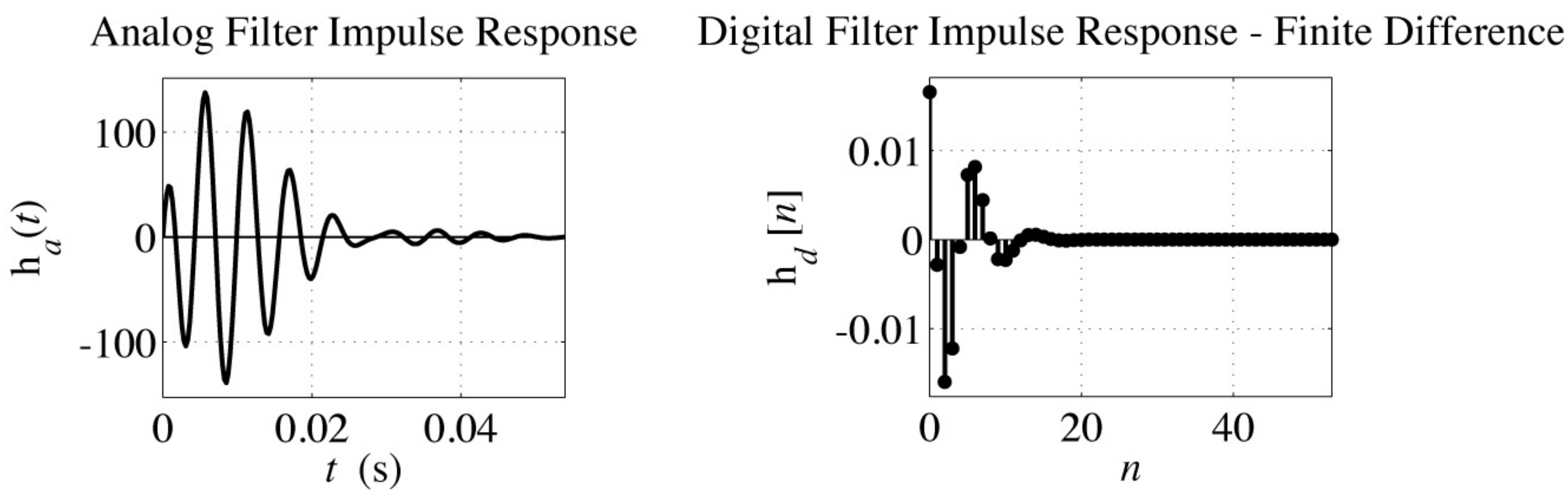
The pole is now at $z = 1/(1+aT_s)$ and is guaranteed to be inside the unit circle in the z plane, making this filter stable for any $a > 0$ and any T_s. Using a backward difference in any digital filter design guarantees stability, but that guarantee comes at a price. The pole locations are restricted to a small region inside the unit circle.

Finite-Difference Design

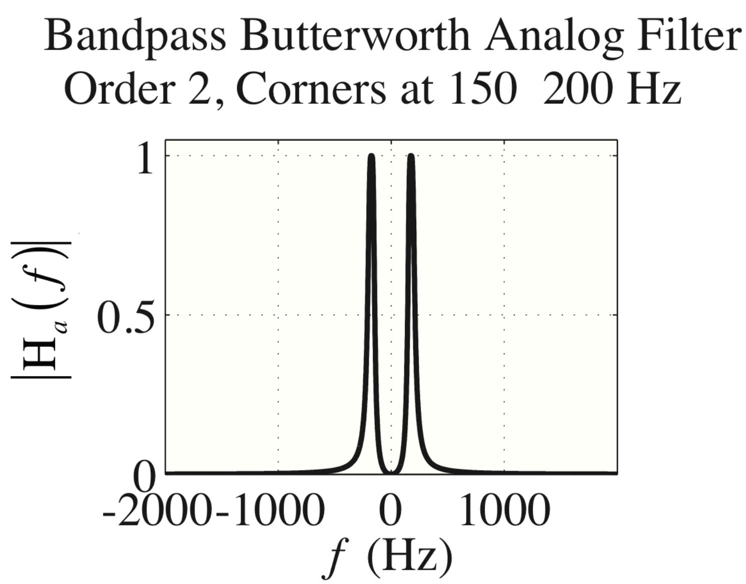
Let $\mathrm{H}_a(s) = \dfrac{9.87\times10^4 s^2}{s^4 + 444.3s^3 + 2.467\times10^6 s^2 + 5.262\times10^8 s + 1.403\times10^{12}}$

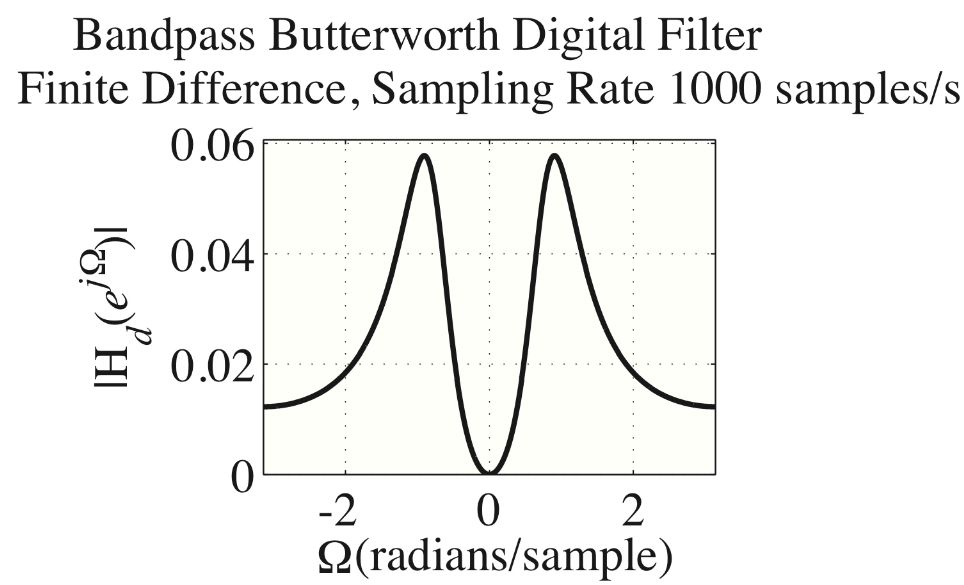
and design a digital filter using the finite-difference technique with $f_s = 1000$.

$$\mathrm{H}_d(z) = \frac{0.169z^2(z-1)^2}{z^4 - 1.848z^3 + 1.678z^2 - 0.7609z + 0.1712}$$

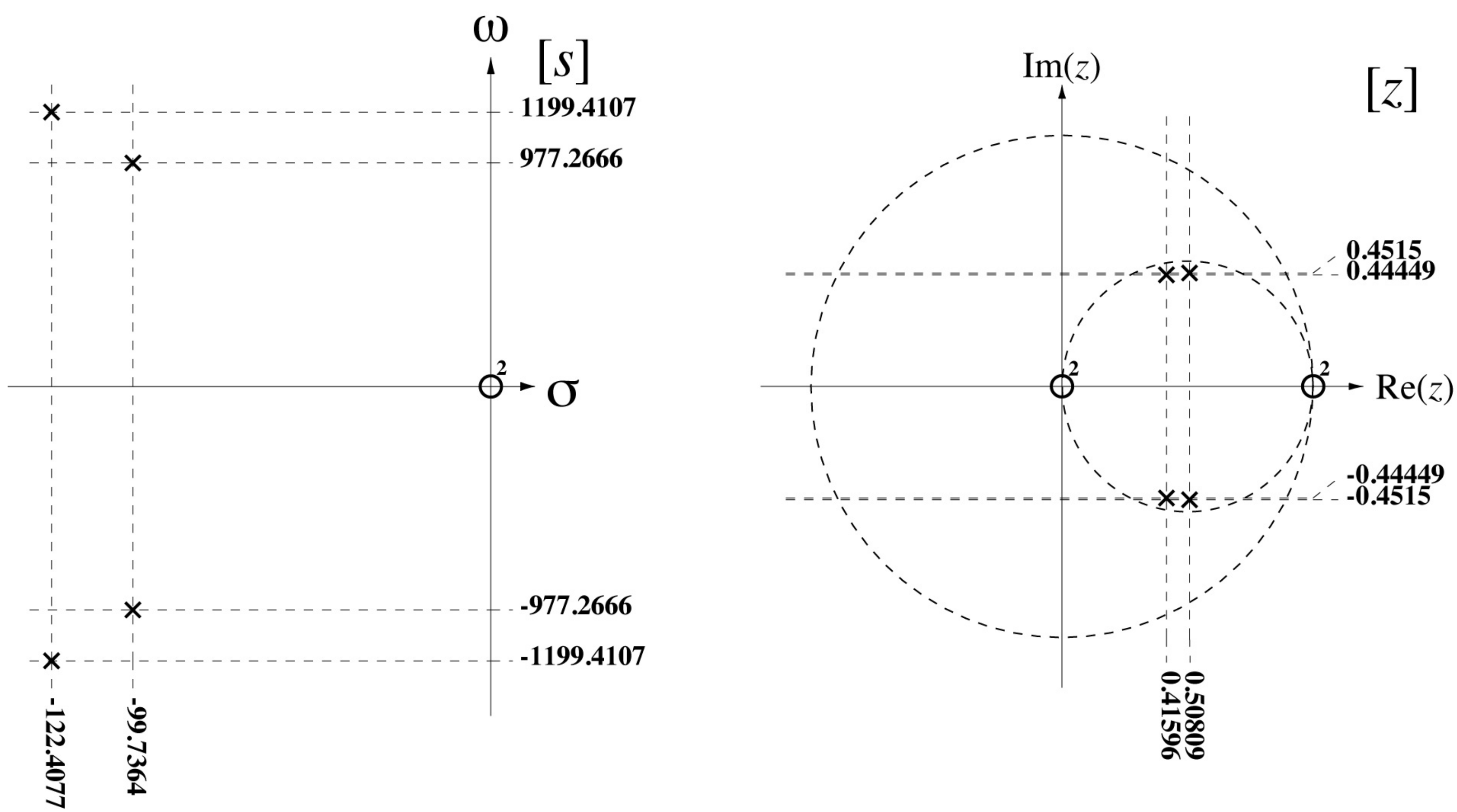


Finite-Difference Design



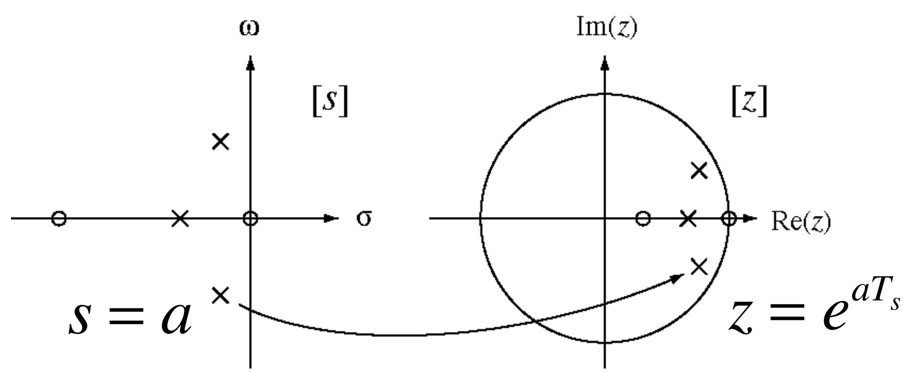


Finite-Difference Design



Matched z and Direct Substitution

The matched-z and direct substitution digital filter design techniques use the correspondence $z = e^{sT_s}$ to map the poles and zeros of the analog filter to corresponding locations in the z plane. If the analog filter has a pole at $s = a$, the corresonding digital filter pole would be at $z = e^{aT_s}$. The matched-z method replaces every factor of the form $s - a$ with the factor $1 - e^{aT_s} z^{-1}$ or $\dfrac{z - e^{aT_s}}{z}$ and the direct substitution method replaces it with the factor $z - e^{aT_s}$. Zeros are mapped in the same way.



Matched z and Direct Substitution

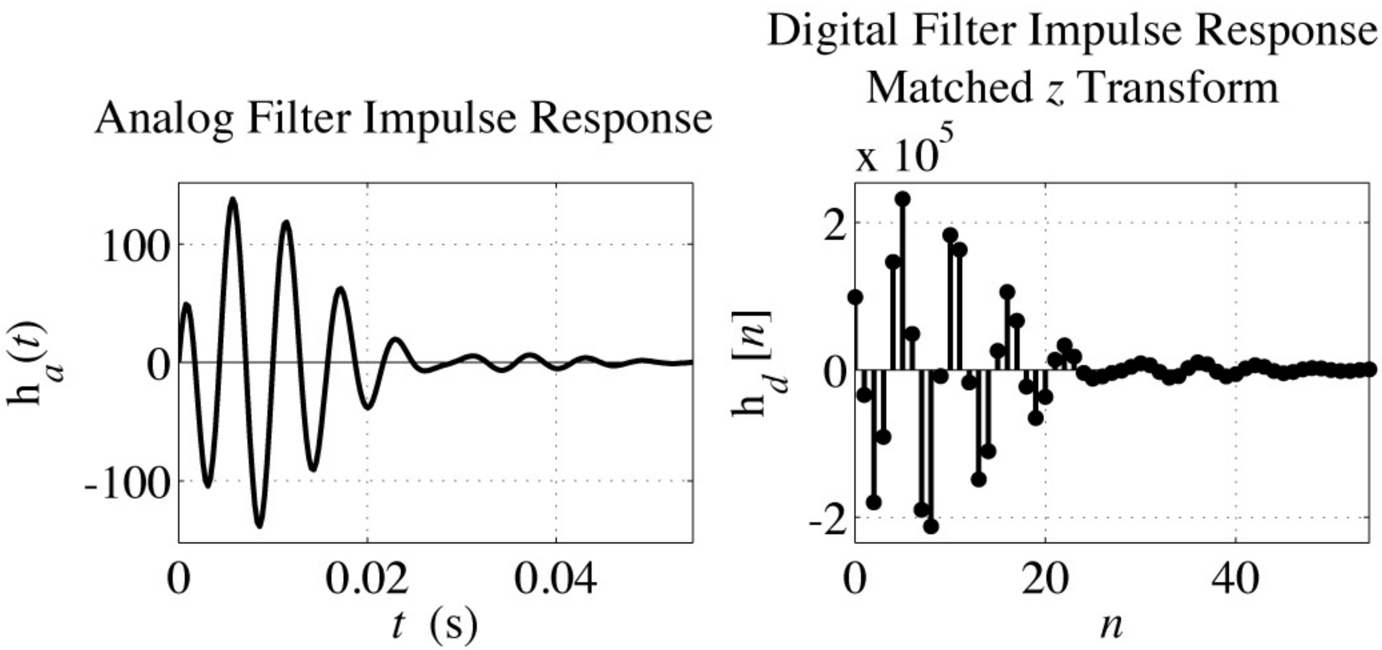
$$\text{Let } \mathrm{H}_a(s) = \frac{9.87\times10^4 s^2}{s^4 + 444.3s^3 + 2.467\times10^6 s^2 + 5.262\times10^8 s + 1.403\times10^{12}} .$$

Then, using the matched-z technique

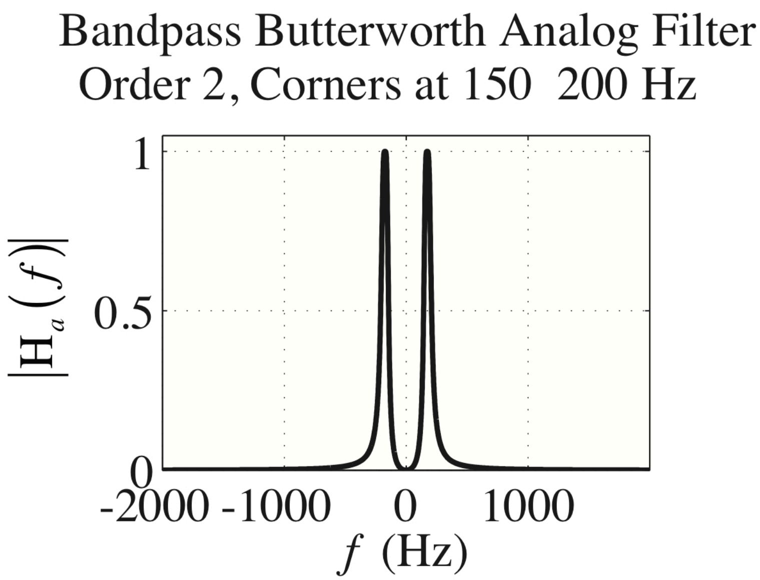
$$\mathrm{H}_d(z) = 98700\frac{z^2(z-1)^2}{z^4 - 1.655z^3 + 2.252z^2 - 1.319z + 0.6413}$$

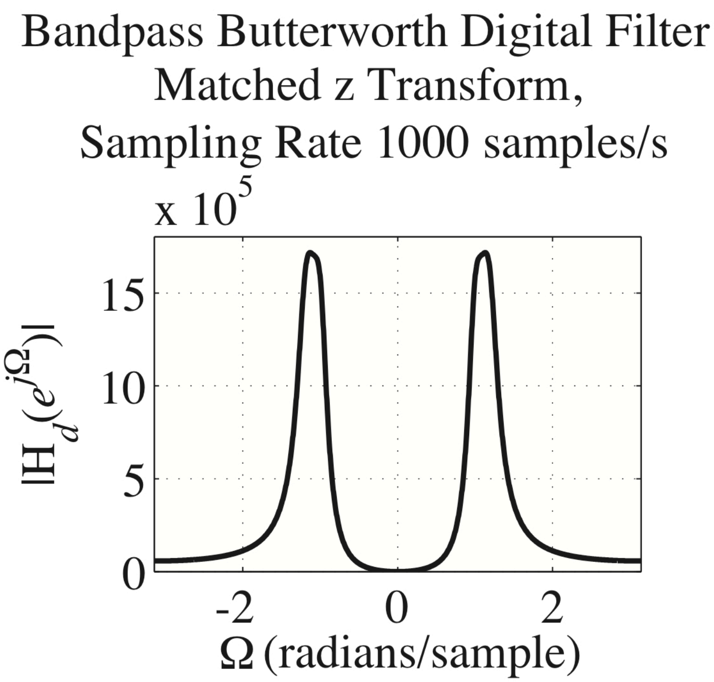
$$\mathrm{y}[n] = 98700\,\mathrm{x}[n] - 197400\,\mathrm{x}[n-1] + 98700\,\mathrm{x}[n-2]$$
$$+1.655\,\mathrm{y}[n-1] - 2.252\,\mathrm{y}[n-2] + 1.319\,\mathrm{y}[n-3] - 0.6413\,\mathrm{y}[n-4]$$

Matched *z* and Direct Substitution

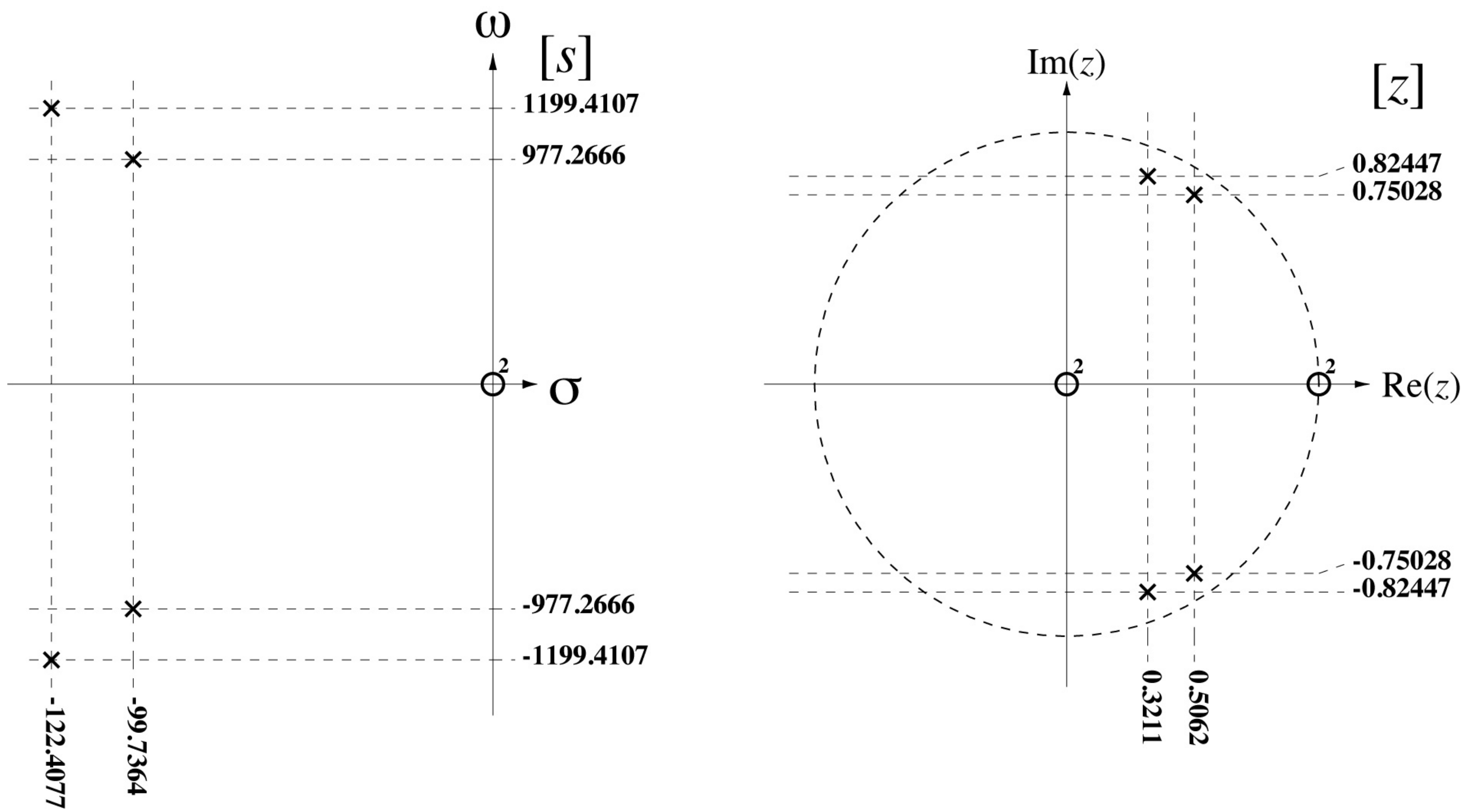


Matched z and Direct Substitution





Matched *z* and Direct Substitution



The Bilinear *z* Method

One approach to digital filter design would be to use the mapping $z = e^{sT_s} \Rightarrow s \to (1/T_s)\ln(z)$ to convert the s-domain expression to the z-domain expression

$$\mathrm{H}_d(z) = \mathrm{H}_a(s)\Big|_{s \to (1/T_s)\ln(z)} .$$

Unfortunately this creates a transcendental function of z with infinitely many poles. We can simplify the result by making an approximation to the exponential function. The exponential function is defined by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The Bilinear z Transform

We can approximate the exponential function by truncating its series definition to two terms $e^x \cong 1 + x$. Then $z = 1 + sT_s$ or $s \to \frac{z-1}{T_s}$. This is exactly the same as the finite difference technique using forward differences and has the same drawback, the possibility of converting a stable analog filter into an unstable digital filter. A simple but very significant variation on this idea is to express e^{sT_s} in the form $\frac{e^{sT_s/2}}{e^{-sT_s/2}}$ and then approximate both exponentials with the first two terms of the series. Then $z = \frac{1 + sT_s/2}{1 - sT_s/2}$ or $s \to \frac{2}{T_s}\frac{z-1}{z+1}$. This is the basis of the **bilinear** digital filter design technique.

The Bilinear z Transform

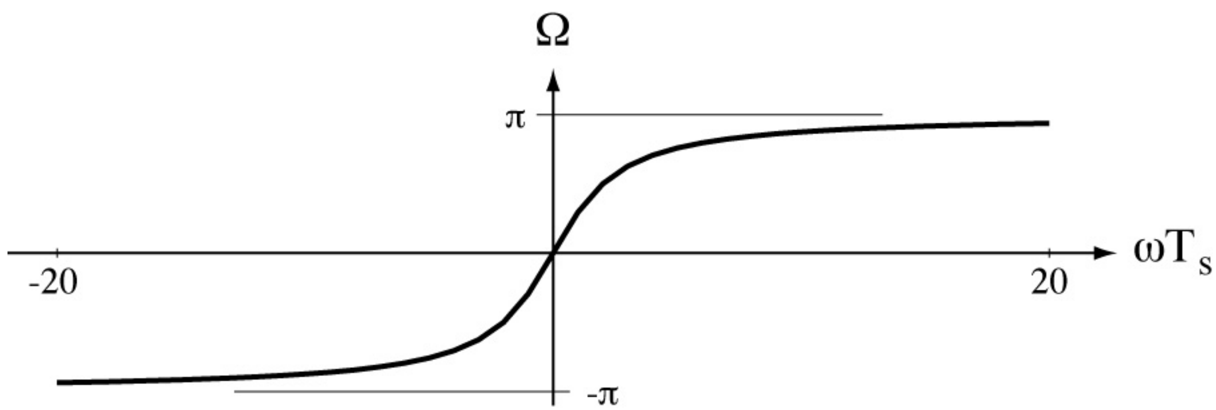
In the mapping $s \rightarrow \frac{2}{T_s}\frac{z-1}{z+1}$ let $s = j\omega$. Then

$$z = \frac{2+sT_s}{2-sT_s} = \frac{2+j\omega T_s}{2-j\omega T_s} = 1\measuredangle \tan^{-1}\left(\omega T_s/2\right) = e^{j2\tan^{-1}\left(\omega T_s/2\right)}$$

The ω axis of the s plane maps into the unit circle of the z plane. If $s = \sigma + j\omega$, $\sigma < 0$, the mapping is from the left half of the s plane to the interior of the unit circle in the z plane. If $s = \sigma + j\omega$, $\sigma > 0$, the mapping is from the right half of the s plane to the exterior of the unit circle in the z plane.

The Bilinear z Transform

For real discrete-time frequencies Ω the correspondence between the s and z planes is $s = \frac{2}{T_s}\frac{e^{j\Omega}-1}{e^{j\Omega}+1} = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right)$. Then, since $s = \sigma + j\omega$, $\sigma = 0$ and $\omega = (2/T_s)\tan(\Omega/2)$ or $\Omega = 2\tan^{-1}(\omega T_s/2)$.

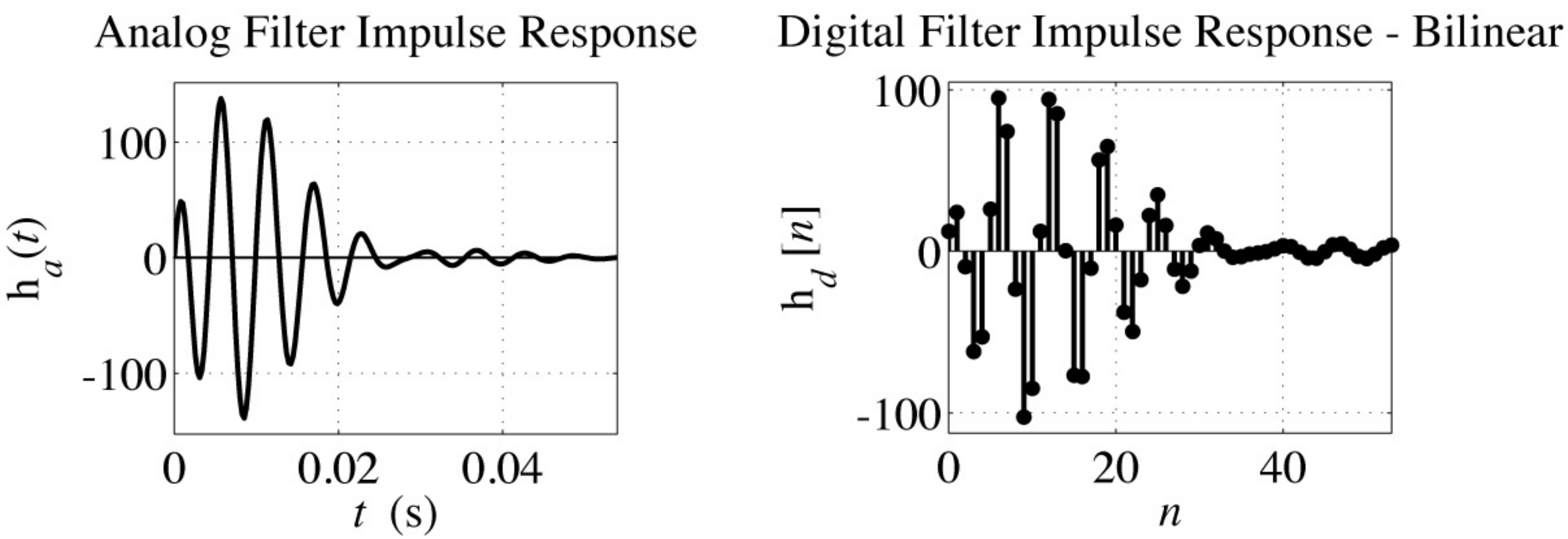


The Bilinear z Transform

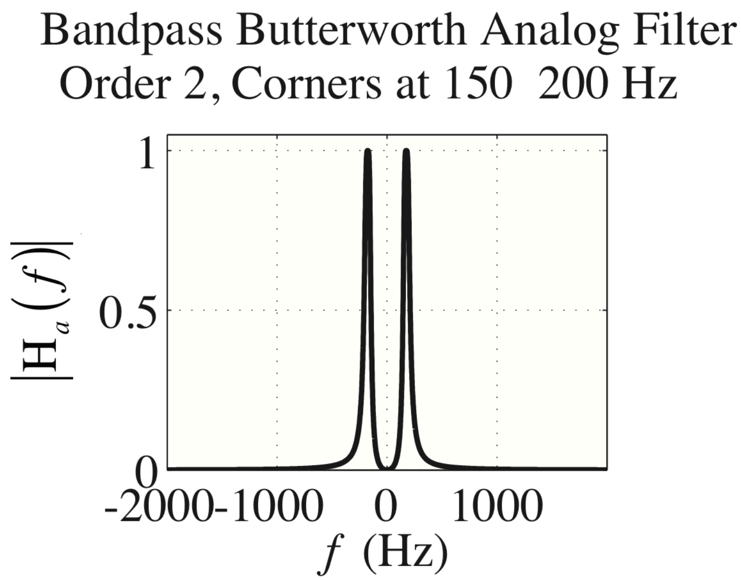
Let $\mathrm{H}_a(s) = \dfrac{9.87 \times 10^4 s^2}{s^4 + 444.3s^3 + 2.467 \times 10^6 s^2 + 5.262 \times 10^8 s + 1.403 \times 10^{12}}$.

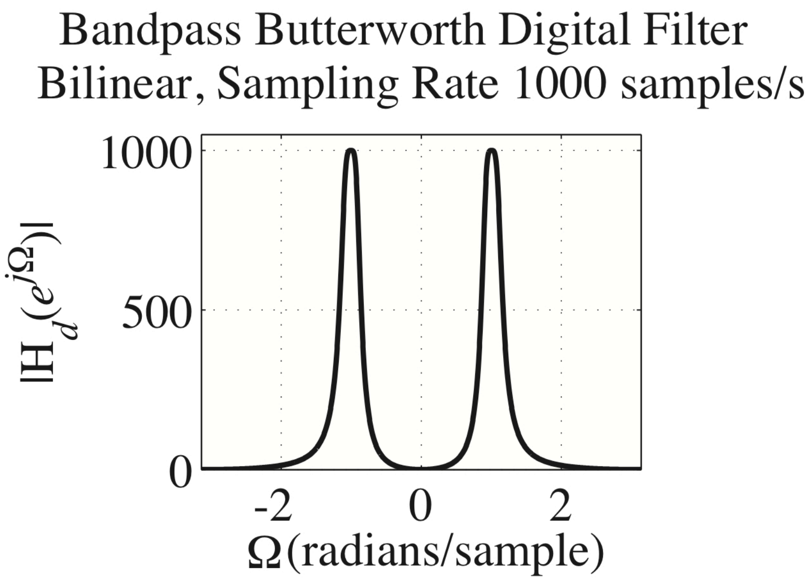
Then $\mathrm{H}_d(z) = 12.38 \dfrac{(z+1)^2 (z-1)^2}{z^4 - 1.989z^3 + 2.656z^2 - 1.675z + 0.711}$

The Bilinear *z* Transform

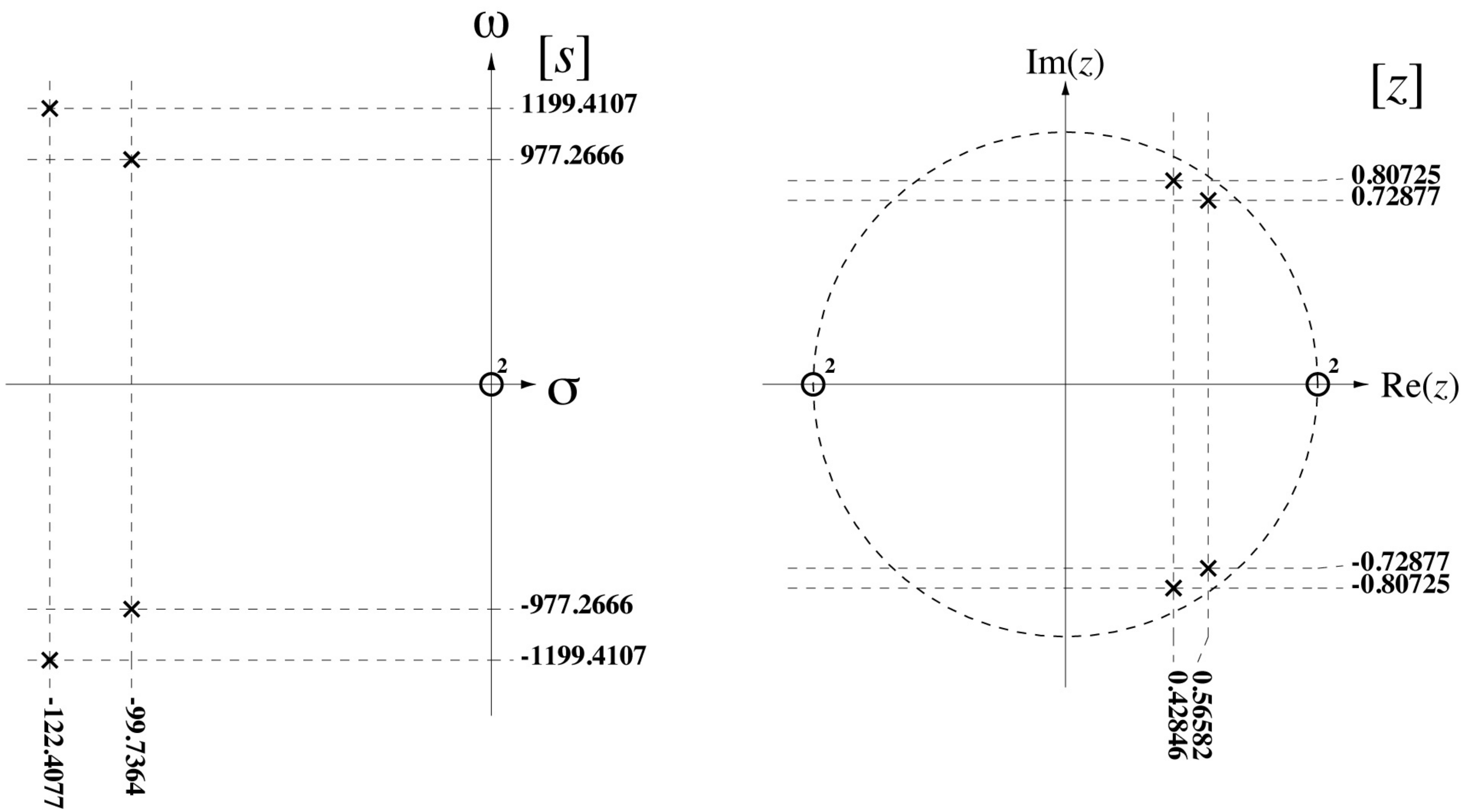


The Bilinear z Transform



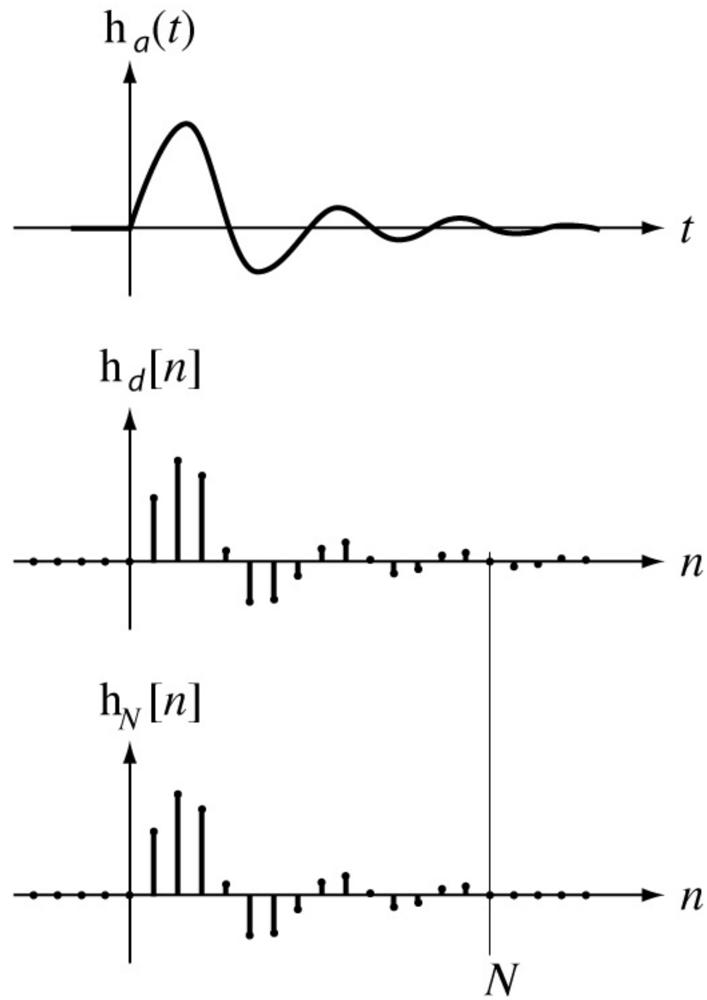


The Bilinear z Transform



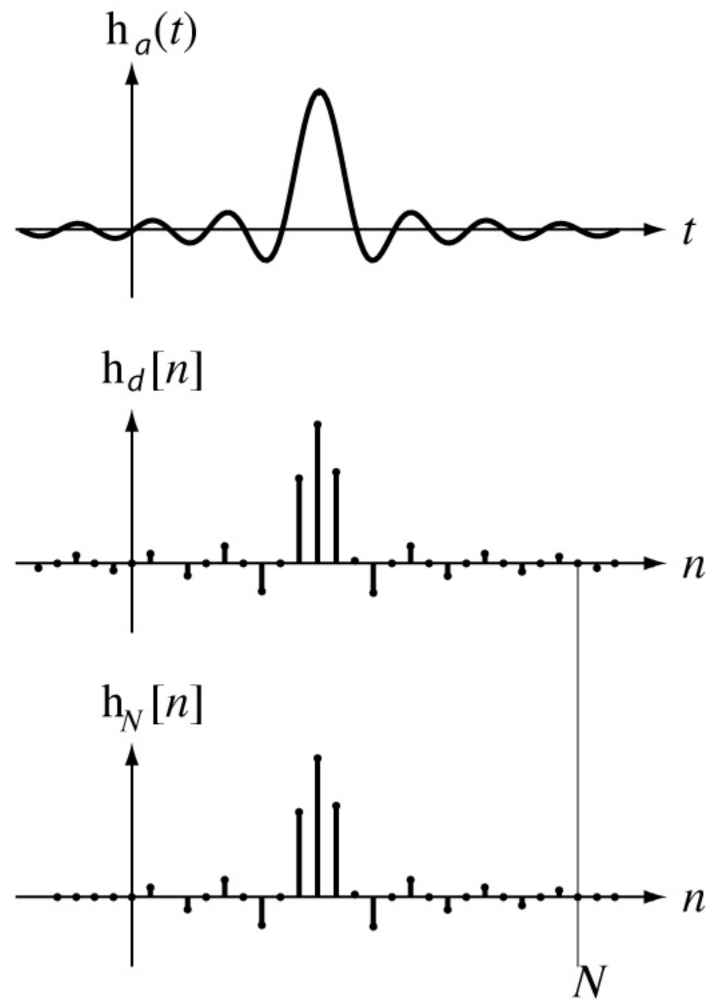
FIR Filter Design

An analog filter can be approximated by an FIR digital filter. The impulse response of the analog filter has infinite duration in time but beyond some time it has dropped to a very low value and the remainder beyond that time can be truncated with little change in the filter characteristics. We can create a digital FIR filter by sampling over that time.



FIR Filter Design

We can also approximate ideal filters, which are non-causal, by truncating that part of the impulse response occurring before time $n = 0$ and later when the impulse response has fallen to a very low value.



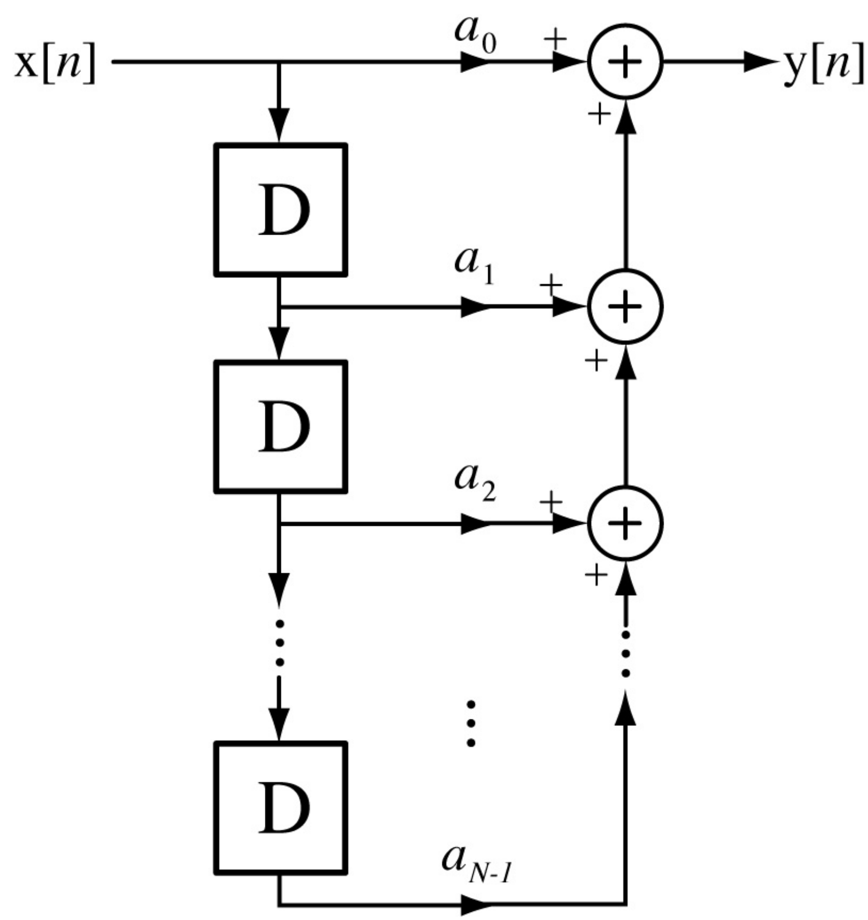
FIR Filter Design

The truncated impulse response has the form

$$h_N[n] = \sum_{m=0}^{N-1} a_m \delta[n-m]$$

where N is the number of samples retained. This type of FIR digital filter can be realized in Direct Form II. Its transfer function is

$$H_d(z) = \sum_{m=0}^{N-1} a_m z^{-m}.$$



FIR Filter Design

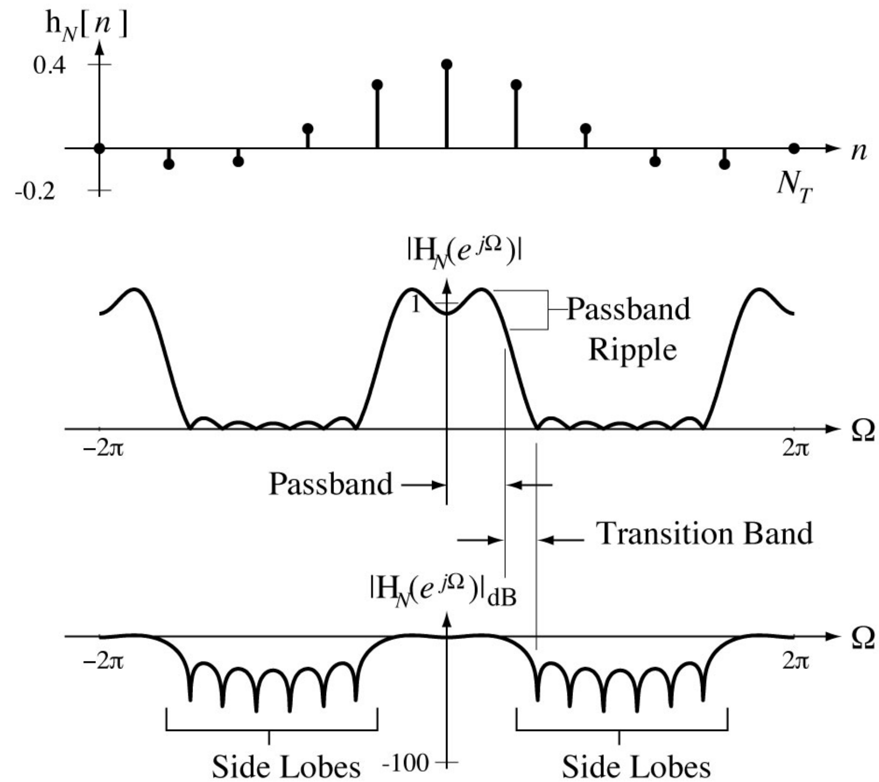
Let $h_N[n] = \begin{cases} h_d[n] & , \; 0 \le n < N \\ 0 & , \; \text{otherwise} \end{cases} = h_d[n]w[n]$ be the impulse response of the digital filter where $h_d[n]$ is the sampled ideal analog filter impulse response and $w[n]$ is a **window** function. Then the frequency response is

$$H_N\left(e^{j\Omega}\right) = H_d\left(e^{j\Omega}\right) \circledast W\left(e^{j\Omega}\right).$$

The periodic convolution with $W\left(e^{j\Omega}\right)$ changes the frequency response from ideal and introduces **ripple** in the frequency response.

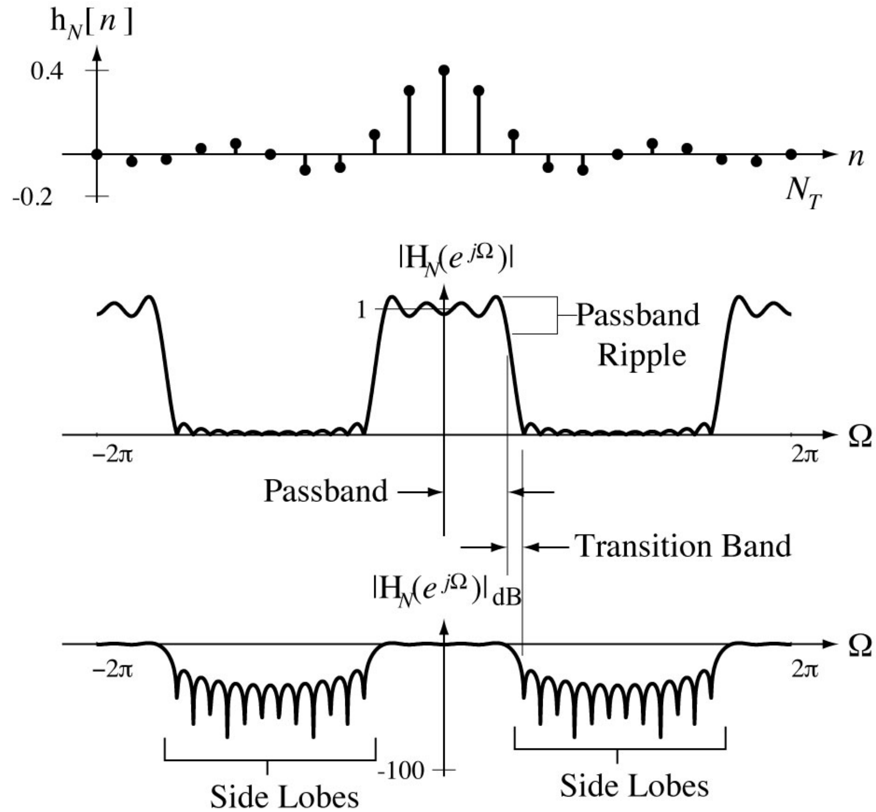
FIR Filter Design

Impulse response and frequency response of an FIR lowpass digital filter designed by truncating the ideal impulse response.



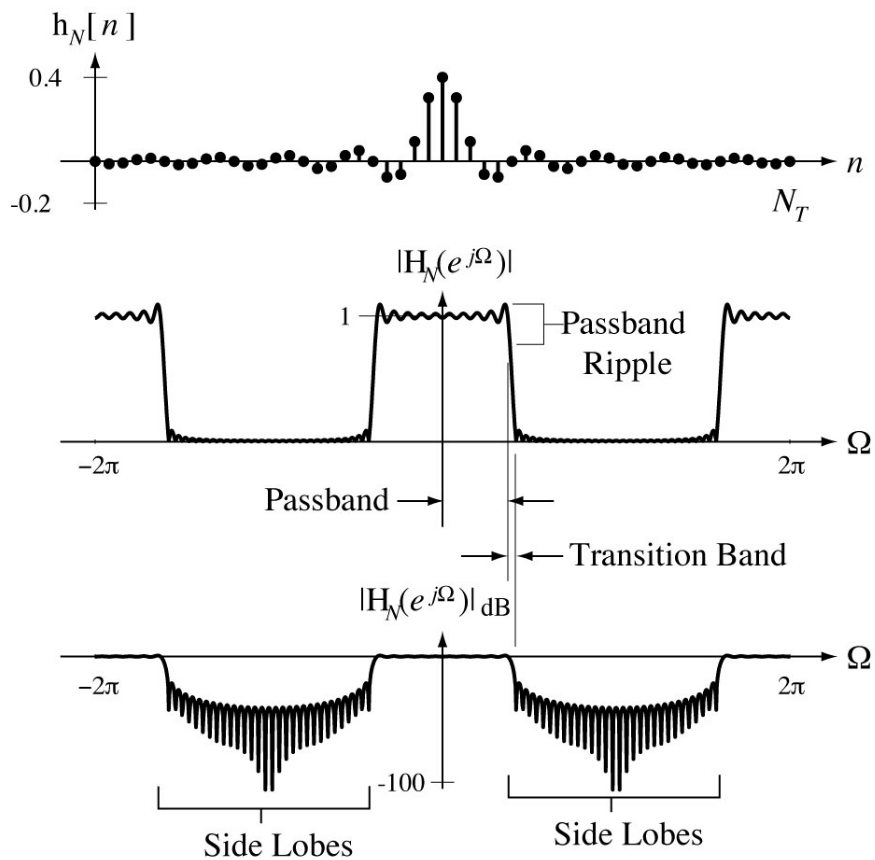
FIR Filter Design

Impulse response and frequency response of an FIR lowpass digital filter designed by truncating the ideal impulse response.



FIR Filter Design

Impulse response and frequency response of an FIR lowpass digital filter designed by truncating the ideal impulse response.



FIR Filter Design

The digital filter impulse and frequency responses on the previous three slides were all formed by using a rectangular window of the form, $\mathrm{w}[n]=\begin{cases}1 \ , \ 0 \le n < N \\ 0 \ , \ \text{otherwise}\end{cases}$. Other window functions could be used to reduce the ripple in the frequency response.

FIR Filter Design

Frequently-Used Window Functions

von Hann

$$\mathrm{w}[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right] \ , \ 0 \le n < N$$

Bartlett

$$\mathrm{w}[n] = \begin{cases} \dfrac{2n}{N-1} & , \ 0 \le n \le \dfrac{N-1}{2} \\ 2 - \dfrac{2n}{N-1} & , \ \dfrac{N-1}{2} \le n < N \end{cases}$$

Hamming

$$\mathrm{w}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \ , \ 0 \le n < N$$

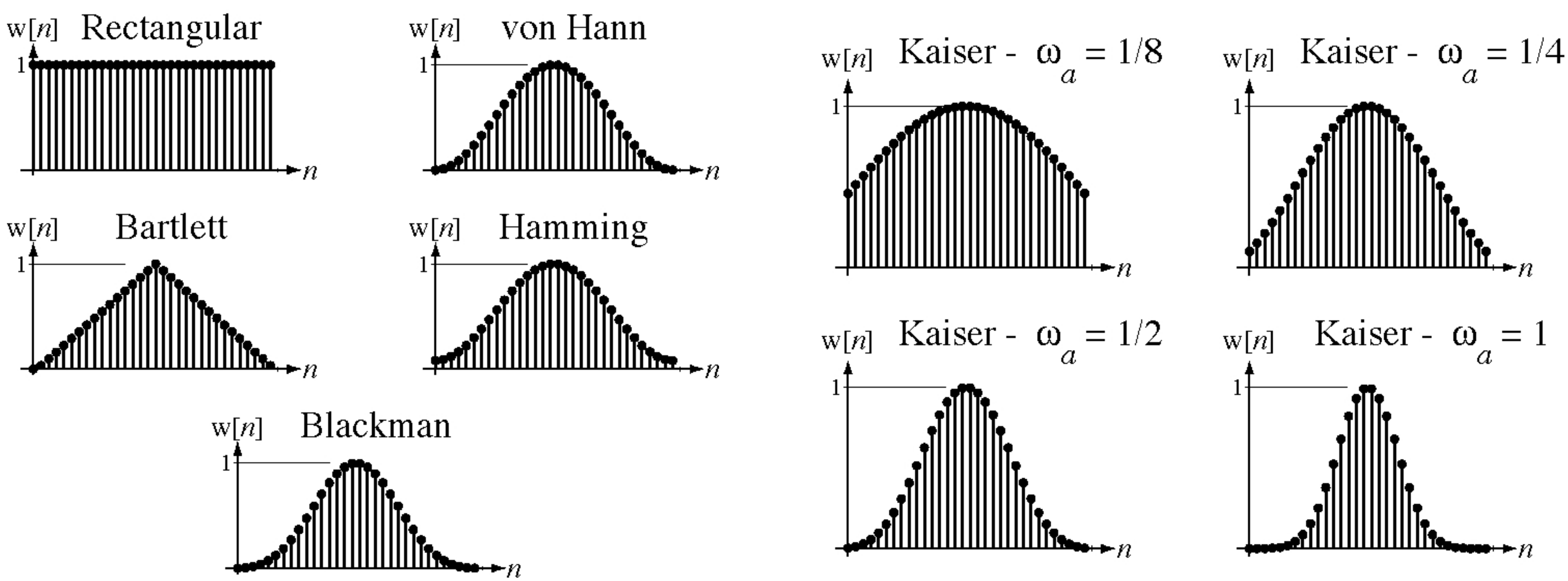
Blackman

$$\mathrm{w}[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) \ , \ 0 \le n < N$$

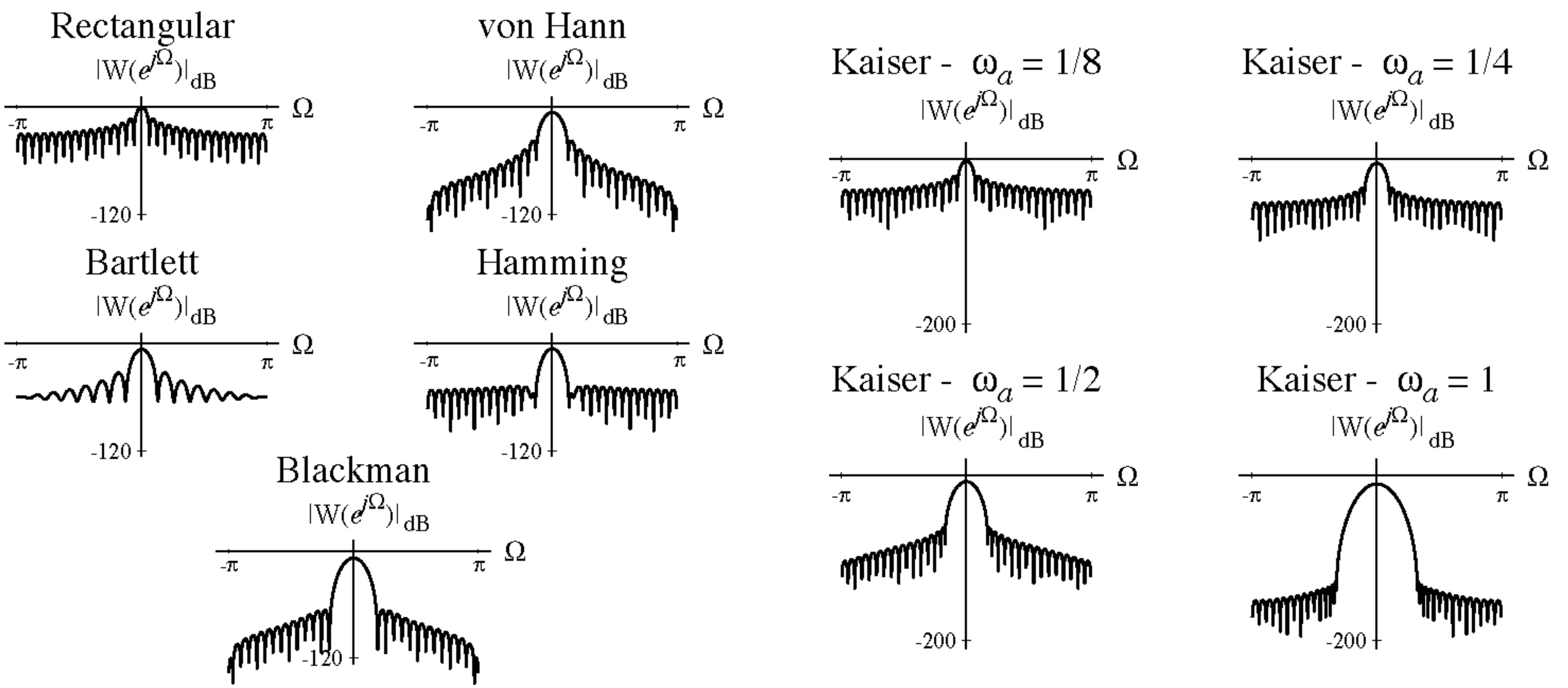
Kaiser

$$\mathrm{w}[n] = \frac{I_0\left(\omega_a\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\right)}{I_0\left(\omega_a \frac{N-1}{2}\right)}$$

FIR Filter Design



FIR Filter Design



FIR Filter Design

In studying the common types of windows in the last two slides it becomes apparent that in designing FIR filters two design goals are in conflict. The ideal filter has no ripple in the passband and the transition from the passband to the stopband is abrupt in frequency; it has no width. For a fixed window width, if we optimize the transition to be as quick as possible we must accept large ripple in the pass and stop bands. If we want to minimize passband and stopband ripple we must accept a slower transition between the passband and stopband.

FIR Filter Design

The general form of an FIR impulse response is

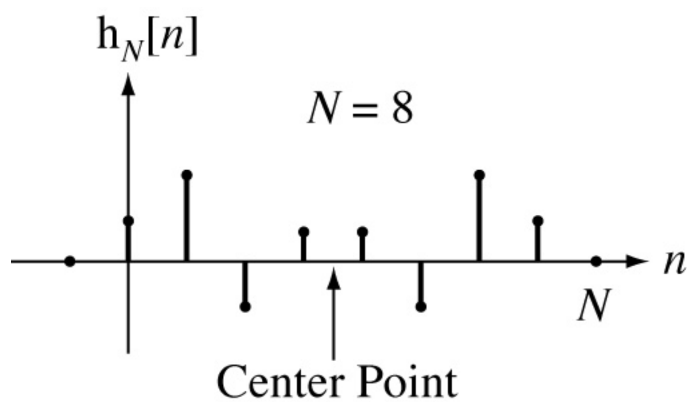
$$h_d[n] = h_d[0]\delta[n] + h_d[1]\delta[n-1] + \cdots + h_d[N-1]\delta[n-(N-1)]$$

and its z transform is

$$H_d(z) = h_d[0] + h_d[1]z^{-1} + \cdots + h_d[N-1]z^{-(N-1)}$$

Let N be an even number and let the coefficients be chosen such that

$$h_d[0] = h_d[N-1] \;,\; h_d[1] = h_d[N-2] \;,\; \cdots \;,\; h_d[N/2-1] = h_d[N/2]$$



FIR Filter Design

The frequency response is

$$H_d\left(e^{j\Omega}\right)=\left\{\begin{array}{l} h_d[0]+h_d[0]e^{-j(N-1)\Omega}+h_d[1]e^{-j\Omega}+h_d[1]e^{-j(N-2)\Omega}+\cdots \\ \quad +h_d[N/2-1]e^{-j(N/2-1)\Omega}+h_d[N/2-1]e^{-jN\Omega/2} \end{array}\right\}$$

which can be expressed in the form

$$H_d\left(e^{j\Omega}\right)=2e^{-j\left(\frac{N-1}{2}\right)\Omega}\left\{\begin{array}{l} h_d[0]\cos\left(\left(\frac{N-1}{2}\right)\Omega\right)+h_d[1]\cos\left(\left(\frac{N-3}{2}\right)\Omega\right)+\cdots \\ \quad +h_d[N/2-1]\cos(\Omega) \end{array}\right\}$$

The phase of the frequency response is linear. This is usually a desirable result because it is the same as an ideal filter. By a similar process it can be shown that if the coefficients are anti-symmetric meaning

$$h_d[0]=-h_d[N-1]\ ,\ h_d[1]=-h_d[N-2]\ ,\ \cdots\ ,\ h_d[N/2-1]=-h_d[N/2]$$

that the phase is also linear and this holds true for N even or odd.

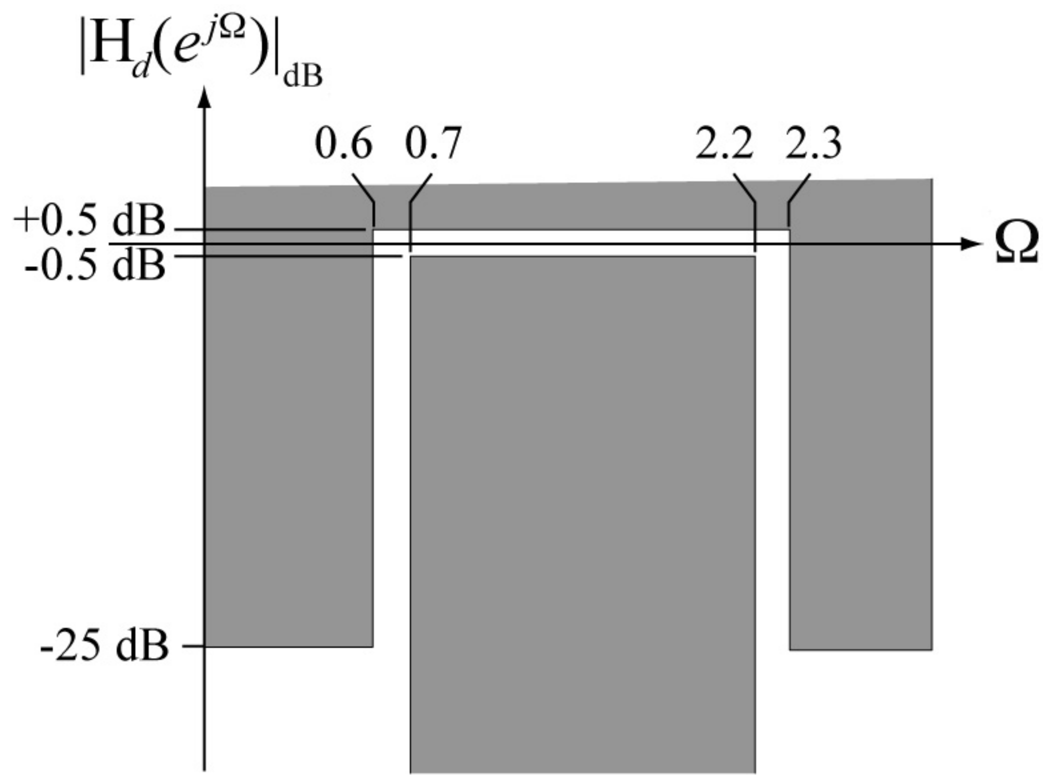
Optimal FIR Filter Design

The Parks-McClellan algorithm is a technique for directly designing a filter by specifying its frequency response by multiple linear functions of frequency in multiple frequency bands separated by transition bands. In MATLAB the syntax is `B = firpm(N,F,A)` where `B` is a vector of $N+1$ real symmetric coefficients in the impulse response of the FIR filter which is the best approximation to the desired frequency response described by `F` and `A`. (See the MATLAB help file for a more detailed description.)

Optimal FIR Filter Design

Example

Design an optimal FIR filter to meet this specification.



Optimal FIR Filter Design

Example

The band edges are $\Omega = \{0, 0.6, 0.7, 2.2, 2.3, \pi\}$ and the desired amplitude responses at those band edges are $\mathsf{A} = \{0,0,1,1,0,0\}$.

The vector F should be $\mathsf{F} = \Omega / \pi = \{0, 0.191, 0.2228, 0.7003, 0.7321, 1\}$.

N is the length of the filter and can be found by trying different values until filter specification is met. In this case $N = 70$ meets the specification.

Optimal FIR Filter Design