Random Signals and Noise

Distribution Functions

The distribution function of a random variable X is the probability that it is less than or equal to some value, as a function of that value.

$$\mathrm{F}_X\left(x\right) = \mathrm{P}\left[X \le x\right]$$

Since the distribution function is a probability it must satisfy the requirements for a probability.

$$0 \le \mathrm{F}_X\left(x\right) \le 1 \quad , \quad -\infty < x < \infty$$

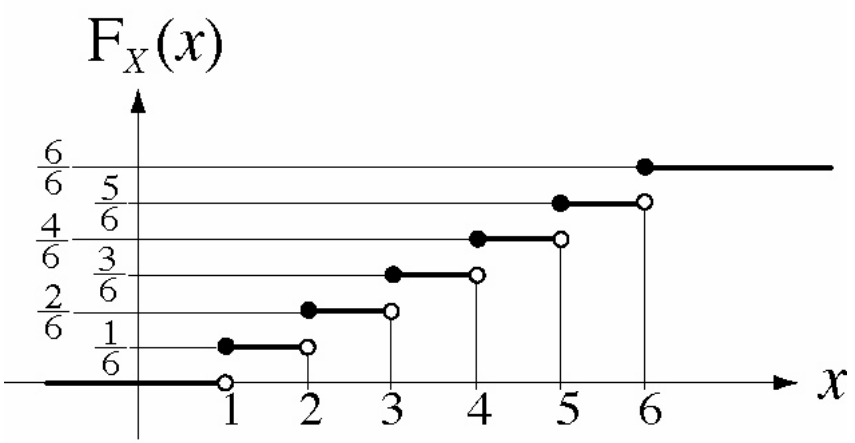
$$\mathrm{P}\left[x_1 < X \le x_2\right] = \mathrm{F}_X\left(x_2\right) - \mathrm{F}_X\left(x_1\right)$$

$\mathrm{F}_X\left(x\right)$ is a monotonic function and its derivative is never negative.

Distribution Functions

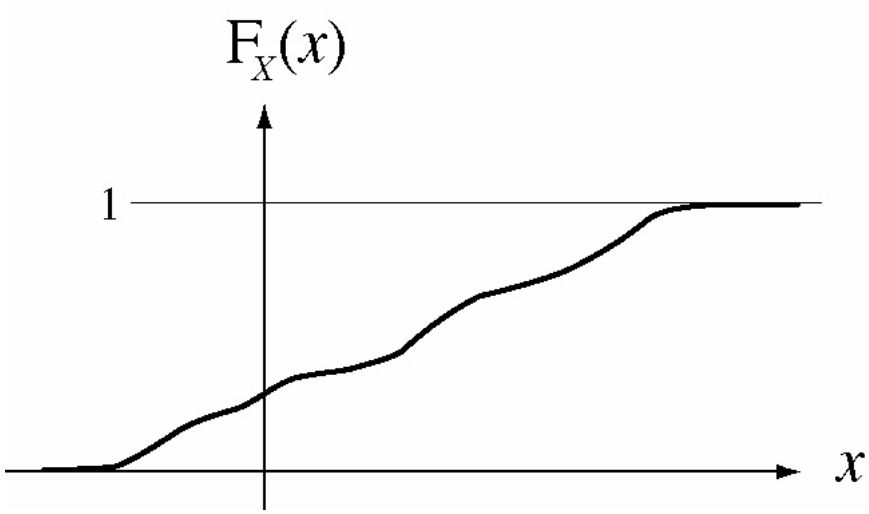
The distribution function for tossing a single die

$$F_X(x) = (1/6)\left[\begin{array}{l} u(x-1)+u(x-2)+u(x-3) \\ +u(x-4)+u(x-5)+u(x-6) \end{array}\right]$$



Distribution Functions

A possible distribution function for a continuous random variable



Probability Density

The derivative of the distribution function is the **probability density function (PDF)**

$$\mathrm{p}_X\left(x\right) \equiv \frac{d}{dx}\left(\mathrm{F}_X\left(x\right)\right)$$

Probability density can also be defined by

$$\mathrm{p}_X\left(x\right)dx = \mathrm{P}\left[\, x < X \le x + dx \,\right]$$

Properties

$$\mathrm{p}_X\left(x\right) \ge 0 \quad , \quad -\infty < x < +\infty \qquad\qquad \int_{-\infty}^{\infty} \mathrm{p}_X\left(x\right)dx = 1$$

$$\mathrm{F}_X\left(x\right) = \int_{-\infty}^{x} \mathrm{p}_X\left(\lambda\right)d\lambda \qquad\qquad \mathrm{P}\left[\, x_1 < X \le x_2 \,\right] = \int_{x_1}^{x_2} \mathrm{p}_X\left(x\right)dx$$

Expectation and Moments

Imagine an experiment with M possible distinct outcomes performed N times. The average of those N outcomes is $\bar{X} = \frac{1}{N}\sum_{i=1}^{M} n_i x_i$ where x_i is the ith distinct value of X and n_i is the number of times that value occurred. Then $\bar{X} = \frac{1}{N}\sum_{i=1}^{M} n_i x_i = \sum_{i=1}^{M} \frac{n_i}{N} x_i = \sum_{i=1}^{M} r_i x_i$.

The **expected value** of X is

$$\mathrm{E}\left(X\right) = \lim_{N\to\infty} \sum_{i=1}^{M} \frac{n_i}{N} x_i = \lim_{N\to\infty} \sum_{i=1}^{M} r_i x_i = \sum_{i=1}^{M} \mathrm{P}\left[X = x_i\right] x_i.$$

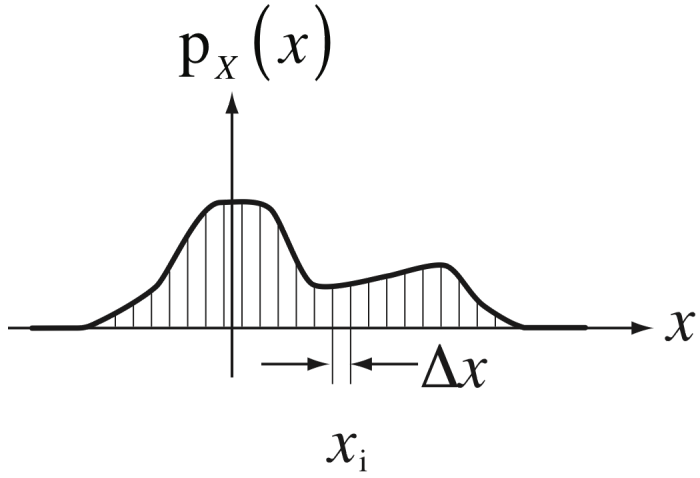
Expectation and Moments

The probability that X lies within some small range can be approximated by $\mathrm{P}\left[x_i - \frac{\Delta x}{2} < X \le x_i + \frac{\Delta x}{2} \right] \cong \mathrm{p}_X\left(x_i\right)\Delta x$

and the expected value is then approximated by

$$\mathrm{E}\left(X\right) = \sum_{i=1}^{M} \mathrm{P}\left[x_i - \frac{\Delta x}{2} < X \le x_i + \frac{\Delta x}{2} \right] x_i \cong \sum_{i=1}^{M} x_i \, \mathrm{p}_X\left(x_i\right)\Delta x$$

where M is now the number of subdivisions of width Δx of the range of the random variable.



Expectation and Moments

In the limit as Δx approaches zero, $\mathrm{E}\left(X\right) = \int_{-\infty}^{\infty} x\,\mathrm{p}_X\left(x\right)dx.$

Similarly $\mathrm{E}\left(\mathrm{g}\left(X\right)\right) = \int_{-\infty}^{\infty} \mathrm{g}\left(x\right)\mathrm{p}_X\left(x\right)dx.$

The nth **moment** of a random variable is $\mathrm{E}\left(X^n\right) = \int_{-\infty}^{\infty} x^n\,\mathrm{p}_X\left(x\right)dx.$

Expectation and Moments

The first moment of a random variable is its expected value $\mathrm{E}\left(X\right) = \int_{-\infty}^{\infty} x\, \mathrm{p}_X\left(x\right) dx$. The second moment of a random variable is its mean-squared value (which is the mean of its square, not the square of its mean).

$$\mathrm{E}\left(X^2\right) = \int_{-\infty}^{\infty} x^2\, \mathrm{p}_X\left(x\right) dx$$

Expectation and Moments

A **central moment** of a random variable is the moment of that random variable after its expected value is subtracted.

$$\mathrm{E}\left(\left[X-\mathrm{E}\left(X\right)\right]^{n}\right)=\int_{-\infty}^{\infty}\left[x-\mathrm{E}\left(X\right)\right]^{n}\,\mathrm{p}_{X}\left(x\right)dx$$

The first central moment is always zero. The second central moment (for real-valued random variables) is the **variance**,

$$\sigma_{X}^{2}=\mathrm{E}\left(\left[X-\mathrm{E}\left(X\right)\right]^{2}\right)=\int_{-\infty}^{\infty}\left[x-\mathrm{E}\left(X\right)\right]^{2}\,\mathrm{p}_{X}\left(x\right)dx$$

The positive square root of the variance is the **standard deviation**.

Expectation and Moments

Properties of Expectation

$$\mathrm{E}\left(a\right)=a \quad , \quad \mathrm{E}\left(aX\right)=a\,\mathrm{E}\left(X\right) \quad , \quad \mathrm{E}\left(\sum_n X_n\right)=\sum_n \mathrm{E}\left(X_n\right)$$

where a is a constant. These properties can be use to prove the handy relationship $\sigma_X^2=\mathrm{E}\left(X^2\right)-\mathrm{E}^2\left(X\right)$. The variance of a random variable is the mean of its square minus the square of its mean.

Joint Probability Density

Let X and Y be two random variables. Their **joint distribution function** is $\mathrm{F}_{XY}\left(x,y\right) \equiv \mathrm{P}\left[X \le x \cap Y \le y\right]$.

$$0 \le \mathrm{F}_{XY}\left(x,y\right) \le 1 \quad , \quad -\infty < x < \infty \quad , \quad -\infty < y < \infty$$

$$\mathrm{F}_{XY}\left(-\infty,-\infty\right) = \mathrm{F}_{XY}\left(x,-\infty\right) = \mathrm{F}_{XY}\left(-\infty,y\right) = 0$$

$$\mathrm{F}_{XY}\left(\infty,\infty\right) = 1$$

$\mathrm{F}_{XY}\left(x,y\right)$ does not decrease if either x or y increases or both increase

$$\mathrm{F}_{XY}\left(\infty,y\right) = \mathrm{F}_{Y}\left(y\right) \text{ and } \mathrm{F}_{XY}\left(x,\infty\right) = \mathrm{F}_{X}\left(x\right)$$

Joint Probability Density

$$\mathrm{p}_{XY}\left(x,y\right)=\frac{\partial^2}{\partial x\partial y}\left(\mathrm{F}_{XY}\left(x,y\right)\right)$$

$$\mathrm{p}_{XY}\left(x,y\right)\geq 0\ \ ,\ -\infty<x<\infty\ \ ,\ -\infty<y<\infty$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{p}_{XY}\left(x,y\right)dx\,dy=1 \qquad \mathrm{F}_{XY}\left(x,y\right)=\int_{-\infty}^{y}\int_{-\infty}^{x}\mathrm{p}_{XY}\left(\alpha,\beta\right)d\alpha\,d\beta$$

$$\mathrm{p}_{X}\left(x\right)=\int_{-\infty}^{\infty}\mathrm{p}_{XY}\left(x,y\right)dy \ \text{ and } \ \mathrm{p}_{Y}\left(y\right)=\int_{-\infty}^{\infty}\mathrm{p}_{XY}\left(x,y\right)dx$$

$$\mathrm{P}\left[x_1<X\leq x_2\ ,\ y_1<Y\leq y_2\right]=\int_{y_1}^{y_2}\int_{x_1}^{x_2}\mathrm{p}_{XY}\left(x,y\right)dx\,dy$$

$$\mathrm{E}\left(\mathrm{g}\left(X,Y\right)\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{g}\left(x,y\right)\mathrm{p}_{XY}\left(x,y\right)dx\,dy$$

Independent Random Variables

If two random variables X and Y are independent the expected value of their product is the product of their expected values.

$$\mathrm{E}\left(XY\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,\mathrm{p}_{XY}\left(x,y\right)dx\,dy = \int_{-\infty}^{\infty} y\,\mathrm{p}_{Y}\left(y\right)dy \int_{-\infty}^{\infty} x\,\mathrm{p}_{X}\left(x\right)dx = \mathrm{E}\left(X\right)\mathrm{E}\left(Y\right)$$

Independent Random Variables

Covariance

$$\sigma_{XY} \equiv \mathrm{E}\left(\left[X - \mathrm{E}\left(X \right) \right] \left[Y - \mathrm{E}\left(Y \right) \right]^{*} \right)$$

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(x - \mathrm{E}\left(X \right) \right) \left(y^{*} - \mathrm{E}\left(Y^{*} \right) \right) \mathrm{p}_{XY}\left(x, y \right) dx\, dy$$

$$\sigma_{XY} = \mathrm{E}\left(XY^{*} \right) - \mathrm{E}\left(X \right) \mathrm{E}\left(Y^{*} \right)$$

If X and Y are independent, $\sigma_{XY} = \mathrm{E}\left(X \right) \mathrm{E}\left(Y^{*} \right) - \mathrm{E}\left(X \right) \mathrm{E}\left(Y^{*} \right) = 0$

Independent Random Variables

Correlation Coefficient

$$\rho_{XY} = \mathrm{E}\left(\frac{X - \mathrm{E}\left(X\right)}{\sigma_X} \times \frac{Y^* - \mathrm{E}\left(Y^*\right)}{\sigma_Y} \right)$$

$$\rho_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{x - \mathrm{E}\left(X\right)}{\sigma_X} \right) \left(\frac{y^* - \mathrm{E}\left(Y^*\right)}{\sigma_Y} \right) \mathrm{p}_{XY}\left(x, y\right) dx\, dy$$

$$\rho_{XY} = \frac{\mathrm{E}\left(XY^*\right) - \mathrm{E}\left(X\right)\mathrm{E}\left(Y^*\right)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

If X and Y are independent $\rho = 0$. If they are perfectly positively correlated $\rho = +1$ and if they are perfectly negatively correlated $\rho = -1$.

Independent Random Variables

If two random variables are independent, their covariance is zero. However, if two random variables have a zero covariance that does not mean they are necessarily independent.

$$\text{Independence} \Rightarrow \text{Zero Covariance}$$

$$\cancel{\text{Zero Covariance} \Rightarrow \text{Independence}}$$

Independent Random Variables

In the traditional jargon of random variable analysis, two "uncorrelated" random variables have a covariance of zero.

Unfortunately, this does not also imply that their correlation is zero. If their correlation is zero they are said to be **orthogonal**.

$$X \text{ and } Y \text{ are "Uncorrelated"} \Rightarrow \sigma_{XY} = 0$$

$$\cancel{X \text{ and } Y \text{ are "Uncorrelated"} \Rightarrow \mathrm{E}(XY) = 0}$$

Independent Random Variables

The variance of a sum of random variables X and Y is

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$$

If Z is a linear combination of random variables X_i

$$Z = a_0 + \sum_{i=1}^{N} a_i X_i$$

then $$\mathrm{E}\left(Z\right) = a_0 + \sum_{i=1}^{N} a_i \,\mathrm{E}\left(X_i\right)$$

$$\sigma_Z^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \sigma_{X_i X_j} = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i\neq j}}^{N}\sum_{j=1}^{N} a_i a_j \sigma_{X_i X_j}$$

Independent Random Variables

If the X's are all independent of each other, the variance of the linear combination is a linear combination of the variances.

$$\sigma_Z^2 = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2$$

If Z is simply the sum of the X's, and the X's are all independent of each other, then the variance of the sum is the sum of the variances.

$$\sigma_Z^2 = \sum_{i=1}^{N} \sigma_{X_i}^2$$

Probability Density of a Sum of Random Variables

Let $Z = X + Y$. Then for Z to be less than z, X must be less than $z - Y$. Therefore, the distribution function for Z is

$$\mathrm{F}_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} \mathrm{p}_{XY}(x, y)\, dx\, dy$$

If X and Y are independent, $\mathrm{F}_Z(z) = \int_{-\infty}^{\infty} \mathrm{p}_Y(y) \left(\int_{-\infty}^{z-y} \mathrm{p}_X(x)\, dx \right) dy$

and it can be shown that $\mathrm{p}_Z(z) = \int_{-\infty}^{\infty} \mathrm{p}_Y(y)\, \mathrm{p}_X(z - y)\, dy = \mathrm{p}_Y(z) * \mathrm{p}_X(z)$

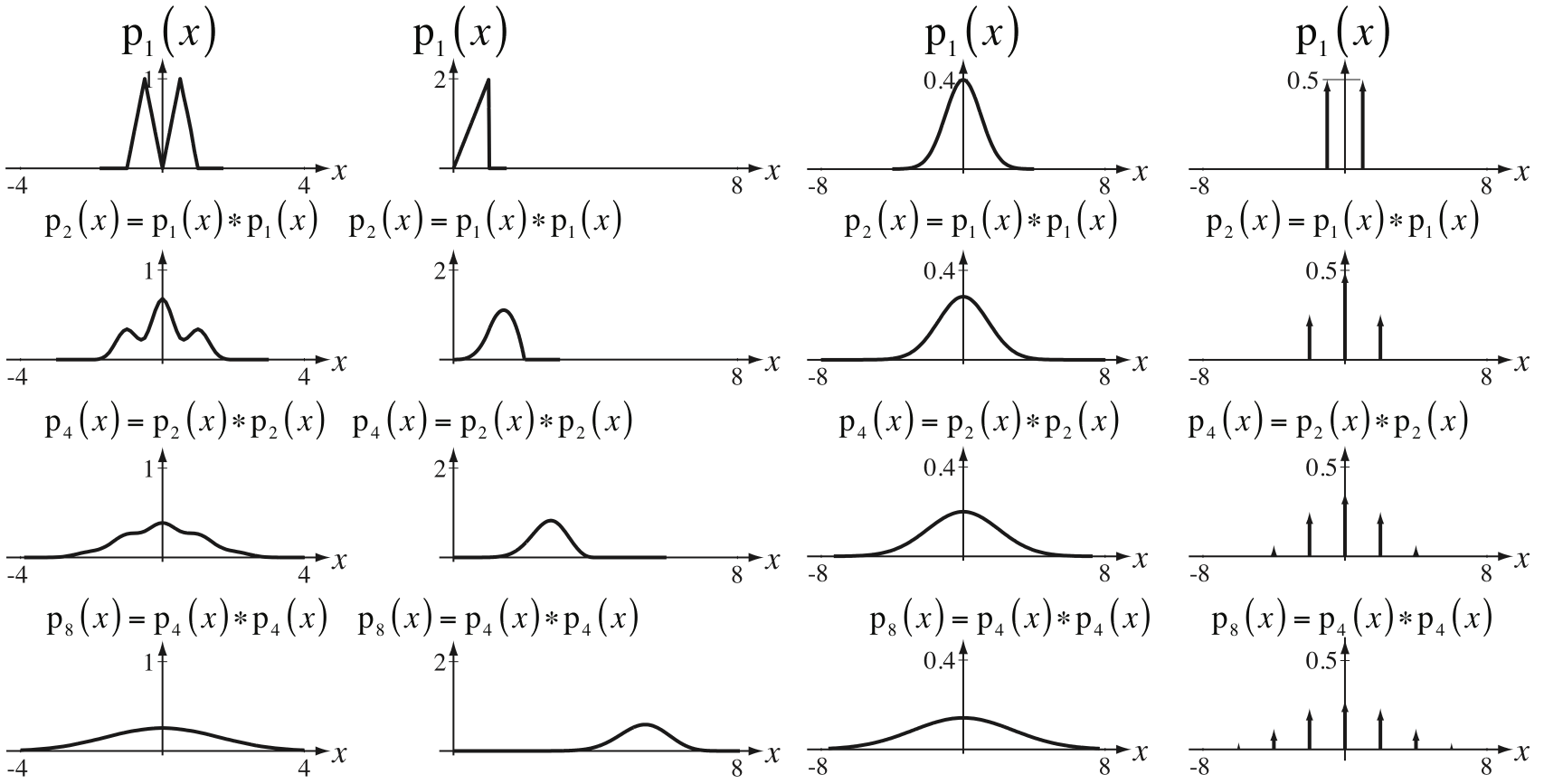
The Central Limit Theorem

If N independent random variables are added to form a resultant random variable $Z = \sum_{n=1}^{N} X_n$ then

$$\mathrm{p}_Z(z) = \mathrm{p}_{X_1}(z) * \mathrm{p}_{X_2}(z) * \mathrm{p}_{X_2}(z) * \cdots * \mathrm{p}_{X_N}(z)$$

and it can be shown that, under very general conditions, the PDF of a sum of a large number of independent random variables with continuous PDF's approaches a limiting shape called the **Gaussian** PDF regardless of the shapes of the individual PDF's.

The Central Limit Theorem

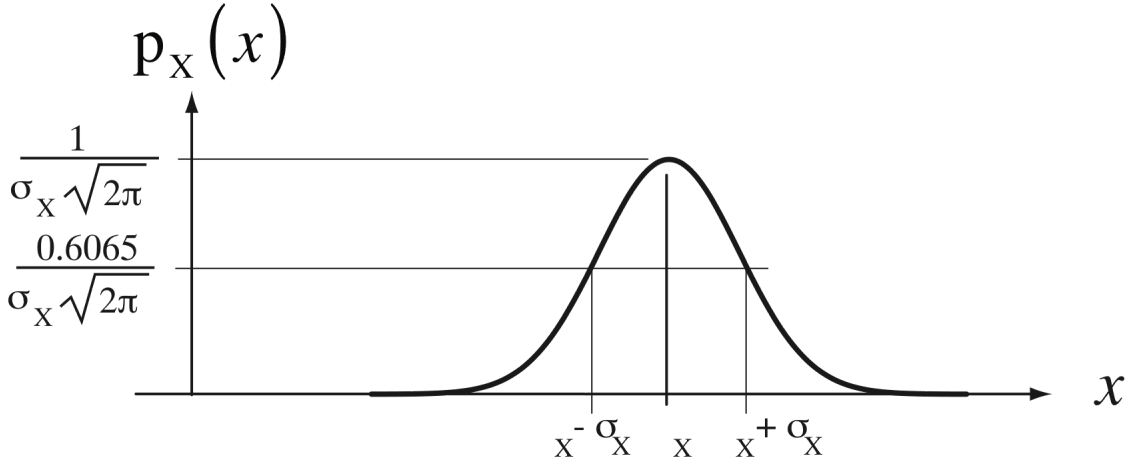


The Central Limit Theorem

The Gaussian pdf

$$p_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

$$\mu_X = E(X) \text{ and } \sigma_X = \sqrt{E\left(\left[X - E(X)\right]^2\right)}$$



The Central Limit Theorem

The Gaussian PDF

Its maximum value occurs at the mean value of its argument.

It is symmetrical about the mean value.

The points of maximum absolute slope occur at one standard deviation above and below the mean.

Its maximum value is inversely proportional to its standard deviation.

The limit as the standard deviation approaches zero is a unit impulse.

$$\delta\left(x-\mu_x\right)=\lim_{\sigma_X\to 0}\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\left(x-\mu_X\right)^2/2\sigma_X^2}$$

Correlation

The **correlation** between two signals is a measure of how similarly shaped they are. The definition of correlation R_{12} for two signals $\mathrm{x}_1(t)$ and $\mathrm{x}_2(t)$, at least one of which is an energy signal, is the area under the product of $\mathrm{x}_1(t)$ and $\mathrm{x}_2^*(t)$

$$R_{12} = \int_{-\infty}^{\infty} \mathrm{x}_1(t)\mathrm{x}_2^*(t)dt.$$

If we applied this definition to two power signals, R_{12} would be infinite. To avoid that problem, the definition of correlation R_{12} for two power signals $\mathrm{x}_1(t)$ and $\mathrm{x}_2(t)$ is changed to the average of the product of $\mathrm{x}_1(t)$ and $\mathrm{x}_2^*(t)$.

$$R_{12} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{x}_1(t)\mathrm{x}_2^*(t)dt.$$

Correlation

For two energy signals notice the similarity of correlation to signal energy.

$$R_{12} = \int_{-\infty}^{\infty} \mathrm{x}_1(t)\,\mathrm{x}_2^*(t)\,dt \quad E_1 = \int_{-\infty}^{\infty} \left|\mathrm{x}_1(t)\right|^2 dt \quad E_2 = \int_{-\infty}^{\infty} \left|\mathrm{x}_2(t)\right|^2 dt$$

In the special case in which $\mathrm{x}_1(t) = \mathrm{x}_2(t)$, $R_{12} = E_1 = E_2$. So for energy signals, correlation has the same units as signal energy.

For power signals,

$$R_{12} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{x}_1(t)\,\mathrm{x}_2^*(t)\,dt \quad P_1 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\mathrm{x}_1(t)\right|^2 dt \quad P_2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\mathrm{x}_2(t)\right|^2 dt$$

In the special case in which $\mathrm{x}_1(t) = \mathrm{x}_2(t)$, $R_{12} = P_1 = P_2$. So for power signals, correlation has the same units as signal power.

Correlation

Consider two energy signals $x_1(t)$ and $x_2(t)$.

If $x_1(t) = x_2(t)$, $R_{12} = E_1 = E_2$

If $x_1(t) = -x_2(t)$, $R_{12} = -E_1 = -E_2$

More generally, if $x_1(t) = a\,x_2(t)$, $R_{12} = E_1 / a = aE_2$.

If R_{12} is positive we say that $x_1(t)$ and $x_2(t)$ are **positively correlated** and if R_{12} is negative we say that $x_1(t)$ and $x_2(t)$ are **negatively correlated**.

If $R_{12} = 0$, what does that imply?

1. One possibility is that $x_1(t)$ or $x_2(t)$ is zero or both are zero.

2. Otherwise $\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$ must be zero with $x_1(t)$ and $x_2(t)$ both non-zero.

In either case, $x_1(t)$ and $x_2(t)$ are **orthogonal**.

Correlation

Consider two energy signals

$$\mathrm{x}_1(t) = \mathrm{x}(t) + \mathrm{y}(t) \text{ and } \mathrm{x}_2(t) = a\,\mathrm{x}(t) + \mathrm{z}(t)$$

and let x, y and z all be mutually orthogonal.

What is R_{12} ?

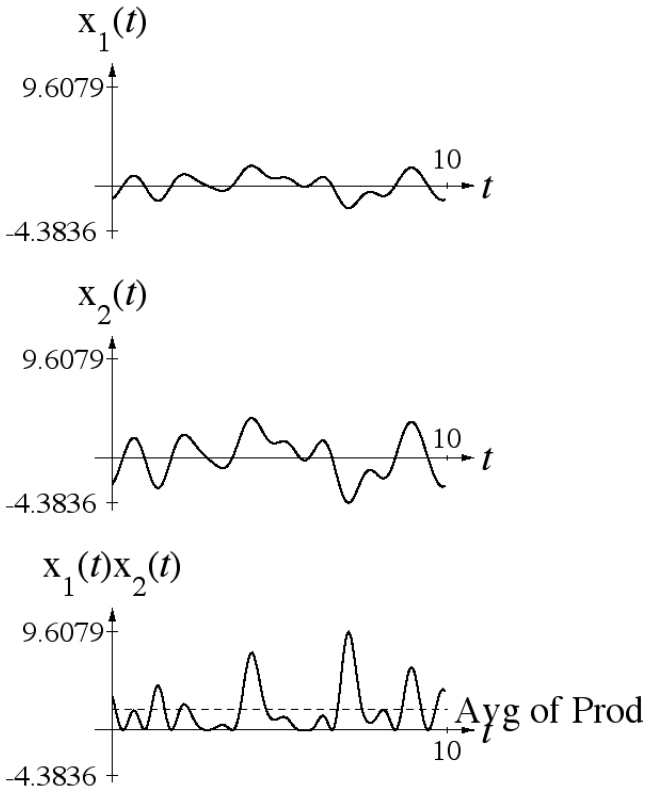
$$R_{12} = \int_{-\infty}^{\infty} \left[\mathrm{x}(t) + \mathrm{y}(t)\right]\left[a\,\mathrm{x}(t) + \mathrm{z}(t)\right]^* dt$$

$$= \int_{-\infty}^{\infty} \left[a\,\mathrm{x}(t)\mathrm{x}^*(t) + \mathrm{x}(t)\mathrm{z}^*(t) + a\,\mathrm{y}(t)\mathrm{x}^*(t) + \mathrm{y}(t)\mathrm{z}^*(t)\right] dt$$

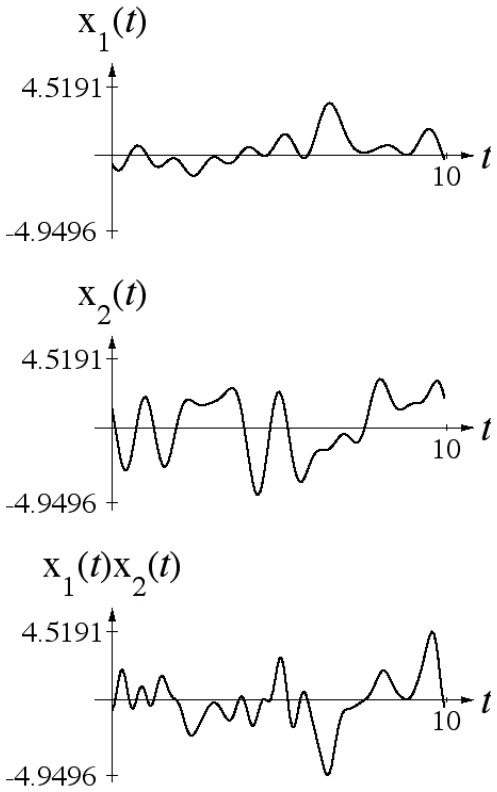
$$= a\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{x}^*(t)\,dt = aR_{11}$$

Correlation

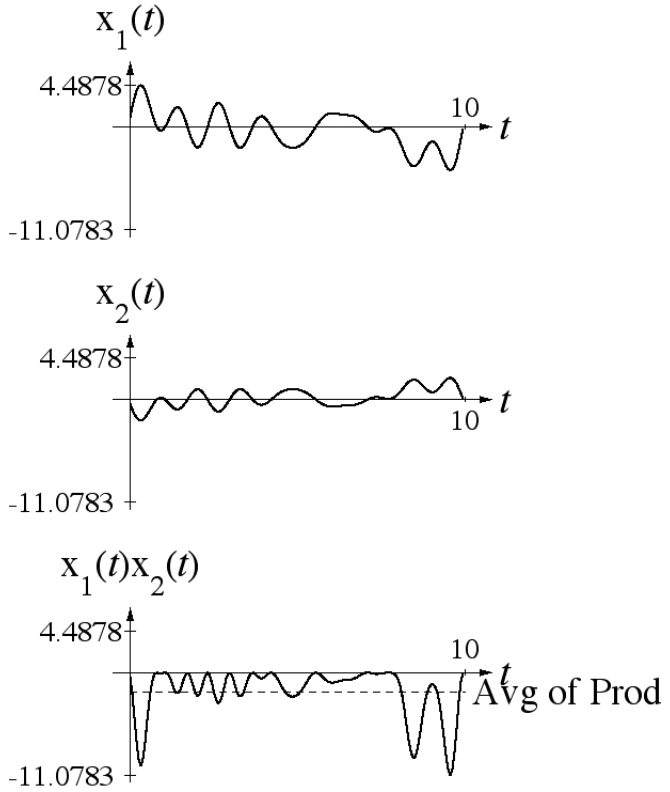
Positively Correlated Random Signals with Zero Mean



Uncorrelated Random Signals with Zero Mean

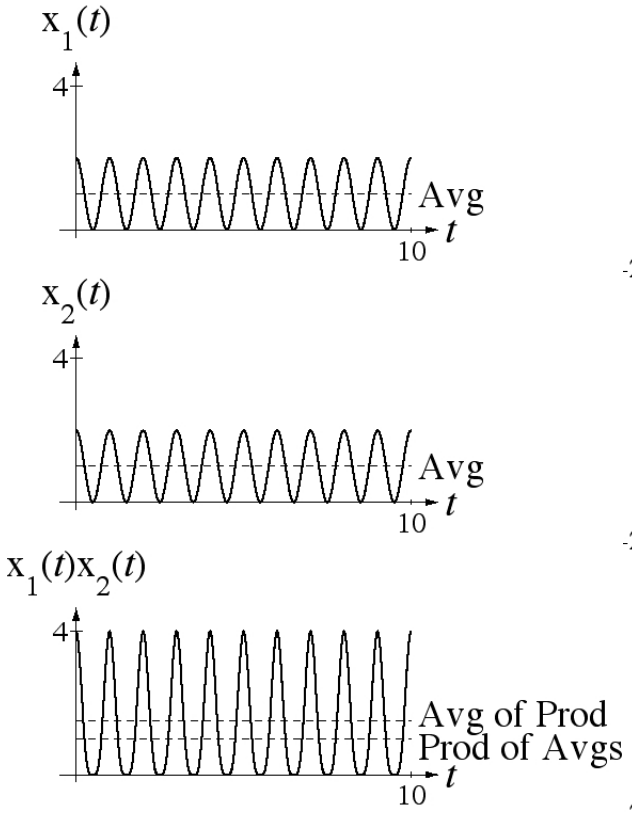


Negatively Correlated Random Signals with Zero Mean

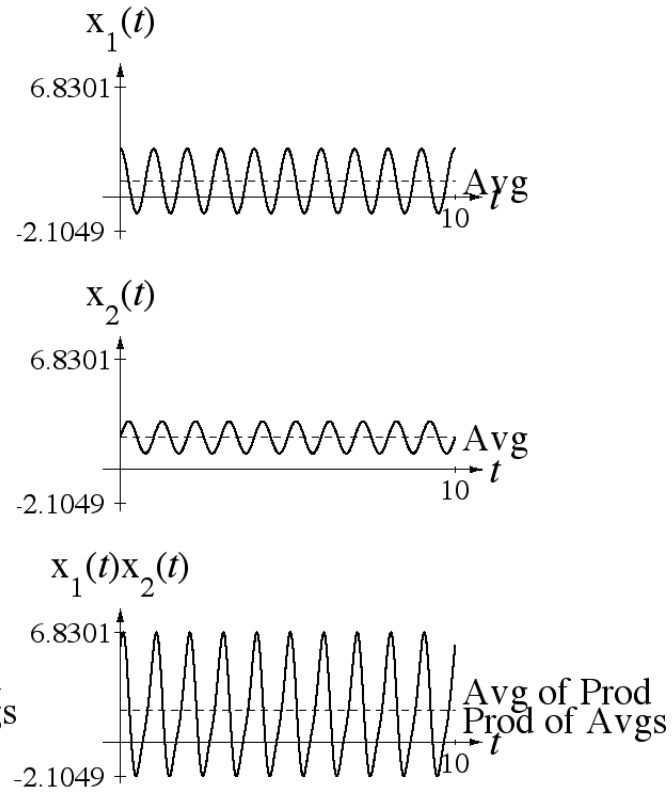


Correlation

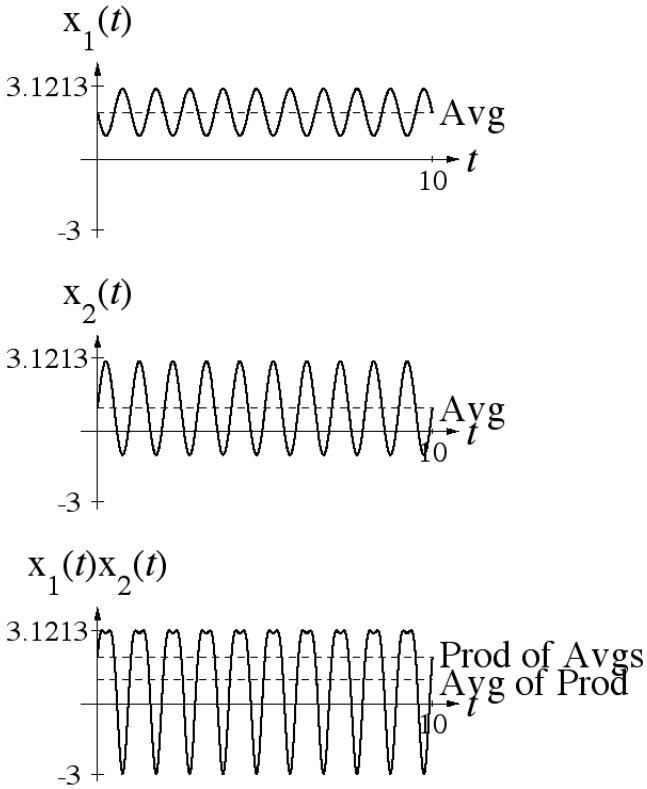
Positively Correlated Sinusoids with Non-Zero Mean



Uncorrelated Sinusoids with Non-Zero Mean



Negatively Correlated Sinusoids with Non-Zero Mean



Correlation

Let $\mathrm{v}(t)$ be a power signal, not necessarily real-valued or periodic, but with a well-defined average signal power

$$P_{\mathrm{v}} \triangleq \left\langle \left|\mathrm{v}(t)\right|^2 \right\rangle = \left\langle \mathrm{v}(t)\mathrm{v}^*(t) \right\rangle \geq 0$$

where $\langle \cdot \rangle$ means "time average of" and mathematically means

$$\left\langle \mathrm{z}(t) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{z}(t)\,dt.$$

Time averaging has the properties $\left\langle \mathrm{z}^*(t) \right\rangle = \left\langle \mathrm{z}(t) \right\rangle^*$, $\left\langle \mathrm{z}(t-t_d) \right\rangle = \left\langle \mathrm{z}(t) \right\rangle$ for any t_d and $\left\langle a_1 \mathrm{z}_1(t) + a_2 \mathrm{z}_2(t) \right\rangle = a_1 \left\langle \mathrm{z}_1(t) \right\rangle + a_2 \left\langle \mathrm{z}_2(t) \right\rangle$. If $\mathrm{v}(t)$ and $\mathrm{w}(t)$ are power signals, $\left\langle \mathrm{v}(t)\mathrm{w}^*(t) \right\rangle$ is the **scalar product** of $\mathrm{v}(t)$ and $\mathrm{w}(t)$. The scalar product is a measure of the similarity between two signals.

Correlation

Let $\mathrm{z}(t) = \mathrm{v}(t) - a\mathrm{w}(t)$ with a real. Then the average power of $\mathrm{z}(t)$ is

$$P_{\mathrm{z}} = \left\langle \mathrm{z}(t)\mathrm{z}^*(t) \right\rangle = \left\langle \left[\mathrm{v}(t) - a\mathrm{w}(t) \right] \left[\mathrm{v}^*(t) - a^*\mathrm{w}^*(t) \right] \right\rangle .$$

Expanding,

$$P_{\mathrm{z}} = \left\langle \mathrm{v}(t)\mathrm{v}^*(t) - a\mathrm{w}(t)\mathrm{v}^*(t) - \mathrm{v}(t)a^*\mathrm{w}^*(t) + a^2\,\mathrm{w}(t)\mathrm{w}^*(t) \right\rangle$$

Using the fact that $a\mathrm{w}(t)\mathrm{v}^*(t)$ and $\mathrm{v}(t)a^*\mathrm{w}^*(t)$ are complex conjugates, and that the sum of a complex number and its complex conjugate is twice the real part of either one,

$$P_{\mathrm{z}} = P_{\mathrm{v}} + a^2 P_{\mathrm{w}} - 2a\,\mathrm{Re}\left[\left\langle \mathrm{v}(t)\mathrm{w}^*(t) \right\rangle \right] = P_{\mathrm{v}} + a^2 P_{\mathrm{w}} - 2aR_{vw}$$

Correlation

$$P_z = P_v + a^2 P_w - 2aR_{vw}$$

Now find the value of a that minimizes P_z by differentiating with respect to a and setting the derivative equal to zero.

$$\frac{\partial}{\partial a} P_z = 2aP_w - 2R_{vw} = 0 \Rightarrow a = \frac{R_{vw}}{P_w}$$

Therefore, to make v and aw as similar as possible (minimizing z) set a to the correlation of v and w divided by the signal power of w.

If $v(t) = w(t)$, $a = 1$. If $v(t) = -w(t)$ then $a = -1$.

If $R_{vw} = 0$, $P_z = P_v + a^2 P_w$.

Correlation

The **correlation** between two energy signals x and y is the area under the product of x and y^*.

$$R_{xy} = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

The **correlation function** between two energy signals x and y is the area under the product as a function of how much y is shifted relative to x.

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t-\tau) dt = \int_{-\infty}^{\infty} x(t+\tau) y^*(t) dt$$

In the very common case in which x and y are both real-valued,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t-\tau) dt = \int_{-\infty}^{\infty} x(t+\tau) y(t) dt$$

Correlation

The correlation function for two real-valued energy signals is very similar to the convolution of two real-valued energy signals.

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\lambda) y(\lambda) d\lambda = \int_{-\infty}^{\infty} x(\lambda) y(t-\lambda) d\lambda$$

Therefore it is possible to use convolution to find the correlation function.

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(\lambda) y(\lambda-\tau) d\lambda = \int_{-\infty}^{\infty} x(\lambda) y(-(\tau-\lambda)) d\lambda = x(\tau) * y(-\tau)$$

(λ is used here as the variable of integration instead of t or τ to avoid confusion among different meanings for t and τ in correlation and convolution formulas.) It also follows that

$$R_{xy}(\tau) \xleftrightarrow{\mathcal{F}} X(f) Y^*(f)$$

Correlation

The correlation function between two power signals x and y is the average value of the product of x and y* as a function of how much y* is shifted relative to x.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T \mathrm{x}(t)\,\mathrm{y}^*(t-\tau)\,dt$$

If the two signals are both periodic and their fundamental periods have a finite least common period, where T is any integer multiple of that least common period.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \frac{1}{T} \int_T \mathrm{x}(t)\,\mathrm{y}^*(t-\tau)\,dt$$

For real-valued periodic signals this becomes

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \frac{1}{T} \int_T \mathrm{x}(t)\,\mathrm{y}(t-\tau)\,dt$$

Correlation

Correlation of real periodic signals is very similar to periodic convolution

$$R_{xy}(\tau) = \frac{x(\tau) \circledast y(-\tau)}{T}$$

where it is understood that the period of the periodic convolution is any integer multiple of the least common period of the two fundamental periods of x and y.

$$R_{xy}(\tau) \xleftrightarrow[T]{\mathcal{FS}} c_x[k] c_y^*[k]$$

Correlation

Find the correlation of $\mathrm{x}(t) = A\cos(2\pi f_0 t)$ with $\mathrm{y}(t) = B\sin(2\pi f_0 t)$.

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{x}_1(t)\mathrm{x}_2(t-\tau)dt = \lim_{T\to\infty}\frac{AB}{T}\int_{-T/2}^{T/2}\cos(2\pi f_0 t)\sin\left(2\pi f_0(t-\tau)\right)dt$$

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{AB}{2T}\int_{-T/2}^{T/2}\left[\sin\left(2\pi f_0(-\tau)\right)+\sin\left(4\pi f_0 t-\tau\right)\right]dt$$

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}-\frac{AB}{2T}\int_{-T/2}^{T/2}\sin(2\pi f_0\tau)dt = \lim_{T\to\infty}-\frac{AB}{2T}\left[t\sin(2\pi f_0\tau)\right]_{-T/2}^{T/2} = -\frac{AB}{2}\sin(2\pi f_0\tau)$$

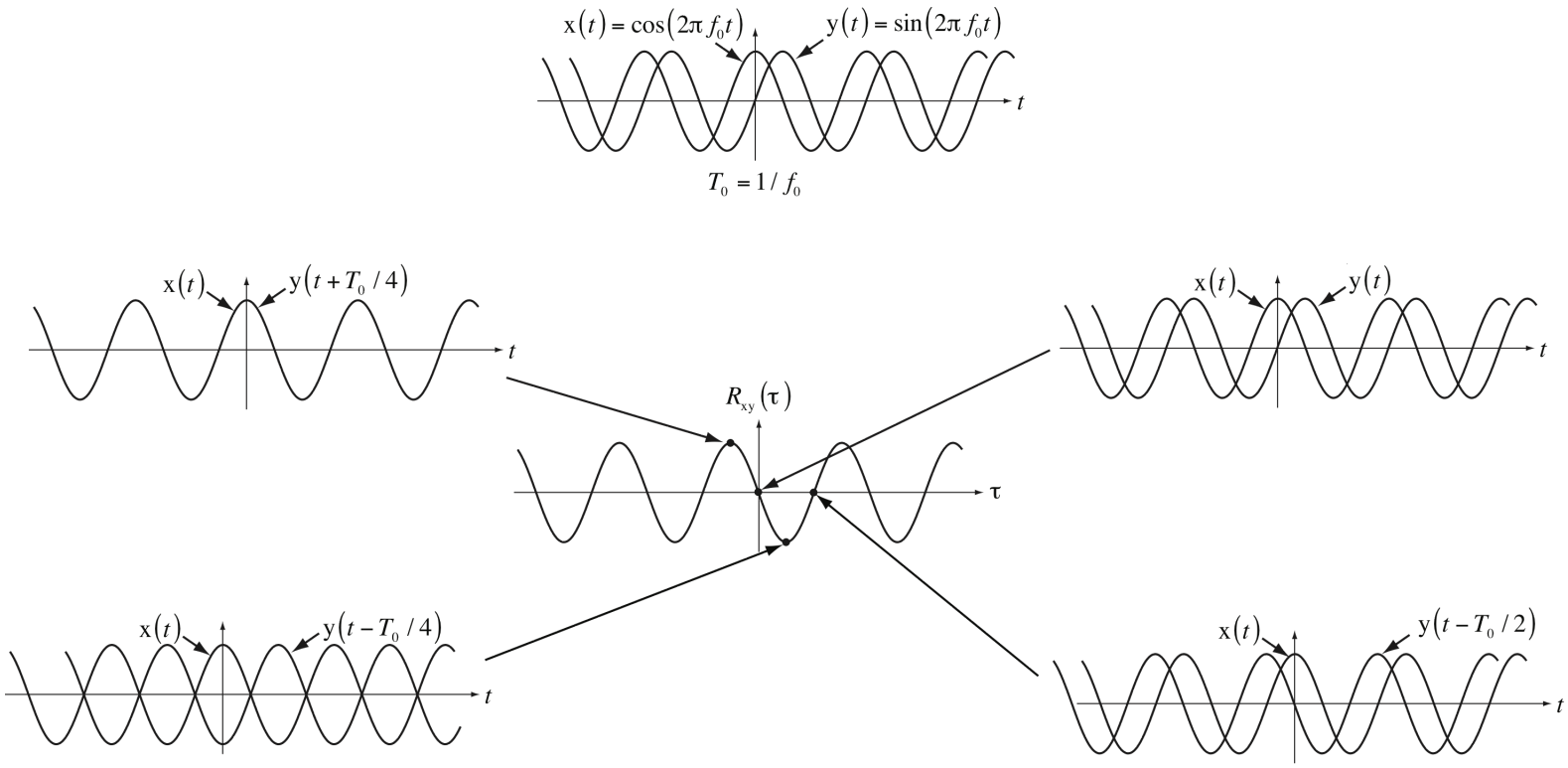
OR

$$\mathrm{R}_{12}(\tau)\xleftrightarrow[T_0]{\mathcal{FS}}\mathrm{c}_{\mathrm{x}}[k]\mathrm{c}_y^*[k] = (A/2)\left(\delta[k-1]+\delta[k+1]\right)(-jB/2)\left(\delta[k+1]-\delta[k-1]\right)$$

$$\mathrm{R}_{12}(\tau)\xleftrightarrow[T_0]{\mathcal{FS}}(-jAB/4)\left(\delta[k+1]-\delta[k-1]\right)$$

$$\mathrm{R}_{12}(\tau) = -\frac{AB}{2}\sin(2\pi f_0\tau)\xleftrightarrow[T_0]{\mathcal{FS}}(-jAB/4)\left(\delta[k+1]-\delta[k-1]\right)$$

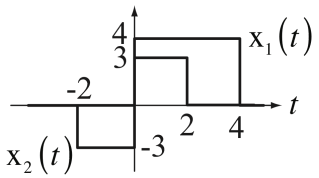
Correlation



Correlation

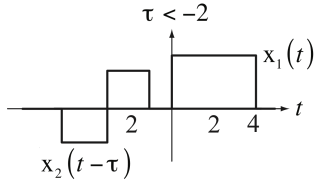
Find the correlation function between these two functions.

$$x_1(t)=\begin{cases}4 & , \ 0<t<4\\ 0 & , \ \text{otherwise}\end{cases} \quad , \quad x_2(t)=\begin{cases}-3 & , \ -2<t<0\\ 3 & , \ 0<t<2\\ 0 & , \ \text{otherwise}\end{cases}$$

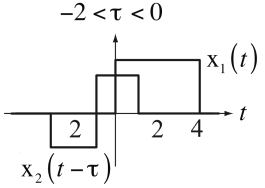


$$R_{12}(\tau)=\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)\,dt$$

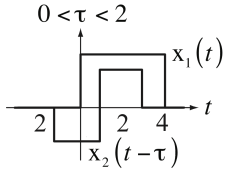
For $\tau<-2$, $x_1(t)x_2(t-\tau)=0$ and $R_{12}(\tau)=0$.



For $-2<\tau<0$, $x_1(t)x_2(t-\tau)=\begin{cases}4\times 3 & , \ 0<t<2+\tau\\ 0 & , \ \text{otherwise}\end{cases}\Rightarrow R_{12}(\tau)=12(2+\tau)$

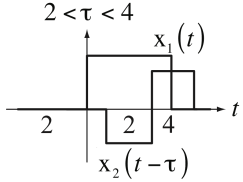


For $0<\tau<2$, $x_1(t)x_2(t-\tau)=\begin{cases}4\times(-3) & , \ 0<t<\tau\\ 4\times 3 & , \ \tau<t<2+\tau\\ 0 & , \ \text{otherwise}\end{cases}\Rightarrow R_{12}(\tau)=-12\tau+24=12(2-\tau)$

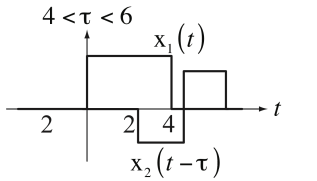


Correlation

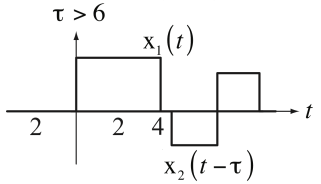
For $2<\tau<4$, $x_1(t)x_2(t-\tau)=\begin{cases}4\times(-3) & ,\ \tau-2<t<\tau\\ 4\times 3 & ,\ \tau<t<4\\ 0 & ,\ \text{otherwise}\end{cases}\Rightarrow R_{12}(\tau)=12(2-\tau)$



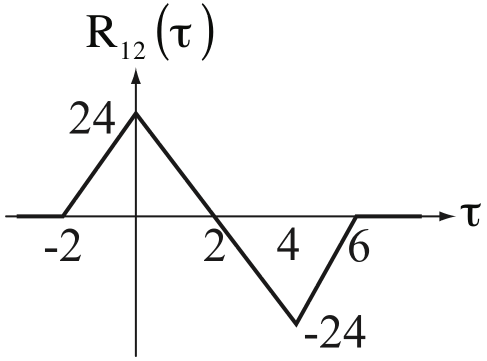
For $4<\tau<6$, $x_1(t)x_2(t-\tau)=\begin{cases}4\times(-3) & ,\ \tau-2<t<4\\ 0 & ,\ \text{otherwise}\end{cases}\Rightarrow R_{12}(\tau)=-12(6-\tau)$



For $\tau>6$, $x_1(t)x_2(t-\tau)=0$ and $R_{12}(\tau)=0$.



τ	–2	–1	0	1	2	3	4	5	6
$R_{12}(\tau)$	0	12	24	12	0	–12	–24	–12	0



Correlation

Find the correlation function between these two functions.

$$x_1(t)=\begin{cases}4 & , \ 0<t<4\\ 0 & , \ \text{otherwise}\end{cases} \quad , \quad x_2(t)=\begin{cases}-3 & , \ -2<t<0\\ 3 & , \ 0<t<2\\ 0 & , \ \text{otherwise}\end{cases}$$

Alternate Solution:

$$x_1(t)=4\,\text{rect}\left(\frac{t-2}{4}\right) \quad , \quad x_2(t)=3\left[-\text{rect}\left(\frac{t+1}{2}\right)+\text{rect}\left(\frac{t-1}{2}\right)\right] \ , \ x_2(-t)=3\left[\text{rect}\left(\frac{t+1}{2}\right)-\text{rect}\left(\frac{t-1}{2}\right)\right]$$

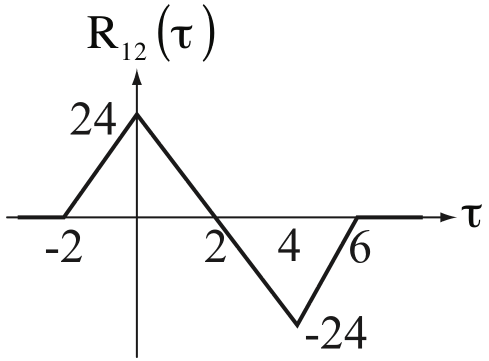
Using $\text{rect}(t/a)*\text{rect}(t/b)=\frac{|a|+|b|}{2}\text{tri}\left(\frac{2t}{|a|+|b|}\right)-\frac{||a|-|b||}{2}\text{tri}\left(\frac{2t}{||a|-|b||}\right)$

$$R_{12}(\tau)=x_1(\tau)*x_2(-\tau)=12\left[3\,\text{tri}\left(\frac{\tau-1}{3}\right)-\text{tri}(\tau-1)-3\,\text{tri}\left(\frac{\tau-3}{3}\right)+\text{tri}(\tau-3)\right]$$

Checking some values of τ:

τ	–2	–1	0	1	2	3	4	5	6
$R_{12}(\tau)$	0	12	24	12	0	–12	–24	–12	0

These answers are the same as in the previous solution.



Autocorrelation

A very important special case of correlation is **autocorrelation**. Autocorrelation is the correlation of a function with a shifted version of itself. For energy signals,

$$R_x(\tau) = R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

At a shift τ of zero,

$$R_x(0) = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = E_x$$

which is the signal energy of the signal.

Autocorrelation

For power signals autocorrelation is

$$\mathrm{R_x}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T \mathrm{x}(t)\,\mathrm{x}^*(t-\tau)\,dt$$

At a shift τ of zero,

$$\mathrm{R_x}(0) = \lim_{T\to\infty} \frac{1}{T} \int_T \left|\mathrm{x}(t)\right|^2 dt$$

which is the average signal power of the signal.

Autocorrelation

For real signals, autocorrelation is an even function.

$$R_x(\tau) = R_x(-\tau)$$

Autocorrelation magnitude can never be larger than it is at zero shift.

$$R_x(0) \geq |R_x(\tau)|$$

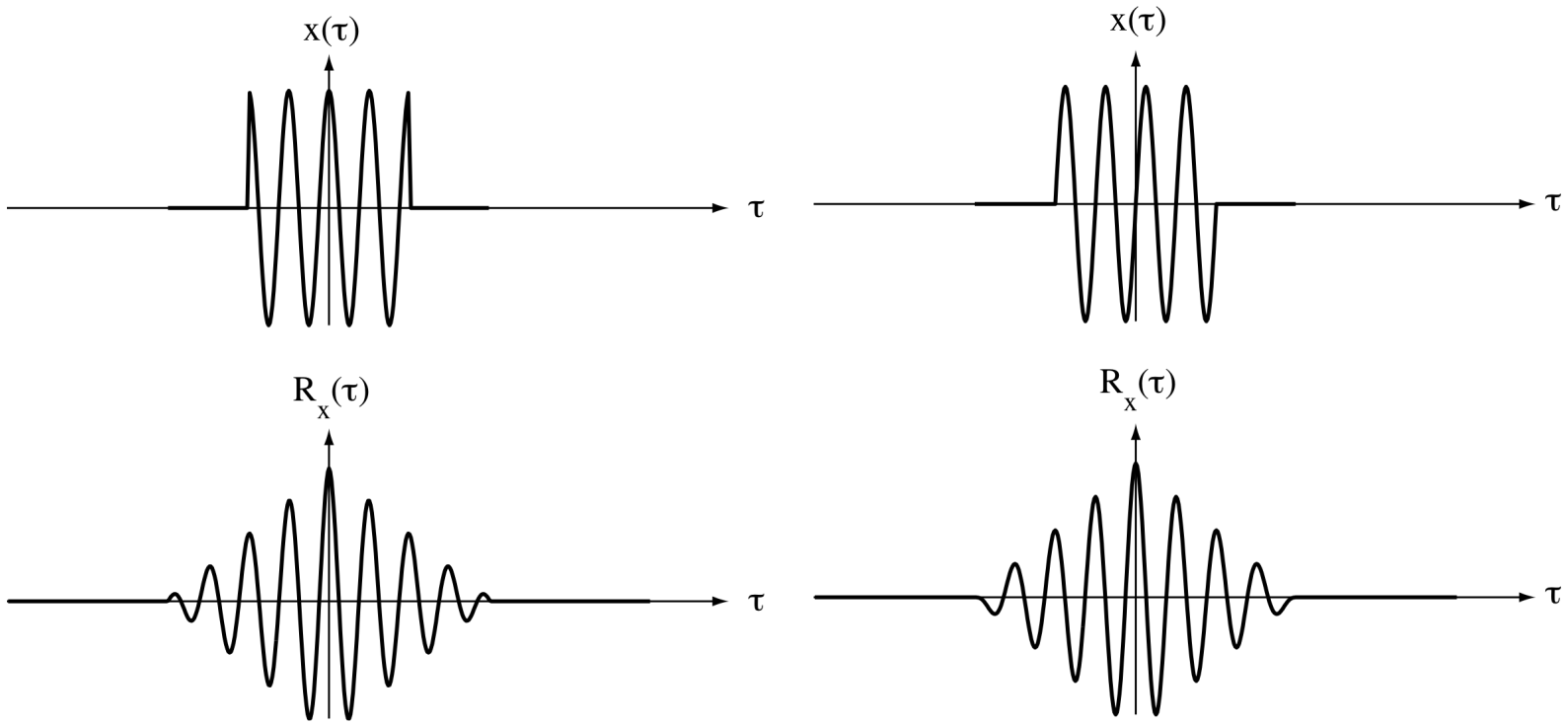
If a signal is time shifted its autocorrelation does not change.

The autocorrelation of a sum of sinusoids of different frequencies is the sum of the autocorrelations of the individual sinusoids.

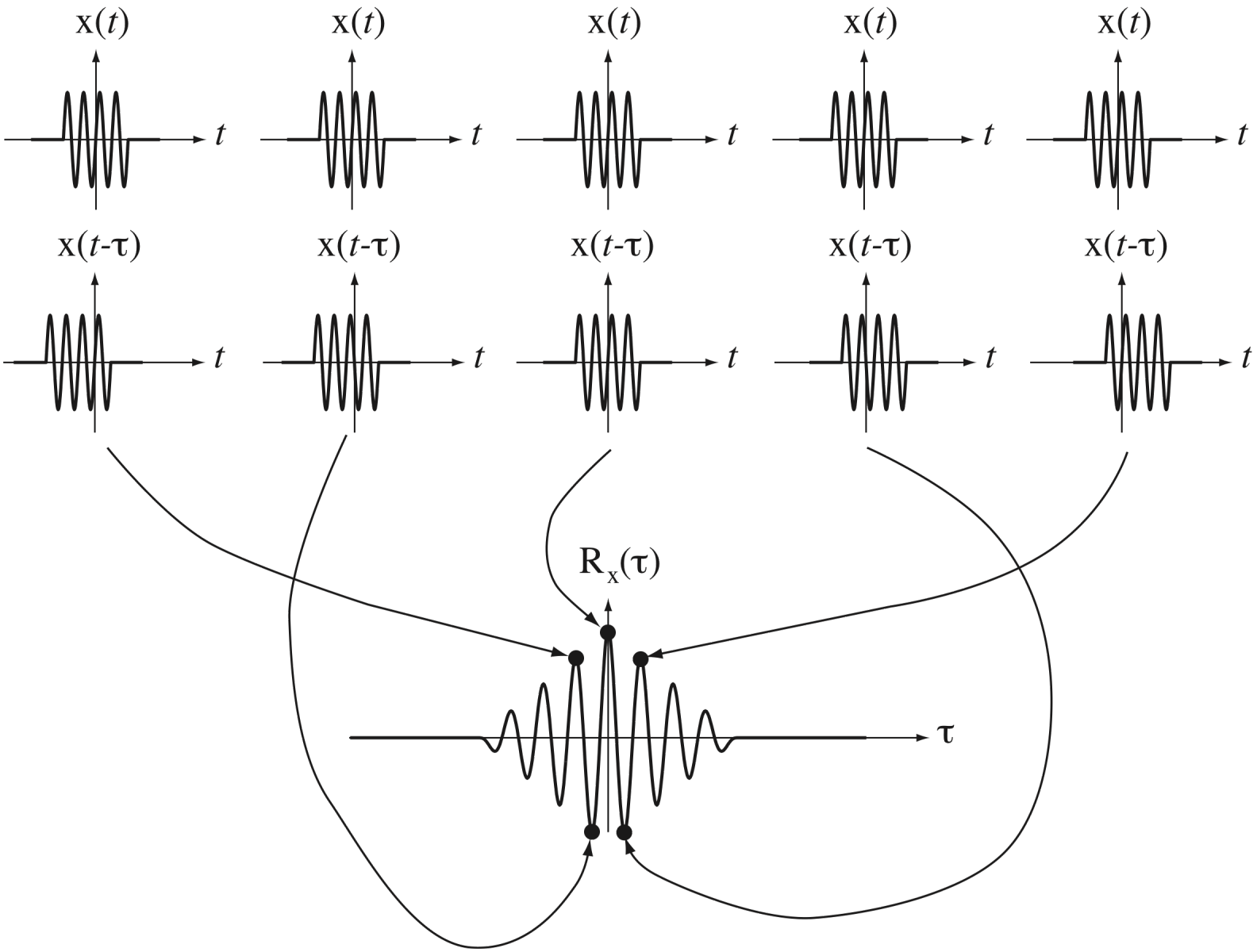
Autocorrelation

Autocorrelations for a cosine “burst” and a sine “burst”.

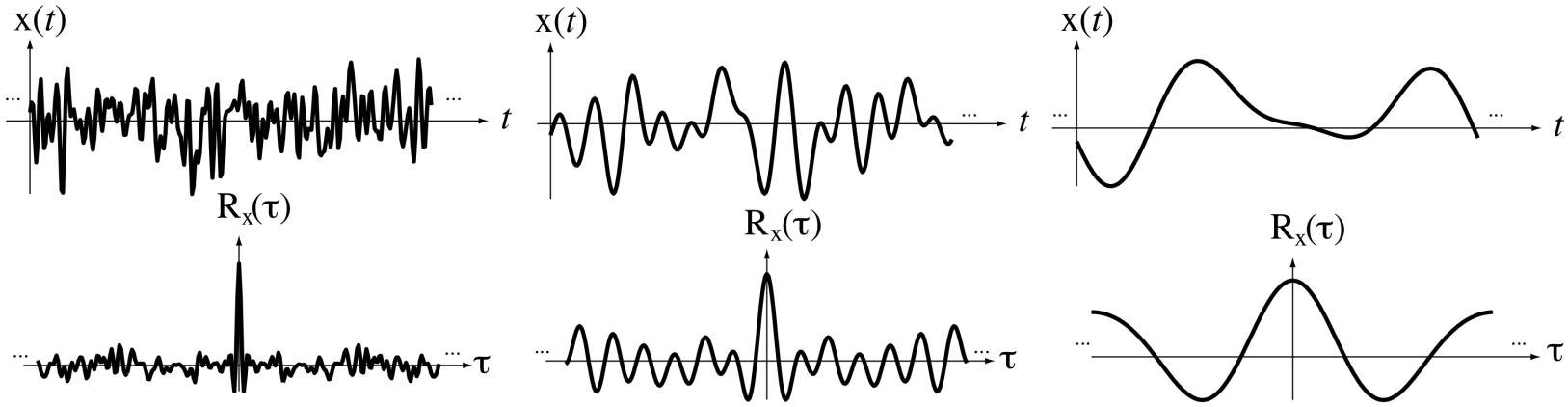
Notice that they are almost (but not quite) identical.



Autocorrelation

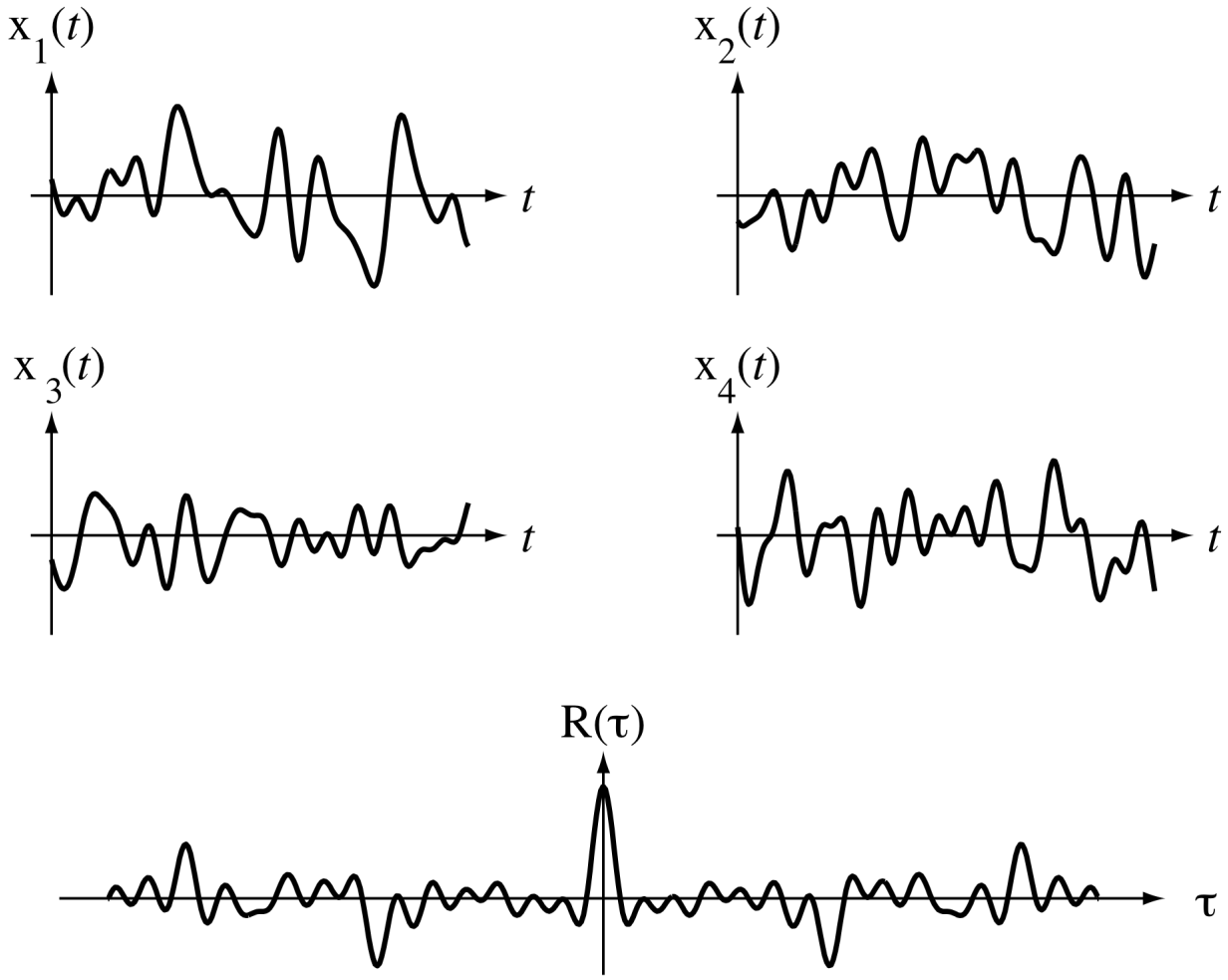


Autocorrelation



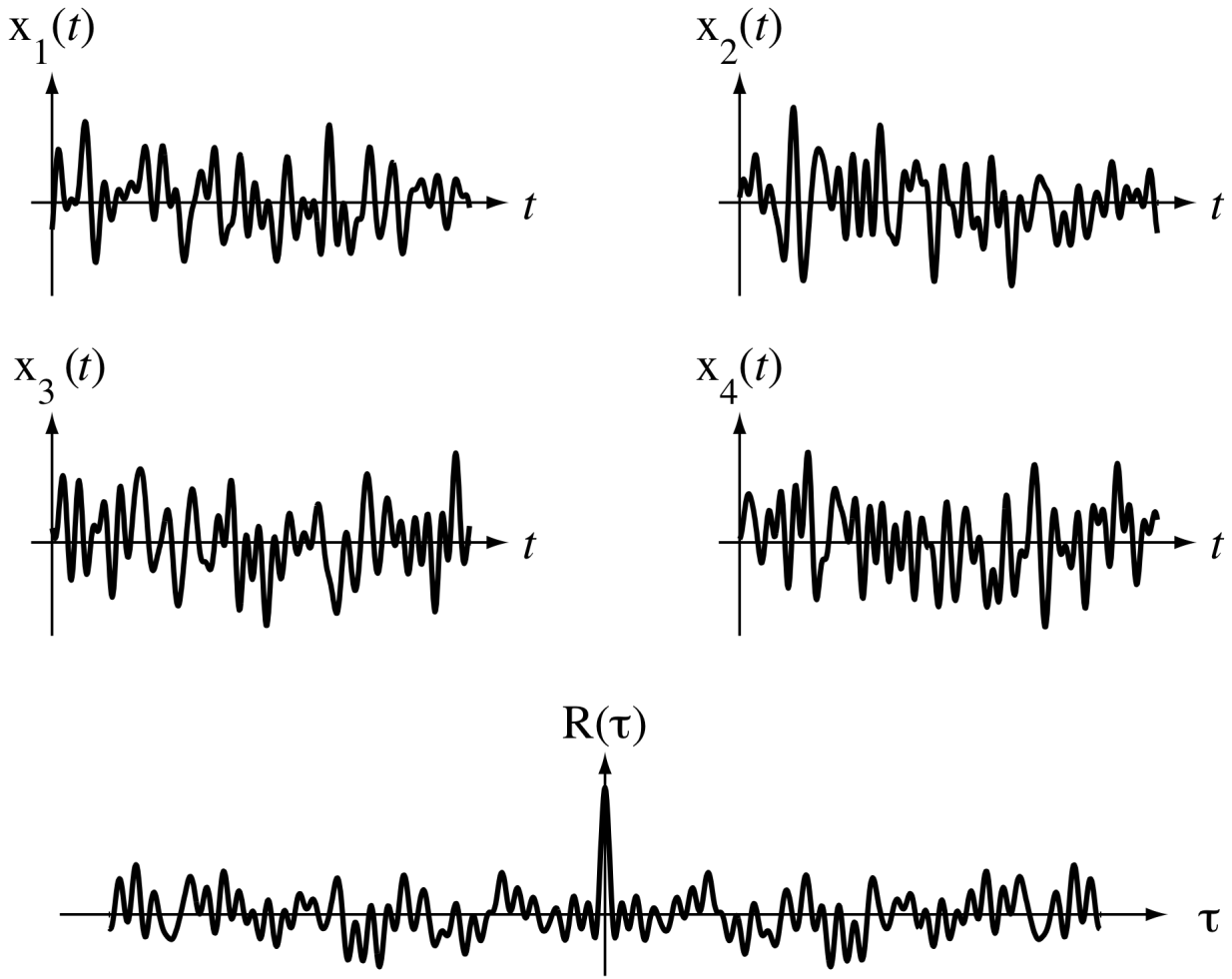
Autocorrelation

Different Signals Can Have the Same Autocorrelation



Autocorrelation

Different Signals Can Have the Same Autocorrelation



Energy Spectral Density

Parseval's theorem says that the total signal energy in an energy signal is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

The quantity $|X(f)|^2$ is called the **energy spectral density** (**ESD**) of the signal x and is conventionally given the symbol $\Psi_x(f)$ ($G_x(f)$ in the book). That is,

$$\Psi_x(f) = |X(f)|^2 = X(f)X^*(f)$$

It can be shown that if x is a real-valued signal that the ESD is even, non-negative and real. In the term "spectral density", "spectral" refers to variation over a "spectrum" of frequencies and "density" refers to the fact that, since the integral if $\Psi_x(f)$ yields signal energy, $\Psi_x(f)$ must be signal energy density in signal energy/Hz.

Energy Spectral Density

It can be shown that, for an energy signal, ESD and autocorrelation form a Fourier transform pair.

$$\mathrm{R}_{\mathrm{x}}(t) \overset{\mathcal{F}}{\longleftrightarrow} \Psi_{\mathrm{x}}(f)$$

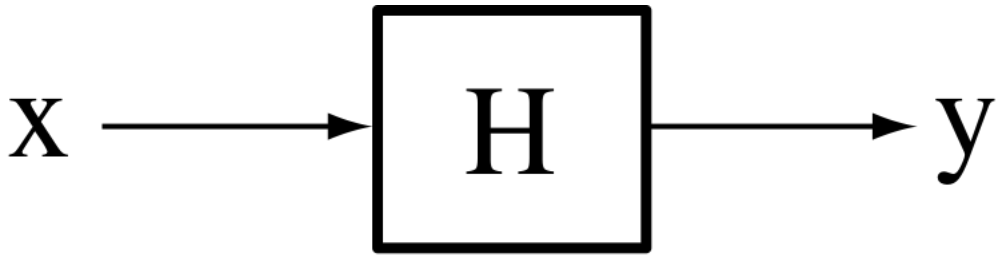
The signal energy of a signal is the area under the energy spectral density and is also the value of the autocorrelation at zero shift.

$$E_{\mathrm{x}} = \mathrm{R}_{\mathrm{x}}(0) = \int_{-\infty}^{\infty} \Psi_{\mathrm{x}}(f)\,df$$

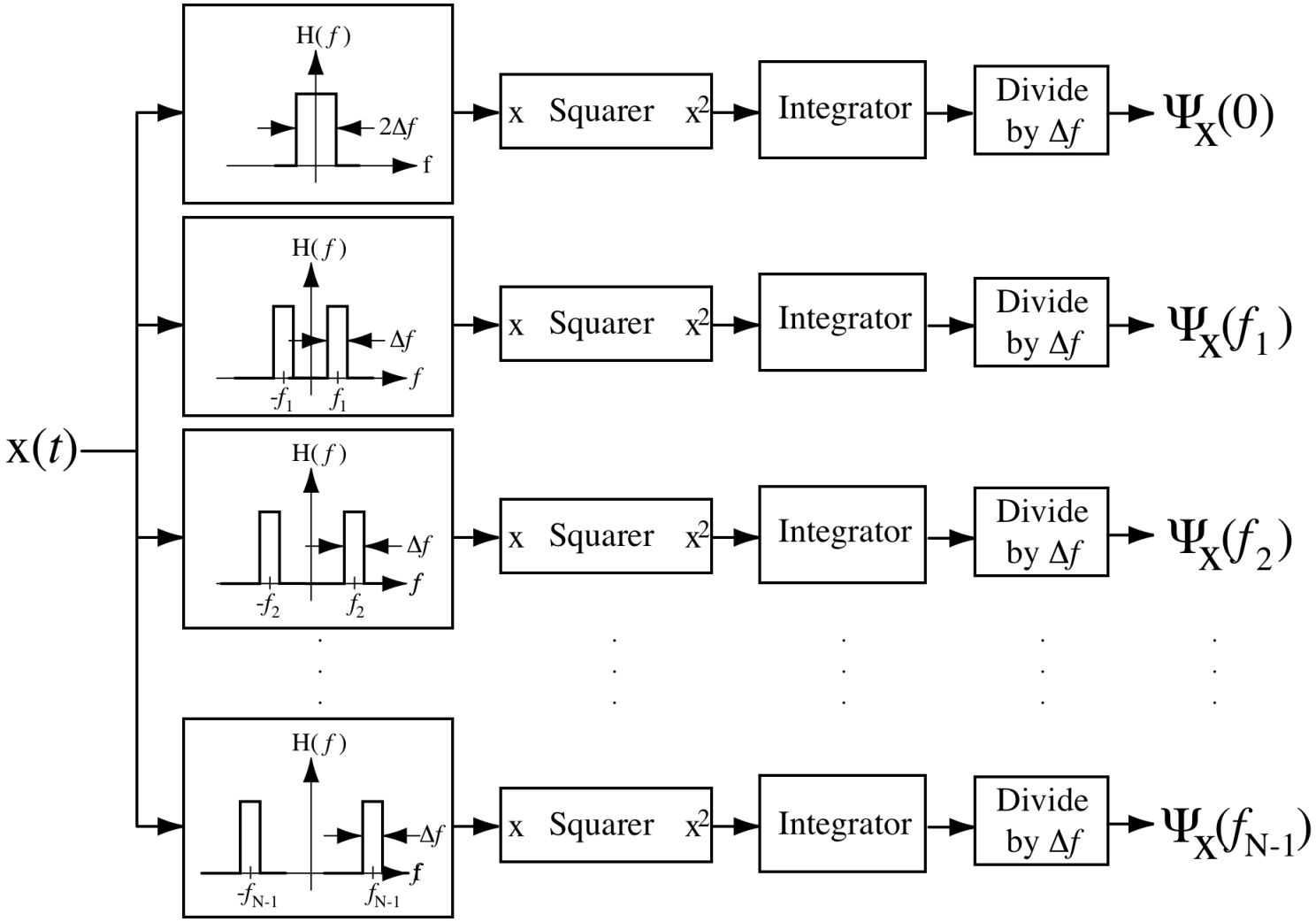
Energy Spectral Density

Probably the most important fact about ESD is the relationship between the ESD of the excitation of an LTI system and the ESD of the response of the system. It can be shown that they are related by

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f)H^*(f)\Psi_x(f)$$



Energy Spectral Density



Energy Spectral Density

Find the energy spectral density of $\mathrm{x}(t) = 10\,\mathrm{rect}\left(\frac{t-3}{4}\right)$.

Using $\Psi_{\mathrm{x}}(f) = \left|\mathrm{X}(f)\right|^2 = \mathrm{X}(f)\mathrm{X}^*(f)$,

$$\mathrm{X}(f) = 40\,\mathrm{sinc}(4f)e^{-j6\pi f}$$

$$\mathrm{X}(f)\mathrm{X}^*(f) = 40\,\mathrm{sinc}(4f)e^{-j6\pi f} \times 40\,\mathrm{sinc}(4f)e^{j6\pi f}$$

$$\mathrm{X}(f)\mathrm{X}^*(f) = 1600\,\mathrm{sinc}^2(4f)$$

Power Spectral Density

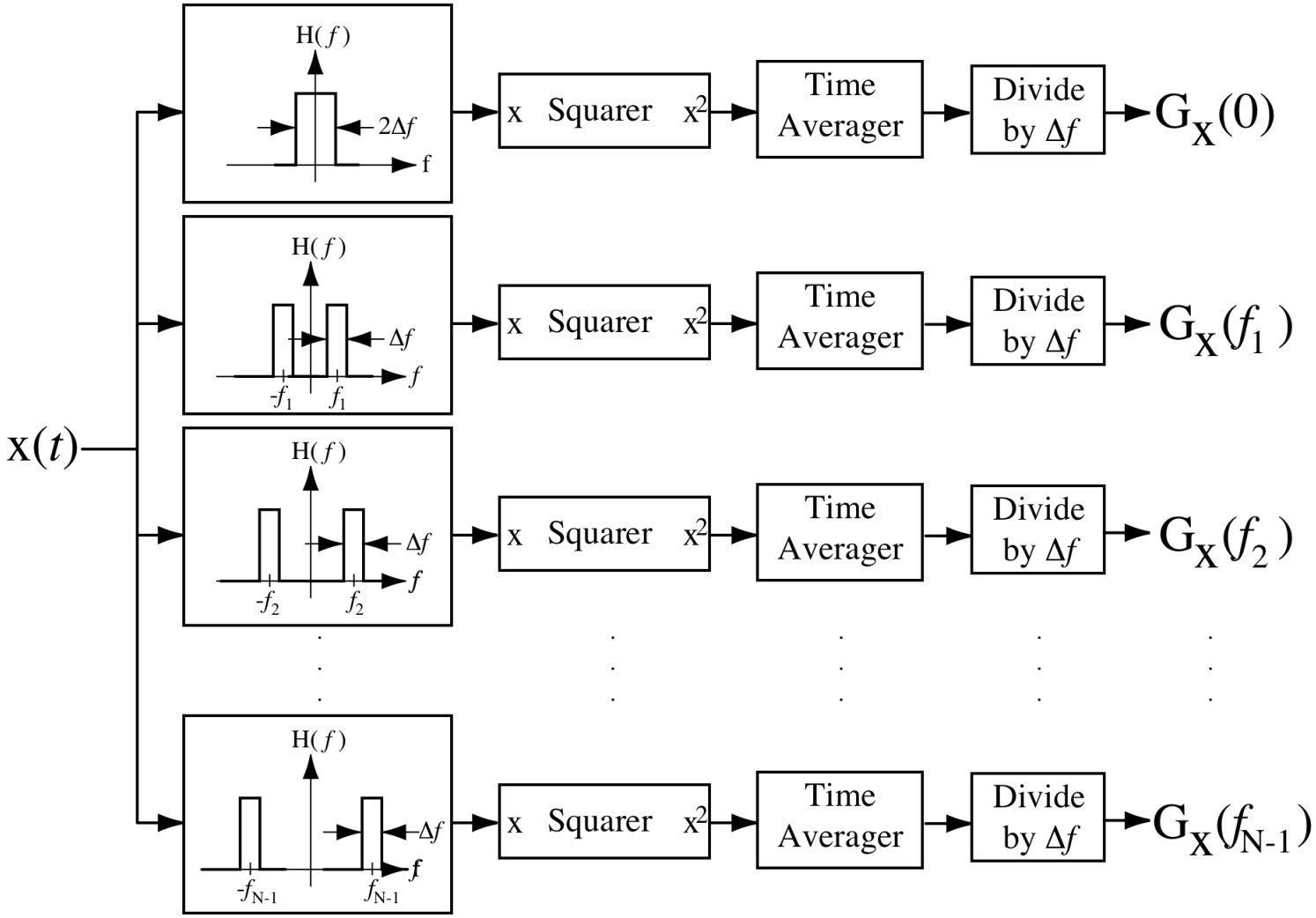
Power spectral density (**PSD**) applies to power signals in the same way that energy spectral density applies to energy signals. The PSD of a signal x is conventionally indicated by the notation $G_x(f)$ whose units are signal power/Hz. In an LTI system,

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

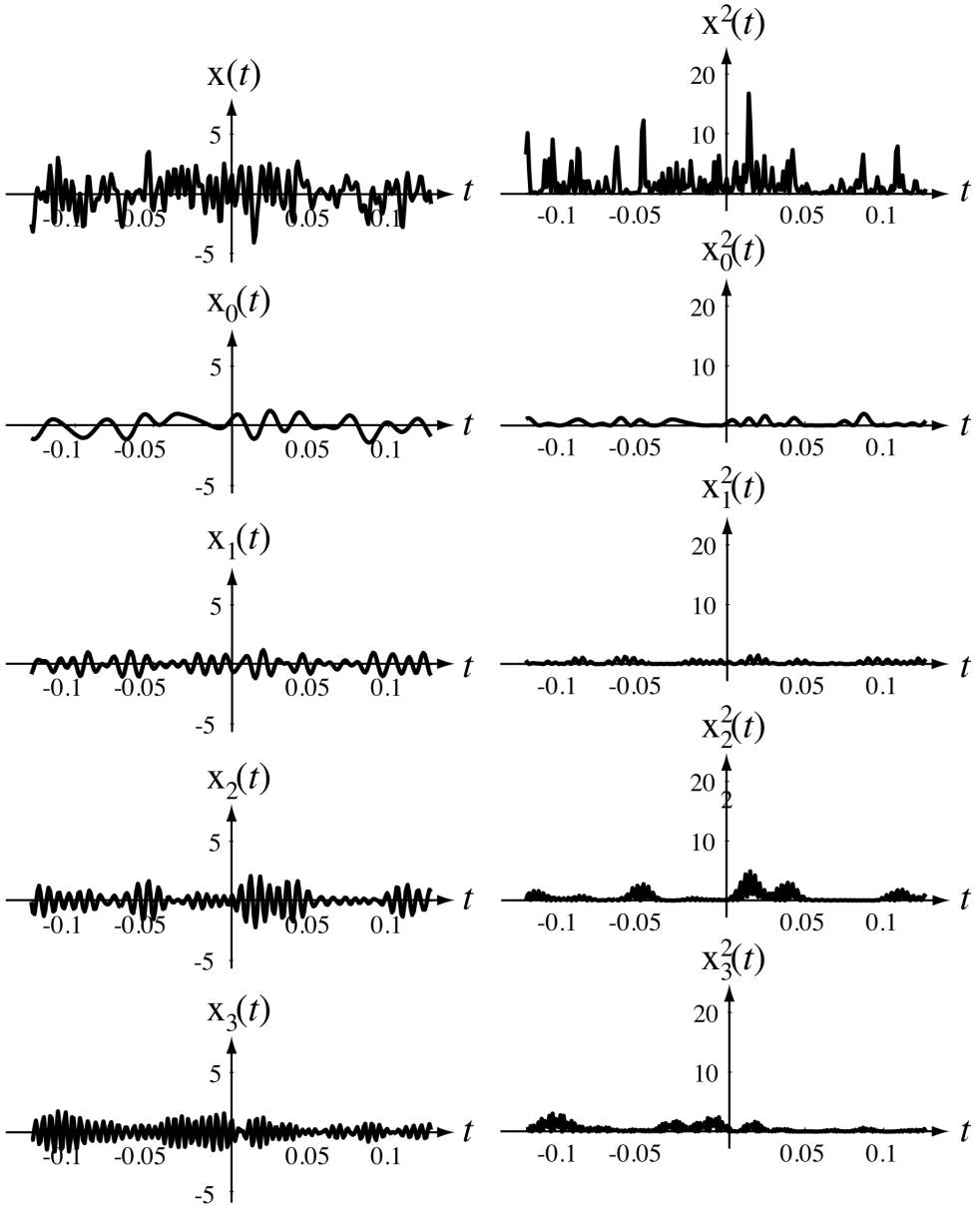
Also, for a power signal, PSD and autocorrelation form a Fourier transform pair.

$$R(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$$

PSD Concept


x(t)
H(f)
2Δf
f
Δf
-f1
f1
-f2
f2
-fN-1
fN-1
x Squarer x2
Time Averager
Divide by Δf
GX(0)
GX(f1)
GX(f2)
GX(fN-1)

Typical Signals in PSD Concept



Power Spectral Density

Find the power spectral density of $\mathrm{x}(t) = 30\sin(200\pi t)\cos(200000\pi t)$.

Using $\mathrm{R}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{G}(f)$ and $\mathrm{R}_{\mathrm{xy}}(\tau) \overset{\mathcal{FS}}{\underset{T}{\longleftrightarrow}} \mathrm{c}_{\mathrm{x}}[k]\mathrm{c}_{\mathrm{y}}^{*}[k]$

$\mathrm{R}_{\mathrm{x}}(\tau) \overset{\mathcal{FS}}{\underset{T}{\longleftrightarrow}} \mathrm{c}_{\mathrm{x}}[k]\mathrm{c}_{\mathrm{x}}^{*}[k]$

Using $T = T_0 = 0.01$ s,

$$\mathrm{c}_{\mathrm{x}}[k] = 30(j/2)\left(\delta[k+1] - \delta[k-1]\right) * (1/2)\left(\delta[k-1000] + \delta[k+1000]\right)$$

$$\mathrm{c}_{\mathrm{x}}[k] = j\frac{15}{2}\left(\delta[k-999] + \delta[k+1001] - \delta[k-1001] - \delta[k+999]\right)$$

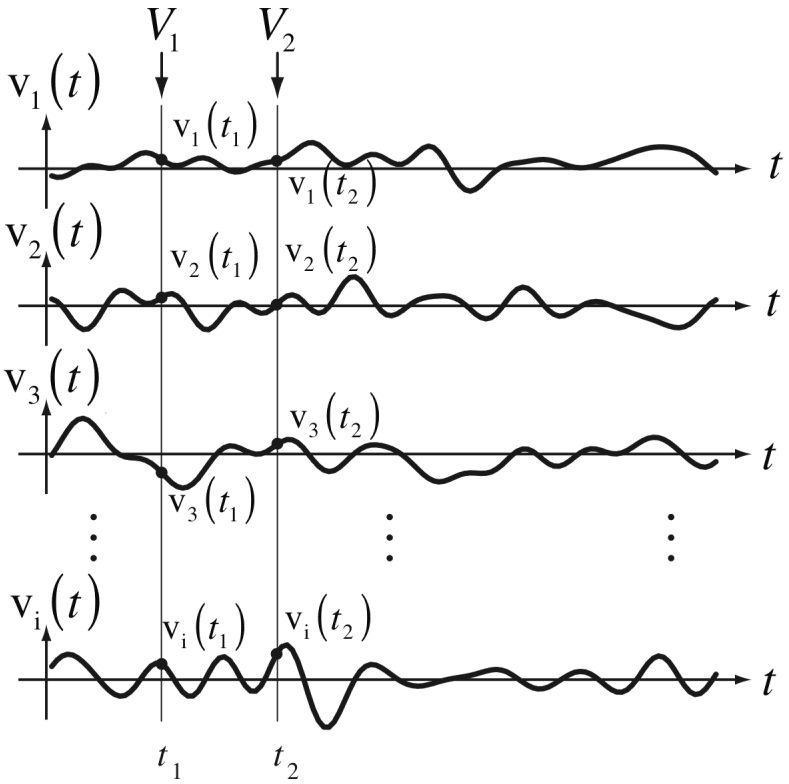
$$\mathrm{c}_{\mathrm{x}}[k]\mathrm{c}_{\mathrm{x}}^{*}[k] = \left(\frac{15}{2}\right)^{2}\left(\delta[k-999] + \delta[k+1001] + \delta[k-1001] + \delta[k+999]\right)$$

$$\mathrm{R}_{\mathrm{x}}(\tau) = \frac{15^2}{2}\left[\cos(199800\pi\tau) + \cos(200200\pi\tau)\right]$$

$$\mathrm{G}(f) = \frac{15^2}{4}\left[\delta(f-99900) + \delta(f+9900) + \delta(f-100100) + \delta(f+100100)\right]$$

Random Processes

A **random process** maps experimental outcomes into real functions of time. The collection of time functions is known as an **ensemble** and each member of the ensemble is called a **sample function**. The ensemble will be represented by the notation $\mathrm{v}(t,s)$ in which t is time and s is the sample function.



Random Processes

To simplify notation, let $\mathrm{v}(t,s)$ become just $\mathrm{v}(t)$ where it will be understood from context that $\mathrm{v}(t)$ is a sample function from a random process. The **mean value** of $\mathrm{v}(t)$ at any arbitrary time t is $\mathrm{E}\big(\mathrm{v}(t)\big)$. This is an **ensemble mean**, not a time average. It is the average of all the sample function values at time t, $\mathrm{E}\big(\mathrm{v}(t_1)\big) = \overline{V}_1$. **Autocorrelation** is defined by $\mathrm{R}_\mathrm{v}(t_1,t_2) \triangleq \mathrm{E}\big(\mathrm{v}(t_1)\mathrm{v}(t_2)\big)$. If V_1 and V_2 are statistically independent then $\mathrm{R}_\mathrm{v}(t_1,t_2) = \overline{V}_1\overline{V}_2$. If $t_1 = t_2$, then $V_1 = V_2$ and $\mathrm{R}_\mathrm{v}(t_1,t_2) = \overline{V_1^2}$ and, in general, $\mathrm{R}_\mathrm{v}(t,t) = \mathrm{E}\big(\mathrm{v}^2(t)\big) = \overline{\mathrm{v}^2(t)}$, the mean-squared value of $\mathrm{v}(t)$ as a function of time.

Random Processes

A generalization of autocorrelation to the relation between two different random processes is **cross - correlation** defined by $R_{vw}(t_1,t_2) \triangleq E\left(v(t_1)w(t_2)\right)$. The **covariance function** is defined by

$$C_{vw}(t_1,t_2) \triangleq E\left(\left[v(t_1)-E\left(v(t_1)\right)\right]\left[w(t_1)-E\left(w(t_1)\right)\right]\right).$$

If, for all t_1 and t_2, $R_{vw}(t_1,t_2) = \overline{v(t_1)} \times \overline{w(t_2)}$ then v and w are said to be **uncorrelated** and $C_{vw}(t_1,t_2) = 0$. So zero covariance implies that two processes are uncorrelated *but not necessarily independent*. Independent random processes are uncorrelated but uncorrelated random processes are not necessarily independent. If, for all t_1 and t_2, $R_{vw}(t_1,t_2) = 0$, the two random processes are said to be **orthogonal**.

Random Processes

A random process is **ergodic** if all time averages of sample functions are equal to the corresponding ensemble averages. If $\mathrm{g}\left(\mathrm{v}_\mathrm{i}\left(t\right)\right)$ is any function of $\mathrm{v}_\mathrm{i}\left(t\right)$, then its time average is

$$\left\langle \mathrm{g}\left(\mathrm{v}_\mathrm{i}\left(t\right)\right)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{g}\left(\mathrm{v}_\mathrm{i}\left(t\right)\right)dt$$

So, for an ergodic process, $\left\langle \mathrm{g}\left(\mathrm{v}_\mathrm{i}\left(t\right)\right)\right\rangle = \mathrm{E}\left(\mathrm{g}\left(\mathrm{v}\left(t\right)\right)\right)$. By definition $\left\langle \mathrm{g}\left(\mathrm{v}_\mathrm{i}\left(t\right)\right)\right\rangle$ is independent of time because it is an average over all time. It then follows that ensemble averages of ergodic processes are independent of time. If a random process $\mathrm{v}\left(t\right)$ is ergodic then $\mathrm{E}\left(\mathrm{v}\left(t\right)\right) = \overline{\mathrm{v}} = m_\mathrm{v}$ and $\mathrm{E}\left(\mathrm{v}^2\left(t\right)\right) = \overline{\mathrm{v}^2} = \sigma_\mathrm{v}^2 + m_\mathrm{v}^2$ where m and σ_v^2 are the mean and variance of $\mathrm{v}\left(t\right)$.

Random Processes

For an ergodic random process representing an electrical signal we can identify some common terms as follows:

Mean value m_v is the "DC" component $\langle v_i(t) \rangle$.

The square of the mean m_v^2 is the "DC" power $\langle v_i(t) \rangle^2$ (the power in the average).

The mean-squared value $\overline{v^2}$ is the total average power $\langle v_i^2(t) \rangle$.

The variance σ_v^2 is the "AC" power (the power in the time-varying part).

The standard deviation σ_v is the RMS value of the time-varying part.

(Be sure not to make the common mistake of confusing "the square of the mean" with "the mean-squared value", which means "the mean of the square". In general the square of the mean and the mean of the square are different.)

Random Processes

Actually proving that a random process is ergodic is usually very difficult, if not impossible. A much more common and useful requirement on a random process that is much easier to prove is that it be **wide-sense stationary** (**WSS**). A random process is wide-sense stationary when the mean $\mathrm{E}\left(\mathrm{v}\left(t\right)\right)$ is independent of time and the autocorrelation function $\mathrm{R}_{\mathrm{v}}\left(t_1,t_2\right)$ depends only on the time difference $t_1 - t_2$. So wide-sense stationarity requires $\mathrm{E}\left(\mathrm{v}\left(t\right)\right) = m_{\mathrm{v}}$ and $\mathrm{R}_{\mathrm{v}}\left(t_1,t_2\right) = \mathrm{R}_{\mathrm{v}}\left(t_1 - t_2\right)$ and we usually write autocorrelation functions with the notation $\mathrm{R}_{\mathrm{v}}\left(\tau\right)$ in which $\tau = t_1 - t_2$. So

$$\mathrm{R}_{\mathrm{v}}\left(\tau\right) = \mathrm{E}\left(\mathrm{v}\left(t\right)\mathrm{v}\left(t-\tau\right)\right) = \mathrm{E}\left(\mathrm{v}\left(t+\tau\right)\mathrm{v}\left(t\right)\right)$$

and $\mathrm{R}_{\mathrm{v}}\left(\tau\right)$ has the properties $\mathrm{R}_{\mathrm{v}}\left(\tau\right) = \mathrm{R}_{\mathrm{v}}\left(-\tau\right)$, $\mathrm{R}_{\mathrm{v}}\left(0\right) = \overline{\mathrm{v}^2} = m_{\mathrm{v}}^2 + \sigma_{\mathrm{v}}^2$ and $\left|\mathrm{R}_{\mathrm{v}}\left(\tau\right)\right| \le \mathrm{R}_{\mathrm{v}}\left(0\right)$.

Random Processes

$\mathrm{R_v}(\tau)$ indicates the similarity of $\mathrm{v}(t)$ and $\mathrm{v}(t \pm \tau)$. If $\mathrm{v}(t)$ and $\mathrm{v}(t \pm \tau)$ are independent of each other as $\tau \to \infty$, then $\lim_{\tau \to \pm\infty} \mathrm{R_v}(\tau) = \overline{\mathrm{v}^2} = m_{\mathrm{v}}^2$. If the sample functions of $\mathrm{v}(t)$ are periodic, then $\mathrm{v}(t)$ and $\mathrm{v}(t \pm \tau)$ do not become independent as $\tau \to \infty$ and $\mathrm{R_v}(\tau \pm nT_0) = \mathrm{R_v}(\tau)$, n an integer.

The **average power** of a random process $\mathrm{v}(t)$ is the ensemble average of $\langle \mathrm{v}^2(t) \rangle$, $P \triangleq \mathrm{E}\left(\langle \mathrm{v}^2(t) \rangle\right) = \langle \mathrm{E}\left(\mathrm{v}^2(t)\right) \rangle$. If the random process is stationary $P = \mathrm{R_v}(0)$.

Random Processes

A very important special case of a random process is the **gaussian** random process. A random process is gaussian if all its marginal, joint and conditional probability density functions (pdf's) are gaussian. Gaussian processes are important because they occur so frequently in nature. If a random process $\mathrm{v}(t)$ is gaussian the following properties apply:

1. The process is completely characterized by $\mathrm{E}(\mathrm{v}(t))$ and $\mathrm{R}_{\mathrm{v}}(t_1,t_2)$.
2. If $\mathrm{R}_{\mathrm{v}}(t_1,t_2)=\mathrm{E}(\mathrm{v}(t_1))\mathrm{E}(\mathrm{v}(t_2))$ then $\mathrm{v}(t_1)$ and $\mathrm{v}(t_2)$ are uncorrelated and statistically independent.
3. If $\mathrm{v}(t)$ is wide-sense stationary it is also strictly stationary and ergodic.
4. Any linear operation on $\mathrm{v}(t)$ produces another gaussian process.

Random Signals

If a random signal $v(t)$ is stationary then its **power spectrum** $G_v(f)$ is defined as the distribution of its power over the frequency domain. The power spectrum (also known as the "power spectral density" (PSD)) is the Fourier transform of the autocorrelation function, $R_v(\tau) \xleftrightarrow{\mathcal{F}} G_v(f)$.

$G_v(f)$ has the properties:

$$\int_{-\infty}^{\infty} G_v(f)\,df = R_v(0) = \overline{v^2} = P \quad , \quad G_v(f) \geq 0 \quad , \quad G_v(f) = G_v(-f)$$

Random Signals

If two random signals $\mathrm{v}(t)$ and $\mathrm{w}(t)$ are jointly stationary such that $\mathrm{R}_{\mathrm{vw}}(t_1,t_2)=\mathrm{R}_{\mathrm{vw}}(t_1-t_2)$ and if $\mathrm{z}(t)=\mathrm{v}(t)\pm\mathrm{w}(t)$, then

$$\mathrm{R}_{\mathrm{z}}(\tau)=\mathrm{R}_{\mathrm{v}}(\tau)+\mathrm{R}_{\mathrm{w}}(\tau)\pm\left[\mathrm{R}_{\mathrm{vw}}(\tau)+\mathrm{R}_{\mathrm{wv}}(\tau)\right]$$

and

$$\mathrm{G}_{\mathrm{z}}(f)=\mathrm{G}_{\mathrm{v}}(f)+\mathrm{G}_{\mathrm{w}}(f)\pm\left[\mathrm{G}_{\mathrm{vw}}(f)+\mathrm{G}_{\mathrm{wv}}(f)\right]$$

where $\mathrm{R}_{\mathrm{vw}}(\tau)\overset{\mathcal{F}}{\longleftrightarrow}\mathrm{G}_{\mathrm{vw}}(f)$ and $\mathrm{G}_{\mathrm{vw}}(f)$ is **cross-spectral density** (also known as "cross power spectral density (CPSD)"). If $\mathrm{v}(t)$ and $\mathrm{w}(t)$ are uncorrelated and $m_{\mathrm{v}}m_{\mathrm{w}}=0$, then $\mathrm{R}_{\mathrm{vw}}(\tau)=\mathrm{R}_{\mathrm{wv}}(\tau)=0$, $\mathrm{R}_{\mathrm{z}}(\tau)=\mathrm{R}_{\mathrm{v}}(\tau)+\mathrm{R}_{\mathrm{w}}(\tau)$, $\mathrm{G}_{\mathrm{z}}(f)=\mathrm{G}_{\mathrm{v}}(f)+\mathrm{G}_{\mathrm{w}}(f)$ and $\overline{\mathrm{z}^2}=\overline{\mathrm{v}^2}+\overline{\mathrm{w}^2}$.

Random Signals

Let $z(t) = v(t)\cos(\omega_c t + \Phi)$ in which $v(t)$ is a stationary random signal and Φ is a random angle independent of $v(t)$ and uniformly distributed over the range $-\pi \le \Phi \le \pi$. Then

$$R_z(t_1,t_2) = E\left(z(t_1)z(t_2)\right) = E\left(v(t_1)\cos(\omega_c t_1 + \Phi)v(t_2)\cos(\omega_c t_2 + \Phi)\right)$$

$$R_z(t_1,t_2) = E\left(v(t_1)v(t_2)(1/2)\left[\cos\left(\omega_c(t_1 - t_2)\right) + \cos\left(\omega_c(t_1 + t_2) + 2\Phi\right)\right]\right)$$

$$R_z(t_1,t_2) = (1/2)\left[\begin{array}{l} E\left(v(t_1)v(t_2)\cos\left(\omega_c(t_1 - t_2)\right)\right) \\ + E\left(v(t_1)v(t_2)\cos\left(\omega_c(t_1 + t_2) + 2\Phi\right)\right) \end{array}\right]$$

$$R_z(t_1,t_2) = (1/2)\left[\begin{array}{l} E\left(v(t_1)v(t_2)\right)\cos\left(\omega_c(t_1 - t_2)\right) \\ + E\left(v(t_1)v(t_2)\right)\underbrace{E\left(\cos\left(\omega_c(t_1 + t_2) + 2\Phi\right)\right)}_{=0} \end{array}\right]$$

$$R_z(t_1,t_2) = (1/2)E\left(v(t_1)v(t_2)\right)\cos\left(\omega_c(t_1 - t_2)\right)$$

Random Signals

In $\mathrm{R_z}(t_1,t_2)=(1/2)\mathrm{E}\left(\mathrm{v}(t_1)\mathrm{v}(t_2)\right)\cos\left(\omega_c(t_1-t_2)\right)$ since $\mathrm{R_v}(t_1,t_2)=\mathrm{R_v}(\tau)$ we can say that $\mathrm{R_z}(\tau)=(1/2)\mathrm{R_v}(\tau)\cos(\omega_c\tau)$.

Then $\mathrm{G_z}(f)=(1/2)\mathrm{G_v}(f)*(1/2)\left[\delta(f-f_c)+\delta(f+f_c)\right]$

$$\mathrm{G_z}(f)=(1/4)\left[\mathrm{G_v}(f-f_c)+\mathrm{G_v}(f+f_c)\right].$$

In general, if $\mathrm{v}(t)$ and $\mathrm{w}(t)$ are independent and jointly stationary and $\mathrm{z}(t)=\mathrm{v}(t)\mathrm{w}(t)$, then

$$\mathrm{R_z}(\tau)=\mathrm{R_v}(\tau)\mathrm{R_w}(\tau) \text{ and } \mathrm{G_z}(f)=\mathrm{G_v}(f)*\mathrm{G_w}(f)$$

Random Signals

When a random signal $x(t)$ excites a linear system with impulse response $h(t)$ the response is another random signal

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau.$$

So if we have a mathematical description of $x(t)$ we can find $y(t)$. But, of course, if $x(t)$ is random we do not have a mathematical description of it and cannot do the convolution integral. So we cannot describe $y(t)$ exactly because we do not have an exact description of $x(t)$. But we can describe $y(t)$ statistically in the same way we describe $x(t)$, through its mean value and autocorrelation.

Random Signals

When a random signal $x(t)$ excites a linear system with impulse response $h(t)$

1. The mean value of the response $y(t)$ is $m_y = m_x \int_{-\infty}^{\infty} h(\lambda)\,d\lambda = H(0)\,m_x$

   where $H(f)$ is the frequency response of the system,
2. The autocorrelation of the response is $R_y(\tau) = h(-\tau) * h(\tau) * R_x(\tau)$, and
3. The power spectrum of the response is

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

Random Signals

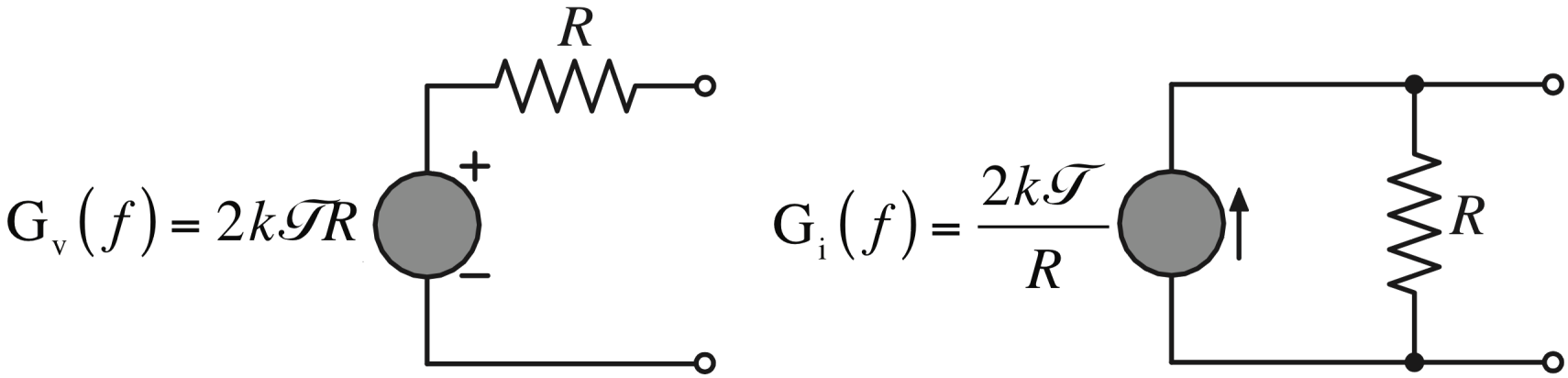
Every signal in every system has **noise** on it and may also have **interference**. Noise is a random signal occurring naturally and interference is a non-random signal produced by another system. In some cases the noise is small enough to be negligible and we need not do any formal analysis of it, but it is never zero. In communication systems the relative powers of the desired signal and the undesirable noise or interference are always important and the noise is often not negligible in comparison with the signal. The most important naturally occurring random noise is **thermal noise** (also called Johnson noise). Thermal noise arises from the random motion of electrons in any conducting medium.

Random Signals

A resistor of resistance R ohms at an absolute temperature of $\mathscr{T}$ kelvins produces a random gaussian noise at its terminals with zero mean and variance $\overline{\mathrm{v}^2} = \sigma_{\mathrm{v}}^2 = \dfrac{2(\pi k \mathscr{T})^2}{3h} R$ V^2 where k is Boltzmann's constant 1.38×10^{-23} J/K and h is Planck's constant 6.62×10^{-34} J$\cdot$s. The power spectrum of this voltage is $\mathrm{G}_{\mathrm{v}}(f) = \dfrac{2Rh|f|}{e^{h|f|/k\mathscr{T}} - 1}$ V^2 / Hz. To get some idea of how this power spectrum varies with frequency let $\mathscr{T} = 290$ K (near room temperature). Then $\mathrm{k}\mathscr{T} = 4 \times 10^{-21}$ J and $h|f| / k\mathscr{T} = |f| / 6.0423 \times 10^{12}$. So at frequencies below about 1 THz, $h|f| / k\mathscr{T} << 1$ and $e^{h|f|/k\mathscr{T}} \cong 1 + h|f| / k\mathscr{T}$. Then $\mathrm{G}_{\mathrm{v}}(f) = \dfrac{2Rh|f|}{h|f| / k\mathscr{T}} \cong 2k\mathscr{T}R$ V^2 / Hz and the power spectrum is approximately constant.

Random Signals

In analysis of noise effects due to the thermal noise of resistors we can model a resistor by a Thevenin equivalent, the series combination of a noiseless resistor of the same resistance and a noise voltage source whose power spectrum is $G_v(f) = 2k\mathcal{T}R$ V^2/Hz. Alternately we could also use a Norton equivalent of a noiseless resistor of the same resistance in parallel with a noise current source whose power spectrum is $G_i(f) = \frac{2k\mathcal{T}}{R}$ A^2/Hz.



Random Signals

We have seen that at frequencies below about 1 THz the power spectrum of thermal noise is essentially constant. Noise whose power spectrum is constant over all frequencies is called **white noise**. (The name comes from optics in which light having a constant spectral density over the visible range appears white to the human eye.) We will designate the power spectrum of white noise as $\mathrm{G}(f) = N_0 / 2$ where N_0 is a density constant. (The factor of 1/2 is there to account for half the power in positive frequencies and half in negative frequencies.) If the power spectrum is constant, the autocorrelation must be $\mathrm{R}(\tau) = \frac{N_0}{2}\delta(\tau)$, an impulse at zero time shift. This indicates that white noise is *completely* uncorrelated with itself at *any* non-zero shift.

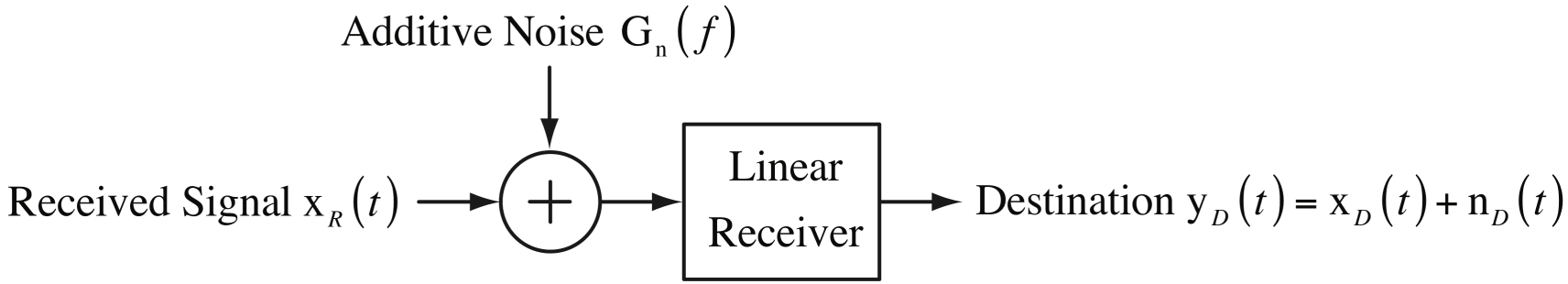
In analysis of communication systems we normally treat thermal noise as white noise because its power spectrum is virtually flat over a very wide frequency range.

Random Signals

Some random noise sources have a power spectrum that is white and unrelated to temperature. But we often assign them a **noise temperature** anyway and analyze them as though they were thermal. This is convenient for comparing white random noise sources. The noise temperature of a non-thermal random noise source is $\mathcal{T}_N = \frac{2\mathrm{G}_a(f)}{k} = \frac{N_0}{k}$. Then, if we know a random noise source's noise temperature its density constant is $N_0 = k\mathcal{T}_N$.

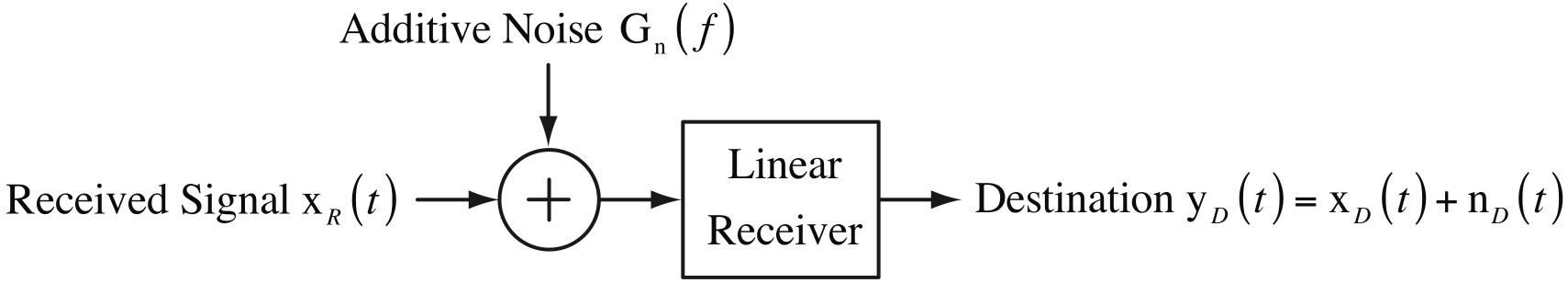
Baseband Signal Transmission with Noise

As a signal propagates from its source to its destination, random noise sources inject noise into the signal at various points. In analysis of noise effects we usually lump all the noise effects into one injection of noise at the input to the receiver that yields equivalent results. As a practical matter the input of the receiver is usually the most vulnerable point for noise injection because the received signal is weakest at this point.



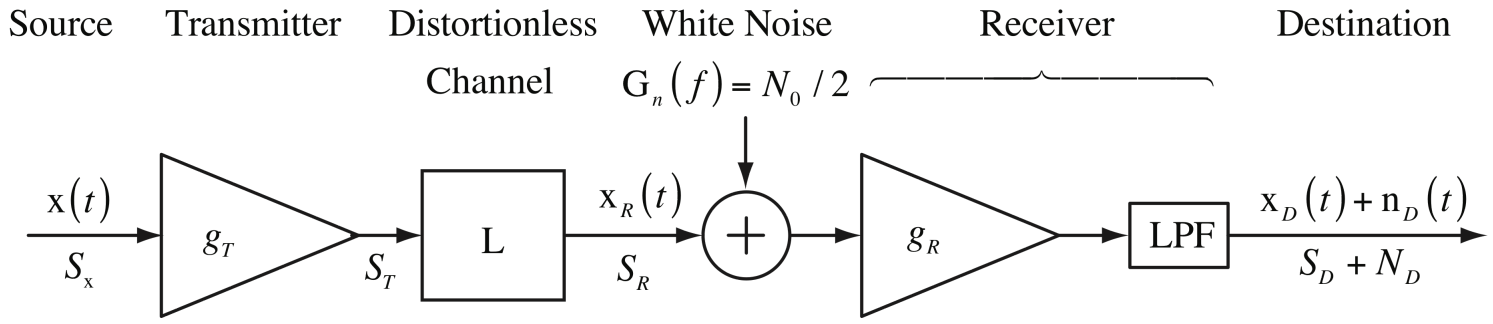
Baseband Signal Transmission with Noise

In analysis we make two reasonable assumptions about the additive noise, it comes from an ergodic source with zero mean and it is physically independent of the signal, therefore uncorrelated with it. Then, the average of the product $\mathrm{x}_D(t)\mathrm{n}_D(t)$ is the product of their averages and, since the average value of $\mathrm{n}_D(t)$ is zero, the product is zero. Also $\mathrm{y}_D^2(t) = \mathrm{x}_D^2(t) + \mathrm{n}_D^2(t)$. Define $S_D \triangleq \overline{\mathrm{x}_D^2}$ and $N_D \triangleq \overline{\mathrm{n}_D^2}$. Then $\mathrm{y}_D^2(t) = S_D + N_D$. There is probably nothing in communication design and analysis more important than **signal-to-noise ratio** (**SNR**). It is defined as the ratio of signal power to noise power $(S/N)_D \triangleq S_D / N_D = \overline{\mathrm{x}_D^2} / \overline{\mathrm{n}_D^2}$.



Baseband Signal Transmission with Noise

In analysis of baseband transmission systems we take $\mathrm{G_n}(f) = N_0/2$. Then the destination noise power is $N_D = g_R N_0 B$ where g_R is the power gain of the receiver amplifier and B is the noise bandwidth of the receiver. $N_0 = k\mathscr{T}_N = k\mathscr{T}_0(\mathscr{T}_N/\mathscr{T}_0) = 4\times10^{-21}(\mathscr{T}_N/\mathscr{T}_0)$ W/Hz where it is understood that $\mathscr{T}_0 = 290$ K. The transmitted signal power is $S_T = g_T S_{\mathrm{x}}$ where g_T is the power gain of the transmitter. The received signal power is $S_R = S_T/L$ where L is the loss of the channel. The signal power at the destination is $S_D = g_R S_R$.



Baseband Signal Transmission with Noise

The signal-to-noise ratio at the destination is

$$S_D / N_D = (S/N)_D = \frac{g_R S_R}{g_R N_0 W} = \frac{S_R}{N_0 W}$$

or, in dB,

$$(S/N)_{D_{\text{dB}}} = 10\log_{10}\left(\frac{S_R}{k\mathscr{T}_N W}\right) = 10\log_{10}\left(\frac{S_R}{k\mathscr{T}_0 W}\times\frac{\mathscr{T}_0}{\mathscr{T}_N}\right)$$

$$= 10\log_{10}(S_R) - 10\log_{10}(k\mathscr{T}_0) - 10\log_{10}\left(\frac{\mathscr{T}_N}{\mathscr{T}_0}W\right)$$

Expressing everything in dBm,

$$(S/N)_{D_{\text{dB}}} = S_{R_{\text{dBm}}} + \left[174 - 10\log_{10}\left(\frac{\mathscr{T}_N}{\mathscr{T}_0}W\right)\right]_{\text{dBm}}$$