The $z$ Transform
Relation to the Laplace Transform

- The $z$ transform is to DT signals and systems what the Laplace transform is to CT signals and systems
Definition

The $z$ transform can be viewed as a generalization of the DTFT or as a natural result of exciting a discrete-time LTI system with its eigenfunction. The DTFT is defined by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\Omega)e^{j\Omega n} d\Omega \leftarrow \mathcal{F} \rightarrow X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

If a strict analogy with the Laplace transform were made $\Omega$ would replace $\omega$, $\Sigma$ would replace $\sigma$, $S$ would replace $s$, a summation would replace the integral and the $z$ transform would be defined by

$$X(S) = \sum_{n=-\infty}^{\infty} x[n]e^{-Sn} = \sum_{n=-\infty}^{\infty} x[n]e^{-(\Sigma+j\Omega)n} = \sum_{n=-\infty}^{\infty} (x[n]e^{-n\Sigma})e^{-j\Omega n}$$
Definition

\[ X(S) = \sum_{n=-\infty}^{\infty} x[n]e^{-Sn} = \sum_{n=-\infty}^{\infty} x[n]e^{-(\Sigma+j\Omega)n} = \sum_{n=-\infty}^{\infty} \left(x[n]e^{-n\Sigma}\right)e^{-j\Omega n} \]

Viewed this way the factor, \( e^{-n\Sigma} \), would be a “convergence” factor in that same way that the factor, \( e^{-\sigma t} \), was for the Laplace transform.

The other approach to defining the \( z \) transform is to excite a DT system with its eigenfunction, \( Az^n \). The response would be

\[ y[n] = x[n]*h[n] = Az^n*h[n] = \sum_{m=-\infty}^{\infty} h[m]Az^{(n-m)} = Az^n \sum_{m=-\infty}^{\infty} h[m]z^{-m} \]

\( z \) transform of \( h[n] \)
Definition

The universally accepted definition of the z transform of a DT function, x, is

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

and x and X form a “z-transform pair”,

\[ x[n] \xrightarrow{z} X(z) \]
Convergence

The DTFT’s of some common functions do not, in the strict sense, converge. The DTFT of the unit sequence would be

\[ X(j\Omega) = \sum_{n=-\infty}^{\infty} u[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-j\Omega n} \]

which does not converge. But the \( z \) transform of the unit sequence does exist. It is

\[ X(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} \]

and the \( z \) transform exists for values of \( z \) whose magnitudes are greater than one. This defines a region of convergence (ROC) for the \( z \) transform of the unit sequence, the exterior of the unit circle in the \( z \) plane.
Convergence

The series, $\sum_{n=0}^{\infty} z^{-n}$, is a geometric series. The general formula for the summation of a finite geometric series is

$$\sum_{n=0}^{N-1} r^n = \begin{cases} 1, & r = 1 \\ \frac{1 - r^N}{1 - r}, & r \neq 1 \end{cases}$$

This formula also applies to the infinite series above if the magnitude of $z$ is greater than one. In that case the $z$ transform of the unit sequence is

$$X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}, \quad |z| > 1$$
Transfer Functions

If \( x \) is the excitation, \( h \) is the impulse response and \( y \) is the system response of a discrete-time LTI system, then

\[
Y(z) = X(z)H(z)
\]

and \( H \) is called the \textit{transfer function} of the system. This is directly analogous to previous transfer functions,

\[
Y(j\omega) = X(j\omega)H(j\omega)
\]

\[
Y(j\Omega) = X(j\Omega)H(j\Omega)
\]

\[
Y(s) = X(s)H(s)
\]
Region of Convergence

Taking a path analogous to that used the development of the Laplace transform, the $z$ transform of the causal DT signal

\[ A\alpha^n u[n], \, |\alpha| > 0 \]

is

\[ X(z) = A \sum_{n=-\infty}^{\infty} \alpha^n u[n]z^{-n} = A \sum_{n=0}^{\infty} \alpha^n z^{-n} = A \sum_{n=0}^{\infty} \left( \frac{\alpha}{z} \right)^n \]

and the series converges if $|z| > |\alpha|$. This defines the region of convergence as the exterior of a circle in the $z$ plane centered at the origin, of radius, $|\alpha|$. The $z$ transform is

\[ X(z) = A \frac{z}{z - \alpha}, \, |z| > |\alpha| \]
Region of Convergence

By similar reasoning, the $z$ transform and region of convergence of the anti-causal signal, $A\alpha^{-n}u[-n]$, $|\alpha| > 0$ are

$$X(z) = \frac{A}{1 - \alpha z} = \frac{A z^{-1}}{z^{-1} - \alpha} , \quad |z| < \frac{1}{|\alpha|}$$
The Unilateral $z$ Transform

Just as it was convenient to define a unilateral Laplace transform it is convenient for analogous reasons to define a unilateral $z$ transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

which will simply be referred to as *the* $z$ transform from this point on.
Properties

If two causal DT signals form these transform pairs,

\[ g[n] \xrightarrow{z} G(z) \quad \text{and} \quad h[n] \xrightarrow{z} H(z) \]

then the following properties hold for the \( z \) transform.

**Linearity**

\[ \alpha g[n] + \beta h[n] \xrightarrow{z} \alpha G(z) + \beta H(z) \]

**Time Shifting**

**Delay:**

\[ g[n - n_0] \xrightarrow{z} z^{-n_0} G(z), \quad n_0 \geq 0 \]

**Advance:**

\[ g[n + n_0] \xrightarrow{z} z^{n_0} \left( G(z) - \sum_{m=0}^{n_0-1} g[m]z^{-m} \right), \quad n_0 > 0 \]
Properties

Change of Scale

\[ \alpha^n g[n] \leftrightarrow_z G\left(\frac{z}{\alpha}\right) \]

Initial Value Theorem

\[ g[0] = \lim_{z \to \infty} G(z) \]

\( z \)-Domain Differentiation

\[ -ng[n] \leftrightarrow_z z \frac{d}{dz} G(z) \]

Convolution in Discrete Time

\[ g[n] * h[n] \leftrightarrow_z H(z)G(z) \]
Properties

Differencing

\[ g[n] - g[n-1] \xlongequal{z} (1 - z^{-1})G(z) \]

Accumulation

\[ \sum_{m=0}^{n} g[m] \xlongequal{z} \frac{z}{z-1}G(z) = \frac{1}{1 - z^{-1}}G(z) \]

Final Value Theorem

\[ \lim_{n \to \infty} g[n] = \lim_{z \to 1} (z - 1)G(z) \]

(if the limit exists)
The Inverse $z$ Transform

There is an inversion integral for the $z$ transform,

$$x[n] = \frac{1}{j2\pi} \oint_{C} X(z)z^{n-1} dz$$

but doing it requires integration in the complex plane and it is rarely used in engineering practice.

There are two other common methods,

Synthetic Division
Partial-Fraction Expansion
Synthetic Division

Suppose it is desired to find the inverse $z$ transform of

$$H(z) = \frac{z^3 - \frac{z^2}{2}}{z^3 - \frac{15}{12} z^2 + \frac{17}{36} z - \frac{1}{18}}$$

Synthetically dividing the numerator by the denominator yields the infinite series

$$1 + \frac{3}{4} z^{-1} + \frac{67}{144} z^{-2} + \cdots$$

This will always work but the answer is not in closed form.
Partial-Fraction Expansion

Algebraically, partial fraction expansion for finding inverse $z$ transforms is identical to the same method applied to inverse Laplace transforms. For example,

$$H(z) = \frac{z^2 \left( z - \frac{1}{2} \right)}{\left( z - \frac{2}{3} \right) \left( z - \frac{1}{3} \right) \left( z - \frac{1}{4} \right)}$$

This fraction is improper in $z$. We could synthetically divide the numerator by the denominator once, yielding a remainder that is proper in $z$ as with the Laplace transform but there is an alternate method that may be preferred in some situations.
Partial-Fraction Expansion

\[ H(z) = \frac{z}{z} = \frac{z \left( z - \frac{1}{2} \right)}{\left( z - \frac{2}{3} \right) \left( z - \frac{1}{3} \right) \left( z - \frac{1}{4} \right)} \]

Dividing both sides by \( z \) makes the fraction proper in \( z \) and partial fraction expansion proceeds normally.

\[ H(z) = \frac{4}{5} + \frac{2}{z - \frac{1}{3}} - \frac{9}{z - \frac{1}{4}} \]

Then

\[ H(z) = \frac{4z}{5} + \frac{2z}{z - \frac{1}{3}} - \frac{9z}{z - \frac{1}{4}} \]
Solving Difference Equations

The unilateral $z$ transform is well suited to solving difference equations with initial conditions. For example,

$$y[n+2] - \frac{3}{2} y[n+1] + \frac{1}{2} y[n] = \left(\frac{1}{4}\right)^n, \quad \text{for } n \geq 0$$

$$y[0] = 10 \quad \text{and} \quad y[1] = 4$$

$z$ transforming both sides,

$$z^2[Y(z) - y[0] - z^{-1} y[1]] - \frac{3}{2} z[Y(z) - y[0]] + \frac{1}{2} Y(z) = \frac{z}{z - \frac{1}{4}}$$

the initial conditions are called for systematically.
Solving Difference Equations

Applying initial conditions and solving,

\[ Y(z) = z \left( \frac{16}{3} z^{-\frac{1}{4}} + \frac{4}{1} z^{-\frac{1}{2}} + \frac{2}{3} \right) \]

and

\[ y[n] = \left[ \frac{16}{3} \left( \frac{1}{4} \right)^n + 4 \left( \frac{1}{2} \right)^n + \frac{2}{3} \right] u[n] \]

This solution satisfies the difference equation and the initial conditions.
Let a signal, $x(t)$, be sampled to form

$$x[n] = x(nT_s)$$

and impulse sampled to form

$$x_\delta(t) = x(t)f_s\comb(f_s t)$$

These two signals are equivalent in the sense that their impulse strengths are the same at corresponding times and the correspondence between times is $t = nT_s$. 

\z Transform - Laplace Transform Relationships
$z$ Transform - Laplace Transform Relationships

Let a DT system have the impulse response, $h[n]$, and let a CT system have the impulse response, $h_\delta(t) = \sum_{n=-\infty}^{\infty} h[n]\delta(t - nT_s)$.

If $x[n]$ is applied to the DT system and $x_\delta(t)$ is applied to the CT system, their responses will be equivalent in the sense that the impulse strengths are the same.
$z$ Transform - Laplace Transform Relationships

The transfer function of the DT system is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

and the transfer function of the CT system is

$$H_\delta(s) = \sum_{n=-\infty}^{\infty} h[n]e^{-nT_ss}$$

The equivalence between them is seen in the transformation,

$$H_\delta(s) = H(z)|_{z \rightarrow e^{sT_s}}$$
The relationship, \( z = e^{sT_s} \), maps points in the \( s \) plane into points in the \( z \) plane and vice versa.

Different contours in the \( s \) plane map into the same contour in the \( z \) plane.
$z$ Transform - Laplace Transform Relationships

\[ \frac{3\pi}{T_s} \]
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\[ \frac{3\pi}{T_s} \]
\[ \sigma \]
\[ \text{Re}(z) \]
\[ \text{Im}(z) \]

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\[ \frac{\pi}{T_s} \]
\[ \frac{3\pi}{T_s} \]
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\[ \sigma \]
\[ \text{Re}(z) \]
\[ \text{Im}(z) \]
The Bilateral $z$ Transform

The bilateral $z$ transform can be used to analyze non-causal signals and/or systems. It is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n} + \sum_{n=-\infty}^{-1} x[n]z^{-n}$$

This can be manipulated into

$$X(z) = X_c(z) - x[0] + X_{ac}(z)$$

where

$$X_c(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad \text{and} \quad X_{ac}(z) = \sum_{n=0}^{\infty} x[-n]z^n$$
The Bilateral $z$ Transform

The bilateral $z$ transform can be found using the unilateral $z$ transform by these four steps.

1. Find the unilateral $z$ transform $X_c(z)$ and its ROC.

2. Find the unilateral $z$ transform, $X_{ac}\left(\frac{1}{z}\right)$, of the discrete-time inverse of the anti-causal part of $x[n]$.

3. Make the change of variable, $z \rightarrow \frac{1}{z}$, in the result of step 2 and in its ROC.

4. Add the results of steps 1 and 3 and subtract $x[0]$ to form $X(z)$. 