The Fourier Series
Representing a Signal

• The convolution method for finding the response of a system to an excitation takes advantage of the linearity and time-invariance of the system and represents the excitation as a linear combination of impulses and the response as a linear combination of impulse responses

• The Fourier series represents a signal as a linear combination of complex sinusoids
Linearity and Superposition

If an excitation can be expressed as a sum of complex sinusoids the response can be expressed as the sum of responses to complex sinusoids.

\[ x(t) = A_1 e^{j2\pi f_1 t} + A_2 e^{j2\pi f_2 t} + A_3 e^{j2\pi f_3 t} \rightarrow h(t) \rightarrow y(t) \]
Real and Complex Sinusoids

\[
\cos(x) = \frac{e^{jx} + e^{-jx}}{2}
\]

\[
\sin(x) = \frac{e^{jx} - e^{-jx}}{j2}
\]
Jean Baptiste Joseph Fourier

3/21/1768 - 5/16/1830
Continuous-Time Fourier Series Concept
Continuous-Time Fourier Series Concept
Continuous-Time Fourier Series Concept
The Fourier series representation, \( x_F(t) \), of a signal, \( x(t) \), over a time, \( t_0 < t < t_0 + T_F \), is

\[
x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(kf_F)t}
\]

where \( X[k] \) is the harmonic function, \( k \) is the harmonic number and \( f_F = 1/T_F \) (pp. 212-215). The harmonic function can be found from the signal as

\[
X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt
\]

The signal and its harmonic function form a Fourier series pair indicated by the notation, \( x(t) \overset{FS}{\longleftrightarrow} X[k] \).
CTFS of a Real Function

It can be shown (pp. 216-217) that the continuous-time Fourier series (CTFS) harmonic function of any real-valued function, $x(t)$, has the property that

$$X[k] = X^*[-k]$$

One implication of this fact is that, for real-valued functions, the magnitude of the harmonic function is an even function and the phase is an odd function.
The Trigonometric CTFS

The fact that, for a real-valued function, $x(t)$,

$$X[k] = X^*[-k]$$

also leads to the definition of an alternate form of the CTFS, the so-called *trigonometric* form.

$$x_F(t) = X_c[0] + \sum_{k=1}^{\infty} \{ X_c[k] \cos(2\pi(kf_F)t) + X_s[k] \sin(2\pi(kf_F)t) \}$$

where

$$X_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \cos(2\pi(kf_F)t) \, dt$$

$$X_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \sin(2\pi(kf_F)t) \, dt$$
The Trigonometric CTFS

Since both the complex and trigonometric forms of the CTFS represent a signal, there must be relationships between the harmonic functions. Those relationships are

\[
\begin{align*}
X_c[0] &= X[0] \\
X_s[0] &= 0 \\
X_c[k] &= X[k] + X^*[k] \\
X_s[k] &= j(X[k] - X^*[k])
\end{align*}
\]

\[
\begin{align*}
X[0] &= X_c[0] \\
X[k] &= \frac{X_c[k] - jX_s[k]}{2} \\
X[-k] &= X^*[k] = \frac{X_c[k] + jX_s[k]}{2}
\end{align*}
\]
Periodicity of the CTFS

It can be shown (pg. 218) that the CTFS representation, $x_F(t)$ of a function, $x(t)$, is periodic with fundamental period, $T_F$. Therefore, if $x(t)$ is also periodic with fundamental period, $T_0$ and if $T_F$ is an integer multiple of $T_0$ then the two functions are equal for all $t$, not just in the interval, $t_0 < t < t_0 + T_F$. 
CTFS Example #1

Let a signal be defined by \( x(t) = 2 \cos(400\pi t) \) and let \( T_F = 5 \) ms which is the same as \( T_0 \)

\[
X_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \cos(2\pi (k f_F) t) \, dt \\
X_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t) \sin(2\pi (k f_F) t) \, dt
\]
CTFS Example #1

Calculation of harmonic amplitude #2

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product

Integral of product
CTFS Example #2

Let a signal be defined by \( x(t) = 2 \cos(400\pi t) \) and let \( T_F = 10 \) ms which is \( 2T_0 \).

Calculation of harmonic amplitude #1

- **x(t) and cosine**
- **x(t) and sine**

Product

Integral of product

\( 0.30003 \)

\( -0.30003 \)

\( 0.087762 \)

\( -0.51229 \)
CTFS Example #2

Calculation of harmonic amplitude #2

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product
CTFS Example #3

Let \( x(t) = \frac{1}{2} - \frac{3}{4} \cos(20\pi t) + \frac{1}{2} \sin(30\pi t) \) and let \( T_F = 200 \text{ ms} \).

Calculation of harmonic amplitude #1

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product
CTFS Example #3

Calculation of Harmonic Amplitude #2

x(t) and Cosine

Product

Integral of Product

x(t) and Sine

Product

Integral of Product
CTFS Example #3
Calculation of harmonic amplitude #3

\[ x(t) \text{ and cosine} \]

\[ x(t) \text{ and sine} \]

Product

Integral of product

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CTFS Example #3

Calculation of harmonic amplitude #4

\[ x(t) \text{ and cosine} \]

\[ x(t) \text{ and sine} \]

Product

Integral of product

\[ 0.19903 \]

\[ 0.064278 \]
Linearity of the CTFS

\[ x(t) = x_1(t) + x_2(t) + \ldots \]

These relations hold only if the harmonic functions, \( X \), of all the component functions, \( x \), are based on the same representation time.
CTFS Example #4

Let the signal be a 50% duty-cycle square wave with an amplitude of one and a fundamental period, $T_0 = 1$

$$x(t) = \text{rect}(2t) * \text{comb}(t)$$

Calculation of harmonic amplitude #1

Product

Integral of product
CTFS Example #4

Calculation of harmonic amplitude #2

x(t) and cosine

Product

Integral of product

x(t) and sine

Product

Integral of product
CTFS Example #4

Calculation of harmonic amplitude #3

x(t) and cosine

Product

Integral of product

x(t) and sine

Product

Integral of product

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CTFS Example #4

Calculation of harmonic amplitude #4

$x(t)$ and cosine

$x(t)$ and sine

Product

Integral of product

0.079555

0.15911
CTFS Example #4

A graph of the magnitude and phase of the harmonic function as a function of harmonic number is a good way of illustrating it.
CTFS Example #5

Let $x(t) = 2\cos(400\pi t)$ and let $T_F = 7.5$ ms which is 1.5 periods of this signal.

Calculation of harmonic amplitude #1
CTFS Example #5
Calculation of harmonic amplitude #2

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product

\( x(t) \) and cosine

\( x(t) \) and sine

Product

Integral of product
CTFS Example #5

Calculation of harmonic amplitude #3

$x(t)$ and cosine

$x(t)$ and sine

Product

Integral of product

Integral of product
CTFS Example #5

The CTFS representation of this cosine is the signal below, which is an odd function, and the discontinuities make the representation have significant higher harmonic content. This is a very inelegant representation.
CTFS of Even and Odd Functions

• For an even function
  – The complex CTFS harmonic function, $X[k]$, is purely real
  – The sine harmonic function, $X_s[k]$, is zero

• For an odd function
  – The complex CTFS harmonic function, $X[k]$, is purely imaginary
  – The cosine harmonic function, $X_c[k]$, is zero
CTFS Example #6

This signal has no known functional description but it can still be represented by a CTFS.
CTFS Example #6
Calculation of Harmonic Amplitude #1

\( x(t) \) and Cosine

Product

Integral of Product

\( x(t) \) and Sine

Product

Integral of Product
CTFS Example #6
Calculation of Harmonic Amplitude #10

\[ x(t) \text{ and Cosine} \]

\[ \begin{align*}
\text{Product} & : 0.64106 \\
\text{Integral of Product} & : 0.044911
\end{align*} \]

\[ x(t) \text{ and Sine} \]

\[ \begin{align*}
\text{Product} & : 0.50717 \\
\text{Integral of Product} & : 0.014042
\end{align*} \]
CTFS Example #6
CTFS Properties

Let a signal, \( x(t) \), have a fundamental period, \( T_{0x} \) and let a signal, \( y(t) \), have a fundamental period, \( T_{0y} \). Let the CTFS harmonic functions, each using the fundamental period as the representation time, \( T_F \), be \( X[k] \) and \( Y[k] \). In the properties which follow the two fundamental periods are the same unless otherwise stated.

\[
\alpha x(t) + \beta y(t) \xrightarrow{\text{FS}} \alpha X[k] + \beta Y[k]
\]

Linearity

\[
x(t) \rightarrow \alpha
\]
\[
y(t) \rightarrow \beta
\]
\[
x(t) \xrightarrow{\text{FS}} X[k] \rightarrow \alpha
\]
\[
y(t) \xrightarrow{\text{FS}} Y[k] \rightarrow \beta
\]
\[
+ \rightarrow Z[k]
\]
CTFS Properties

Time Shifting

\[ x(t - t_0) \overset{FS}{\leftrightarrow} e^{-j2\pi(kf_0)t_0} X[k] \]

\[ x(t - t_0) \overset{FS}{\leftrightarrow} e^{-j(\omega_0)t_0} X[k] \]
CTFS Properties

Frequency Shifting (Harmonic Number Shifting)

\[ e^{j2\pi(k_0f_0)t} x(t) \xrightarrow{FS} X[k - k_0] \]

\[ e^{j(k_0\omega_0)t} x(t) \xrightarrow{FS} X[k - k_0] \]

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal

\[ x(-t) \xrightarrow{FS} X[-k] \]
CTFS Properties

Time Scaling

Let \( z(t) = x(at) , \ a > 0 \)

Case 1. \( T_F = \frac{T_{0x}}{a} = T_{0z} \) for \( z(t) \)

\[
Z[k] = X[k]
\]

Case 2. \( T_F = T_{0x} \) for \( z(t) \)

If \( a \) is an integer,

\[
Z[k] = \begin{cases} 
X\left[ \frac{k}{a} \right] , & \frac{k}{a} \text{ an integer} \\
0 , & \text{otherwise}
\end{cases}
\]
CTFS Properties

Time Scaling (continued)

\[ |X[k]| \]

\[ |Z[k]| \]

\[ a = 2 \]
CTFS Properties

Change of Representation Time

With $T_F = T_{0x}$, $x(t) \xleftarrow{FS} X[k]$

With $T_F = mT_{0x}$, $x(t) \xleftarrow{FS} X_m[k]$

$$X_m[k] = \begin{cases} X\left[\frac{k}{m}\right], & \frac{k}{m} \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

($m$ is any positive integer)
CTFS Properties

Change of Representation Time

\[ T_F = T_0 \]

\[ |X[k]| \]

\[ T_F = 2T_0 \]

\[ |X_2[k]| \]
CTFS Properties

Time Differentiation

\[
\frac{d}{dt}(x(t)) \xrightarrow{\mathcal{F}_S} j2\pi(kf_0)X[k]
\]

\[
\frac{d}{dt}(x(t)) \xrightarrow{\mathcal{F}_S} j(k\omega_0)X[k]
\]
CTFS Properties

Time Integration

Case 1. \( X[0] = 0 \)

\[
\int_{-\infty}^{t} x(\lambda) d\lambda \overset{FS}{\longleftrightarrow} X[k] \frac{1}{j2\pi(kf_{0})} \]

\[
\int_{-\infty}^{t} x(\lambda) d\lambda \overset{FS}{\longleftrightarrow} X[k] \frac{1}{j(k\omega_{0})} \]

Case 2. \( X[0] \neq 0 \)

\[
\int_{-\infty}^{t} x(\lambda) d\lambda \text{ is not periodic} \]
CTFS Properties

Multiplication-Convolution Duality

\[ x(t)y(t) \xrightarrow{FS} X[k] \ast Y[k] \]

(The harmonic functions, \(X[k]\) and \(Y[k]\), must be based on the same representation period, \(T_F\).)

\[ x(t) \otimes y(t) \xrightarrow{FS} T_0 X[k] Y[k] \]

The symbol, \(\otimes\), indicates periodic convolution. Periodic convolution is defined mathematically by

\[ x(t) \otimes y(t) = \int_{T_0} x(\tau)y(t - \tau) d\tau \]

\[ x(t) \otimes y(t) = x_{ap}(t) \ast y(t) \quad \text{where } x_{ap}(t) \text{ is any single period of } x(t) \]
CTFS Properties

\[ x(t) \]

\[ T_{0x} \]

\[ x(t)y(t) \]

\[ y(t) \]

\[ T_{0y} \]

\[ T_0 \]

\[ |X[k]| \]

\[ |X[k]|^2 \]

\[ |X[k]|^2 + |Y[k]|^2 \]

\[ |X[k] \ast Y[k]| \]

\[ |X[k] \ast Y[k]|^2 \]

\[ |X[k] \ast Y[k]|^2 + |Y[k]|^2 \]
CTFS Properties

Conjugation

\[ x^*(t) \xrightarrow{FS} X^*[-k] \]

Parseval’s Theorem

\[
\frac{1}{T_0} \int_{T_0} x(t)^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2
\]

The average power of a periodic signal is the sum of the average powers in its harmonic components.
Convergence of the CTFS

For continuous signals, convergence is exact at every point.

A Continuous Signal

\[
x_N(t) = \sum_{k=-N}^{N} X[k] e^{j2\pi(kf_0)t}
\]
Convergence of the CTFS

For discontinuous signals, convergence is exact at every point of continuity.

Discontinuous Signal
Convergence of the CTFS

The Gibbs Phenomenon

\[ N = 199 \]
\[ N = 59 \]
\[ N = 19 \]

A

A

\( \frac{A}{2} \)

0

\[ t \]
Discrete-Time Fourier Series Concept

$\{x[n]\}$ Original Signal

- **Harmonics**
  - $k = 0$
  - $k = 1$
  - $k = -1$
  - $k = 2$

- **Real**
  - $k = 0$
  - $k = 1$
  - $k = -1$
  - $k = 2$

- **Imaginary**
  - $k = 0$
  - $k = 1$
  - $k = -1$
  - $k = 2$

- **Sum of Harmonics**
  - $k = 0$
  - $k = 1$
  - $k = -1$
  - $k = 2$
The Discrete-Time Fourier Series

The formal derivation of the discrete-time Fourier series (DTFS) is on pages 259-262. The results are

\[ x_F[n] = \sum_{k=\langle N_F \rangle} X[k] e^{j2\pi(kF_F)n} \quad X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] e^{-j2\pi(kF_F)n} \]

where \( N_F \) is the representation time, \( F_F = \frac{1}{N_F} \), and the notation,

\[ \sum_{k=\langle N_F \rangle} \]

means a summation over any range of consecutive \( k \)’s exactly \( N_F \) in length.
The Discrete-Time Fourier Series

Notice that in

\[ x_F[n] = \sum_{k=\langle N_F \rangle} X[k] e^{j2\pi(kF_F)n} \]

the summation is over exactly one period, a finite summation. This is because of the periodicity of the complex sinusoid,

\[ e^{-j2\pi(kF_F)n} \]

in harmonic number, \( k \). That is, if \( k \) is increased by any integer multiple of \( N_F \) the complex sinusoid does not change.

\[ e^{-j2\pi(kF_F)n} = e^{-j2\pi((k+mN_F)F_F)n} \quad (m \text{ an integer}) \]

This occurs because discrete time, \( n \), is always an integer.
The Discrete-Time Fourier Series

In the very common case in which the representation time is taken as the fundamental period, $N_0$, the DTFS is

$$x[n] = \sum_{k=\langle N_0 \rangle} X[k]e^{j2\pi(kF_0)n} \xleftarrow{\text{FS}} X[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]e^{-j2\pi(kF_0)n}$$

or in terms of radian frequency

$$x[n] = \sum_{k=\langle N_0 \rangle} X[k]e^{j(k\Omega_0)n} \xleftarrow{\text{FS}} X[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]e^{-j(k\Omega_0)n}$$

where $\Omega_0 = 2\pi F_0 = \frac{2\pi}{N_0}$
DTFS Example

DT Signal, $x[n]$

$|X[k]|$

Phase of $X[k]$

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DTFS Properties

Let a signal, \( x[n] \), have a fundamental period, \( N_{0x} \), and let a signal, \( y[n] \), have a fundamental period, \( N_{0y} \). Let the DTFS harmonic functions, each using the fundamental period as the representation time, \( N_F \), be \( X[k] \) and \( Y[k] \). In the properties to follow the two fundamental periods are the same unless otherwise stated.

**Linearity**

\[
\alpha x(t) + \beta y(t) \xrightarrow{FS} \alpha X[k] + \beta Y[k]
\]
DTFS Properties

Time Shifting

\[ x[n - n_0] \leftrightarrow^{FS} e^{-j2\pi(kF_0)n_0} X[k] \]

\[ x[n - n_0] \leftrightarrow^{FS} e^{-j(k\Omega_0)n_0} X[k] \]
DTFS Properties

Frequency Shifting (Harmonic Number Shifting)

\[ e^{j2\pi (k_0 F_0)n} x[n] \xleftrightarrow{\text{FS}} X[k - k_0] \]

\[ e^{j(k_0 \Omega_0)n} x[n] \xleftrightarrow{\text{FS}} X[k - k_0] \]

Conjugation

\[ x^*[n] \xleftrightarrow{\text{FS}} X^*[-k] \]

Time Reversal

\[ x[-n] \xleftrightarrow{\text{FS}} X[-k] \]
DTFS Properties

Time Scaling

Let \( z[n] = x[an] , \ a > 0 \)

If \( a \) is not an integer, some values of \( z[n] \) are undefined and no DTFS can be found. If \( a \) is an integer (other than 1) then \( z[n] \) is a decimated version of \( x[n] \) with some values missing and there cannot be a unique relationship between their harmonic functions. However, if

\[
z[n] = \begin{cases} 
    x \left( \frac{n}{m} \right) , & n \text{ an integer} \\
    0 , & \text{otherwise}
\end{cases}
\]

then

\[
Z[k] = \frac{1}{m} X[k] , \ N_F = mN_0
\]
DTFS Properties

\[ x[n] \]

\[ 1 \]

\[ \cdots \]

\[ N_F \]

\[ \cdots \]

\[ n \]

\[ \mathcal{F}_S \]

\[ |X[k]| \]

\[ \cdots \]

\[ 1 \]

\[ \cdots \]

\[ k \]

Phase of \( X[k] \)

\[ \cdots \]

\[ \pi \]

\[ \cdots \]

\[ k \]

\[ -N_F \]

\[ -\pi N_F \]

\[ \cdots \]

\[ N_F \]

\[ \cdots \]

\[ \pi \]

\[ \cdots \]

\[ k \]
DTFS Properties

Change of Representation Time

With $N_F = N_{x0}$, $x[n] \overset{FS}{\longrightarrow} X[k]$

With $N_F = qN_{x0}$, $x[n] \overset{FS}{\longrightarrow} X_q[k]$

$$X_q[k] = \begin{cases} X \left[ \frac{k}{q} \right] , & \frac{k}{q} \text{ an integer} \\ 0 , & \text{otherwise} \end{cases}$$

$q$ is any positive integer
DTFS Properties
DTFS Properties

Accumulation

\[ \sum_{m=-\infty}^{n} x[m] \leftrightarrow_{FS} \frac{X[k]}{1 - e^{-j2\pi(kF_{0})}}, \quad k \neq 0 \]

\[ \sum_{m=-\infty}^{n} x[m] \leftrightarrow_{FS} \frac{X[k]}{1 - e^{-j(k\Omega_{0})}}, \quad k \neq 0 \]

Parseval’s Theorem

\[ \frac{1}{N_{0}} \sum_{n=\langle N_{0} \rangle} |x[n]|^{2} = \sum_{k=\langle N_{0} \rangle} |X[k]|^{2} \]
DTFS Properties

Multiplication-Convolution Duality

\[ x[n]y[n] \overset{\text{FS}}{\longleftrightarrow} Y[k] \otimes X[k] = \sum_{q=\langle N_0 \rangle} Y[q]X[k-q] \]

\[ x[n] \otimes y[n] \overset{\text{FS}}{\longleftrightarrow} N_0 Y[k]X[k] \]

First Backward Difference

\[ x[n] - x[n-1] \overset{\text{FS}}{\longleftrightarrow} (1 - e^{-j2\pi(kF_0)}) X[k] \]

\[ x[n] - x[n-1] \overset{\text{FS}}{\longleftrightarrow} (1 - e^{-j(k\Omega_0)}) X[k] \]
DTFS Properties

\[ x_1[n] \oplus x_2[n] \]

\[ |X_1[k]| \]

\[ |X_1[k]X_2[k]| \]

\[ |X_2[k]| \]

\[ \text{Phase of } X_1[k] \]

\[ \text{Phase of } X_1[k]X_2[k] \]

\[ \text{Phase of } X_2[k] \]
Convergence of the DTFS

• The DTFS converges exactly with a finite number of terms. It does not have a “Gibbs phenomenon” in the same sense that the CTFS does
LTI Systems with Periodic Excitation

The differential equation describing an RC lowpass filter is

\[ RC v'_\text{out}(t) + v_\text{out}(t) = v_\text{in}(t) \]

If the excitation, \( v_\text{in}(t) \), is periodic it can be expressed as a CTFS,

\[ v_\text{in}(t) = \sum_{k=-\infty}^{\infty} V_\text{in}[k] e^{j2\pi(kf_0)t} \]

The equation for the \( k \)th harmonic alone is

\[ RC v'_{\text{out},k}(t) + v_{\text{out},k}(t) = v_{\text{in},k}(t) = V_\text{in}[k] e^{j2\pi(kf_0)t} \]
LTI Systems with Periodic Excitation

If the excitation is periodic, the response is also, with the same fundamental period. Therefore the response can be expressed as a CTFS also.

\[ v_{out,k}(t) = V_{out}[k]e^{j2\pi(kf_0)t} \]

Then the equation for the \( k \)th harmonic becomes

\[ j2k\pi f_0 RC V_{out}[k]e^{j2\pi(kf_0)t} + V_{out}[k]e^{j2\pi(kf_0)t} = V_{in}[k]e^{j2\pi(kf_0)t} \]

Notice that what was once a differential equation is now an algebraic equation.
LTI Systems with Periodic Excitation

Solving the \( k \)th-harmonic equation,

\[
V_{out}[k] = \frac{V_{in}[k]}{j2k\pi f_0 RC + 1}
\]

Then the response can be written as

\[
v_{out}(t) = \sum_{k=-\infty}^{\infty} V_{out}[k] e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{\infty} \frac{V_{in}[k]}{j2k\pi f_0 RC + 1} e^{j2\pi(kf_0)t}
\]
LTI Systems with Periodic Excitation

The ratio, \( \frac{V_{out}[k]}{V_{in}[k]} \), is the frequency response of the system.

\[ R = 1\Omega, \ C = 1 \text{ F}, \ f_0 = 0.05 \text{ Hz} \]