Answers to Exercises in Chapter 1 - Probability

Experiments, Outcomes and Events

1-1. For each of the following experiments, list all of the possible outcomes and state whether these outcomes are equally likely.

(a) While blindfolded selecting two marbles from a bowl which contains two red and two green marbles all of which feel the same. $\{RR, RG, GR, GG\}$
All these outcomes are equally likely.

(b) While blindfolded selecting two marbles from a bowl that contains two black, two white and two blue marbles all of which feel the same and observing whether or not the two marbles are the same color. $\{\text{same}, \text{different}\}$
These outcomes are not equally likely.

$$\mathrm{P}\left[\text{same}\right] = 0.2$$
$$\mathrm{P}\left[\text{different}\right] = 0.8$$

Set Theory and Probability

1-2. A probability space S is given as

$$S = \{2\ ,\alpha\ ,\ y\}\ .$$

Enumerate all the events in this space.

$$\{\varnothing\},\{2\},\{\alpha\},\{y\},\{2,\alpha\},\{2,y\},\{\alpha,y\},\{2,\alpha,y\}$$

1-3. A space S is defined as $S = \{a,b,e,k,q,t\}$ and three subsets of S are defined as

$$A = \{a,e\}\quad,\quad B = \{b,e,k,q\}\quad\text{and}\quad C = \{k,q,t\}\quad.$$

Find the following sets.

$$\begin{array}{ccc} A+B & AB & A+C \\ AC & B+C & BC \\ A-B & A-C & B-A \\ B-C & C-B & \bar{A} \\ \bar{B}-C & \bar{C}B & (C-A)\bar{B} \\ B\bar{A} & A\bar{B} & \overline{(A-B)}+C \end{array}$$

$$\begin{array}{ccc} A+B=\{a,b,e,k,q\} & AB=\{e\} & A+C=\{a,e,k,q,t\} \\ AC=\{\varnothing\} & B+C=\{b,e,k,q,t\} & BC=\{k,q\} \\ A-B=\{a\} & A-C=\{a,e\} & B-A=\{b,k,q\} \\ B-C=\{b,e\} & C-B=\{t\} & \bar{A}=\{b,k,q,t\} \\ \bar{B}-C\{k,q,t\}=\{a\} & \bar{C}B=\{b,e\} & (C-A)\bar{B}=\{t\} \\ B\bar{A}=\{b,k,q\} & A\bar{B}=\{a\} & \overline{(A-B)}+C=\{b,e,k,q,t\} \end{array}$$

1-4. At a blood bank, all blood is checked for the HIV virus and is found to be positive or negative for the virus. To increase the chance of a correctly detecting the virus the blood is tested three times by lab technicians *a*, *b*, and *c*. An example of an outcome for this experiment is *ppn* which indicates that *a* and *b* think the blood sample is positive for the HIV virus and *c* thinks it is negative.

(a) Enumerate all the outcomes for the this experiment.

$$\left\{ \begin{matrix} nnn & nnp & npn & pnn \\ ppn & pnp & npp & ppp \end{matrix} \right\}$$

(b) What are the outcomes in the events

$$A = \{ a \text{ classifies sample HIV positive} \}$$

and

$$B = \{ c \text{ classifies sample HIV negative} \}$$

$$A = \{ pnn \quad ppn \quad pnp \quad ppp \}$$

$$B = \{ pnn \quad ppn \quad npn \quad nnn \}$$

(c) Are *A* and *B* mutually exclusive?

$$AB = \{ pnn \quad ppn \} \neq \{ \varnothing \}$$

A and *B* are not mutually exclusive.

(d) Do *A* and *B* form a partition of the probability space *S*? No.

(e) Enumerate the outcomes in the events

$$C = \{ \text{at least two HIV positive results} \}$$

$$D = \{ \text{more than one HIV negative} \}$$

$$C = \{ ppn \quad pnp \quad npp \quad ppp \}$$

$$D = \{ nnp \quad npn \quad pnn \quad nnn \}$$

(f) Are *C* and *D* mutually exclusive? Yes

(g) Do *C* and *D* form a partition of the probability space *S*? Yes.

1-5. In Exercise 1-3 let $\mathrm{P}[a] = 1/6$, $\mathrm{P}[b] = \mathrm{P}[e] = 1/10$, $\mathrm{P}[k] = \mathrm{P}[q] = 1/5$. Find the following probabilities.

(a) $\mathrm{P}[A] = \dfrac{4}{15}$

(b) $\mathrm{P}[B] = \dfrac{3}{5}$

(c) $\mathrm{P}[C] = \dfrac{19}{30}$

(d) $\mathrm{P}\left[AB\right]=\dfrac{1}{10}$

(e) $\mathrm{P}\left[\bar{A}C\right]=\dfrac{19}{30}$

(f) $\mathrm{P}\left[\overline{(A-B)}+B\right]=\dfrac{11}{15}$

1-6. The city of Trenton has 1000 homes. 600 of the homes subscribe to the Trenton Herald newspaper (Event *A*) and 400 of the homes subscribe to the Trenton Tribune newspaper (Event *B*). 200 homes subscribe to both newspapers.

(a) How many homes do not subscribe to either of these newspapers and how can you describe this event in terms of *A* and *B* and the set operations, union, intersection, complement and difference?

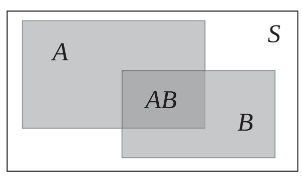
The number of homes not subscribing to any newspaper is 200.

(b) How many of the homes subscribe to exactly one newspaper (which could be either one, the Herald or the Tribune, but not both newspapers) and how can you describe this event in terms of *A* and *B* and the set operations, union, intersection, complement and difference?

The number of homes subscribing to exactly one newspaper is 600.

(c) If a home is selected at random, what is the probability that it subscribes to both newspapers given that we know it subscribes to at least one? 0.25

(d) Draw a Venn diagram to illustrate the sets in this exercise.



1-7. If a pair of dice is tossed, find the probabilities of the following events.

(a) Two even numbers showing. 1/4

(b) A sum less than 8 0.583

(c) An odd sum 1/2

(d) The two numbers differ by 2 or less. 2/3

1-8. The colors of eyes in humans are determined by the genes of the father and mother. Each person has two eye-color genes. Let the gene for brown eyes be *B* and let the gene for blue eyes be *b*. The only people who have blue eyes are those with the gene pair *bb* and everyone else has brown eyes. Assume that the gene inherited from each parent is equally likely to be either of the two genes that a parent has.

(a) Suppose a man with genes *BB* and a woman with genes *bb* produce a child. What is the probability that the child has blue eyes? 0

(b) Suppose a man with genes *Bb* and woman with genes *Bb* produce a child. What is the probability that the child has brown eyes? 3/4

(c) Suppose a man with genes *Bb* and woman with genes *bb* produce a child. What is the probability that the child has brown eyes? 1/2

(d) M_1 with genes *BB* and W_1 with genes *Bb* produce a boy C_1. M_2 with genes *Bb* and W_2 with genes *Bb* produce a girl C_2. When C_1 and C_2 reach adulthood they produce a child C_3. What is the probability that C_3 has brown eyes? 0.875

1-9. The `rand` function in MATLAB generates pseudo-random numbers between zero and one with equal probability of being anywhere in that range. The syntax is `rand(r,c)` where `r` is the number of rows and `c` is the number of columns in a matrix of random numbers. The flipping of a fair coin one time can be simulated by the instruction,

```
outcome = rand(1,1) > 0.5 ;
```

where the outcome `1` represents a head and the outcome `0` represents a tail. Suppose the following instructions are executed.

```
f1 = rand(3,1) > 0.5 ;
f2 = rand(3,1) > 0.5 ;
```

(a) What is the probability that the vector `f1` represents two heads and one tail, in any order and the vector `f1` also represents two heads and one tail, in any order? 0.1406

(b) What is the probability that either `f1` or `f2` represents no heads and the other vector represents three heads? 0.03125

1-10. An experiment has three outcomes, $\{-5,3,8\}$, and they are all equally likely to occur. When the experiment is carried out multiple times, the outcome of any particular trial has no effect on the outcome of any other trial. Find the following probabilities.

(a) The probability that, in three consecutive trials of the experiment, all three outcomes are 8. 1/27

(b) The probability that, in three consecutive trials of the experiment, the outcome 3 does not occur. 8/27

(c) The probability that, in two consecutive trials of the experiment, the sum of the two outcomes is less than 5. 5/9

(d) The probability that, in three consecutive trials of the experiment, the product of the three outcomes is negative. 13/27

1-11. A complex number *z* has a real part *x* an imaginary part *y* a magnitude *r* and an angle θ. Let the range of angles be from -π to +π. In Figure E1-11 is pictured the *z* plane with the real part being the *x* axis and the imaginary part being the *y* axis. An arrow whose tip is always somewhere on the unit circle is mounted on an axis at the origin and can be spun. On any spin there is equal probability of stopping at any angular position. When the arrow stops spinning, the position of the tip determines an exact value of *z*.

(a) What is the probability that $-\pi/2 < \theta < 3\pi/4$? 5/8

(b) What is the probability that the real part of z is greater than 0.5 ? 1/3

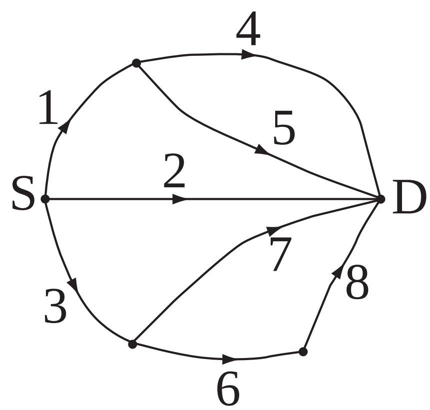
(c) What is the probability that z is purely imaginary ($x = 0$) ? 0

(d) What is the probability that on two spins both angles are in the first quadrant ($0 < \theta < \pi/2$) ? 1/16

(e) What is the probability that on two spins the product of the two *z*'s will have a magnitude of one? 1

1-12. (from Cooper and McGillem) Two solid-state diodes are connected in series without regard to polarity. Each diode has a probability of 0.05 that it has failed as an open circuit and a probability of 0.1 that it has failed as a short circuit. If the two diodes are independent, what is the probability that the series connection will operate as a diode? (from Cooper and McGillem) 0.53125

1-13. A diagram of a communication network is shown below.



The probability that each link is working is 0.8. What is the probability of being able to transmit a message from the source S to the destination D? 0.988

Conditional Probability

1-14. In Figure E1-14 is a "wheel of fortune". When it is spun, the ending angular position is random with all angular positions having equal probability. The outcome is the number pointed to by the arrow when the wheel stops. The outcome of any spin is independent of the outcome of any other spin.

(a) What is the probability, on a single spin, that the outcome will be less than 8? 0.6875

(b) What is the probability that, on any two spins, both outcomes are even numbers? 0.1914

(c) If someone spins the wheel and you cannot see the outcome and you are told (correctly) that the outcome is a number greater than 4, what is the numerical probability that the outcome is 9, given that condition? 0.375

1-15. In Figure E1-15 is an illustration of another "wheel of fortune" which has two pointers, a long one and a short one. The two pointers rotate independently. When both pointers are spun each has an equal probability of stopping at any angular position. The experiment is to spin both pointers and wait until they stop. The outcome of an experiment is the two numbers pointed to by the two pointers.

(a) What is the probability in one trial that both pointers will point to an even number? 1/4

(b) What is the probability in one trial that both pointers will point to a prime number? 35/128

(c) Given that the long pointer is pointing at a prime number, what is the probability in one trial that the short pointer is pointing at a number greater than 3? 0.625

(d) What is the probability in one trial that both pointers will point to the same number? 1/16

(e) What is the probability in ten trials they will both point to the same number exactly two of the ten times? 0.1049

1-16. At an office supply store all the ballpoint pens that have been in stock for more than one year are dumped into a large box. They all look the same but there are four ink colors. There are 30 black ink, 18 blue, 24 red and 10 green.

(a) If a pen is selected at random what is the probability that it writes black? 15/41

(b) If a pen is selected at random what is the probability that it does not write red? 29/41

(c) If the first pen selected writes blue and is not replaced what is the probability that the second pen also writes blue? 0.0461

What is the probability if the first pen is replace before selecting again? 0.0482

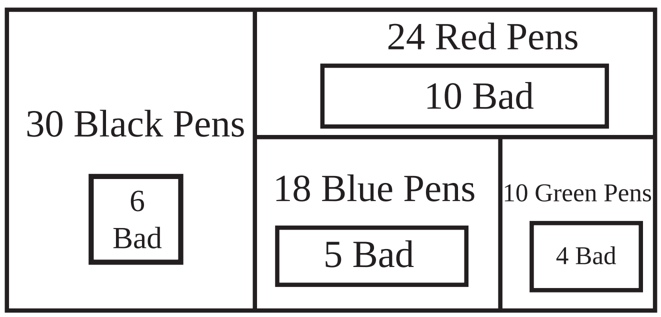
1-17. In Exercise 1-16, 6 of the black pens are defective (won't write), 5 of the blue pens are defective, 10 of the red pens are defective and 4 of the green pens are defective.

(a) If a pen is selected at random what is the probability that it is not defective? 57/82

(b) If a pen is selected at random and has blue ink what is the probability that it is not defective? 13/18

(c) If a pen is selected at random and does write, what is the probability that it writes green? 6/57

(d) Draw a Venn diagram for these pens.



1-18. A smuggler is approaching a border crossing with two open gates. At gate 1 is a guard who catches 70% of all the smugglers that come to his gate. At gate 2 is a guard who catches 90% of all smugglers that come to his gate. The smuggler flips a fair coin to decide which gate to choose. The smuggler gets caught. What is the probability that he chose gate 1? 7/16

1-19. In a digital communication system which transmits 1's and 0's the probability of a 0 being transmitted is 0.45 and the probability of a 1 being transmitted be 0.55. Because of noise in the system sometimes a bit is detected wrong. Let the probability of a transmitted 0 being received as a 1 be 0.1 and the probability of a transmitted 1 being received as a 0 be 0.04.

(a) If a 0 is received, what is the probability that that bit was sent as a 0? 0.948

(b) If a 1 is received, what is the probability that that bit was sent as a 1? 0.921

(c) In a very long message, what fraction of the bits is received wrong? 0.067

1-20. A photon arrives at a photodetector at time t, where t is a random time in the interval $0 < t < 20$ ps. All times are equally likely.

(a) Find $\mathrm{P}\left[9 \text{ ps} < t < 12 \text{ ps}\right]$. 3/20

(b) Find $\mathrm{P}\left[9 \text{ ps} < t < 12 \text{ ps} \mid t > 8 \text{ ps}\right]$. 1/4

1-21. On a TV quiz show a contestant is shown 5 identical boxes and is allowed to pick one and wins whatever is in the box. One box contains a prize and one other box contains a message saying that the contestant gets another chance to pick a box and try to win the prize. The rest of the boxes are empty.

(a) What is the probability that the contestant wins the prize? 1/4

(b) If the contestant wins, what is the probability that the contestant wins on the first pick? 4/5

(c) If the contestant loses, what is the probability that the contestant gets a second pick? 1/5

1-22. (From Cooper and McGillem) A candy machine has 10 buttons of which one never works, two work half the time and the rest work all the time. A coin is inserted and a button is pushed at random.

(a) What is the probability that no candy is received? 1/5

(b) If no candy is received, what is the probability that the button pushed is the one which never works? 1/2

(c) If candy is received, what is the probability that the button pushed is one of the ones which only work half the time? 1/8

1-23. A ball is dropped into the structure pictured in Figure E1-23. It starts at position 1 or 2 or 3 and exits the structure at position A, B or C. As the ball drops through the structure when it strikes the top point of a black diamond shape it has equal probability of bouncing to the right or left. The ball cannot ever move up through the structure, only down.

(a) Find the conditional probabilities $\mathrm{P}[A|1]$ $\mathrm{P}[A|3]$ $\mathrm{P}[C|2]$

$\mathrm{P}[A|1]=3/4$ $\mathrm{P}[A|3]=0$ $\mathrm{P}[C|2]=1/4$.

(b) Assuming that all three beginning points 1, 2 and 3 are equally probable, if the ball exits the structure at position A, what is the numerical probability that it began at position 2? 1/4

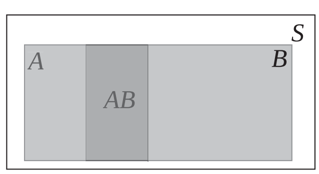
(c) The ball is dropped from high above the structure and the probabilities of arriving at the positions, 1, 2 and 3 are $\mathrm{P}[1]=1/3$, $\mathrm{P}[2]=1/2$ and $\mathrm{P}[3]=1/6$. Find the probability that the ball will exit at B. 3/8

Independence

1-24. A probability space has three independent events *A*, *B* and *C*. $\mathrm{P}[A]=0.6$, $\mathrm{P}[B]=0.2$ and $\mathrm{P}[AC]=0.4$. Find $\mathrm{P}[C]$, $\mathrm{P}[AB]$, $\mathrm{P}[BC]$, $\mathrm{P}[ABC]$.

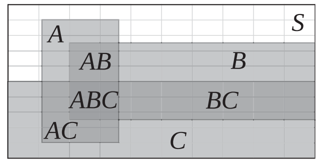
$\mathrm{P}[C]=2/3$ $\mathrm{P}[AB]=0.12$ $\mathrm{P}[BC]=2/15$ $\mathrm{P}[ABC]=0.08$

1-25. Draw a Venn diagram with an event *A* whose probability is 0.3 and an event *B* whose probability is 0.5. Let the areas representing the two events be proportional to their probability and draw them such that their intersection area is correct for two independent events.

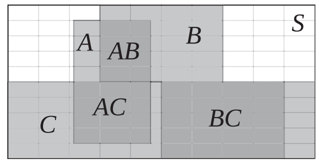


1-26. Draw a Venn diagram with an event *A* whose probability is 0.2, an event *B* whose probability is 0.4 and an event *C* whose probability is 0.5. Let the areas representing the two events be proportional to their probability.

(a) Draw them such that they are independent.



(b) Draw them such that they are pairwise independent but not independent.



Permutations and Combinations

1-27. How many distinguishable permutations of the set of letters, $\{a,b,c,c,d,d,d\}$, can be formed? 420

1-28. You have a set of 26 tiles, each with a different letter of the English alphabet. Assuming each unique permutation of 5 letters is a "word", how many unique words can you form using that set of tiles? 7,893,600

1-29. A child has five balls, all identical except for color. One is red, one is blue and three are white. Assuming there are five positions in a horizontal row, each of which can hold one ball, how many distinguishable arrangements of the five balls can he possibly make? 20

1-30. You have five cards with letters on them, two identical *x*'s, two identical *y*'s and a *z*. There are three positions in which to put the cards. How many distinguishable arrangements are there? 18

1-31. A random English prose selection of several thousand letters is examined and it is found that the relative frequencies of occurrence of the letters is as illustrated and tabulated below.

(a) Based on these relative frequencies, what is the probability that a page of text containing 1000 letters will not contain a *z*. What is the probability that it will contain exactly four *z*'s? What is the probability that it will contain more than 20 *e*'s?

$$\mathrm{P}\left[\text{no } z \text{ in 1000 letters}\right] = 0.2992$$

$$\mathrm{P}\left[\text{exactly 4 } z\text{'s in 1000 letters}\right] = 0.02622$$

$$\mathrm{P}\left[>20\ e\text{'s in 1000 letters}\right] \cong 1$$

(b) A page of prose contains the sentence, Fall frost is on your pumpkin, run fast my girl, run fast. The sentence has 12 words, containing a total of 45 letters, none of which is an *e*. What is the probability that any randomly selected sentence containing 45 letters will not contain an *e*? 0.0008665

(c) A message is encrypted using a simple substitutional cipher in which each letter is substituted by another letter in a "shifted" alphabet. An example of a shift letter substitution with a shift of 3 would be

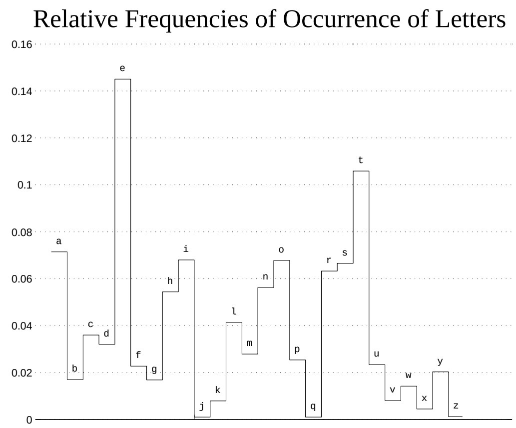
$$A \to D\,,\, B \to E\,,\, C \to F\,,\, \cdots X \to Z\,,\, Y \to A\,,\, Z \to B\,.$$

Using this cipher the message, "Beethoven's Ninth" would become "Ehhwkryhq'v Qlqwk". An encrypted message is received. It is known to be encrypted using a shifted alphabet but the shift amount is not known. The encrypted message is

Ijyjwrnsnsl bmjymjw tw sty f xnlsfq nx wfsitr nx f ywnhpd gzxnsjxx. Xywnhyqd xujfpnsl ymjwj nx st bfd yt gj xzwj bmjymjw tw sty f xnlsfq nx wfsitr bnymtzy tgxjwansl fqq tk ny; ymfy nx, fs nsknsnyj wjhtwi. Bmjymjw tw sty f umdxnhfq umjstrjsts nx fhyzfqqd "wfsitr" nx tkyjs rtwj f umnqtxtumnhfq ymfs f xhnjsynknh vzjxynts. Ns jslnsjjwnsl uwfhynhj ymjwj fwj rfsd umjstrjsf bmnhm fwj xt htruqnhfyji fsi zsuwjinhyfgqj ns fsd uwfhynhfq xjsxj ymfy ymjd fwj ywjfyji fsi fsfqdeji fx ymtzlm ymjd fwj wfsitr, bmjymjw tw sty ymjd wjfqqd fwj. Ns ymnx yjcy, ymj zxj tk ymj btwi "wfsitr" bnqq nruqd ymfy xtrjymnsl nx jnymjw fhyzfqqd wfsitr tw yt gj ywjfyji fx wfsitr, jajs ymtzlm ny rfd sty gj.

What is the original message?

Determining whether or not a signal is random is a tricky business. Strictly speaking there is no way to be sure whether or not a signal is random without observing all of it; that is, an infinite record. Whether or not a physical phenomenon is actually "random" is often more a philosophical than a scientific question. In engineering practice there are many phenomena which are so complicated and unpredictable in any practical sense that they are treated and analyzed as though they are random, whether or not they really are. In this text, the use of the word "random" will imply that something is either actually random or to be treated as random, even though it may not be.



a , 0.071469	b , 0.017043	c , 0.036016	d , 0.032077	e , 0.14503
f , 0.022751	g , 0.016882	h , 0.054426	i , 0.068012	j ,0.0010451
k , 0.0079588	l , 0.041402	m , 0.027896	n , 0.056275	o , 0.067771
p , 0.025404	q , 0.0010451	r , 0.063269	s , 0.066565	t , 0.10588
u , 0.023394	v , 0.0081196	w , 0.014229	x , 0.004502	y , 0.020339
z , 0.0012059				

1-32. A box contains 14 balls numbered 1 through 14. Suppose 5 balls are selected without replacement.

(a) What is the probability that 9 is the largest number drawn? 0.035

(b) What is the probability that the largest number drawn is less than or equal to 9? 0.0629

1-33. Suppose k identical boxes each contain n balls numbered 1 through n. One ball is drawn from each box.

(a) What is the probability that exactly one ball is m and all the rest are less than m?

$$\mathrm{P}\left[\text{all numbers } \le m \text{ and exactly one } m\right] = \frac{k!}{(k-1)!}\left(\frac{m-1}{n}\right)^{k-1}\left(\frac{1}{n}\right)$$

(b) What is the probability that multiple balls may be m but none more than m.

$$\mathrm{P}\left[\text{all numbers } \le m\right] = \sum_{q=1}^{k}\binom{k}{q}\left(\frac{m-1}{n}\right)^{k-q}\left(\frac{1}{n}\right)^{q}$$

1-34. In the world population about 15% of all men are left-handed and about 10% of all women are left-handed. (These are approximations. The exact percentages depend on the definition of “left-handedness”.)

(a) In a typical college undergraduate engineering class of 30 students, 90% are men and 10% are women. What is the most probable number of left-handed people in such a class? 4

(b) What is the most probable number of left-handed people in a nursing class of 120 people, 90% women and 10% men? 12

1-35. In a group of 50 people, what is the probability that there is at least one pair of people with the same birthdate? ("Same birthdate" means the same day of the same month, but not necessarily the same year. Ignore leap-year implications.) 0.9704

1-36. In the digital communication system of Exercise 1-19, assume that the event of an error occurring in one binary symbol is statistically independent of an error occurring in any other binary symbol. Find:
(From Exercise 1-19 Let the probability of a 0 being transmitted be 0.45 and the probability of a 1 being transmitted be 0.55. Let the probability of a transmitted 0 being received as a 1 be 0.1 and the probability of a transmitted 1 being received as a 0 be 0.04.

(a) The probability of receiving 5 consecutive symbols without error. 0.707

(b) The probability of receiving 4 consecutive symbols with exactly one error. 0.2177

(c) The probability of receiving 7 consecutive symbols with more than one error. 0.0752

1-37. Let an experiment be to choose the last digit of a randomly chosen phone number in a phone book.

(a) What is the probability that, in 5 trials, each of the numbers 0 through 4 occurs exactly once? 0.0012

(b) What is the probability that, in 20 trials, each of the numbers 0 through 9 occurs exactly twice? 2.376×10^{-5}

(c) What is the probability that, in 100 trials, each of the numbers 0 through 9 occurs exactly 10 times? 2.357×10^{-8}

1-38. A 12 second message arrives at a receiver at random in the time interval $0 < t < 1$ minute. Another message also arrives at random during that same time interval. Let the duration of the second message be T_2. ("Arrival" in this case means begins and ends within the 1 minute time.)

(a) What value of T_2 yields a probability of 0.3 that the two message times overlap? 7.693 seconds

(b) If the second message is very long (approaching infinity) what is the probability that the two messages overlap? 0.68

1-39. A coin with $\mathrm{P}[H] = p$ is tossed n times. What is the probability that the number of heads is even?

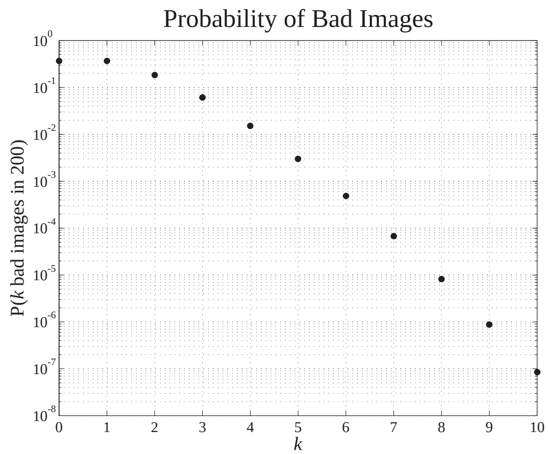
$$\mathrm{P}[\text{even no of heads}] = 0.5\left[1 + (1 - 2p)^n\right]$$

1-40. A television station is broadcasting a program during a thunderstorm which causes, on average, one image in 200 to fall below its standard for picture quality.

(a) In broadcasting 500 images what is the probability that exactly 5 of them do not meet standards? 1/15

(b) In broadcasting 500 images what is the probability that all of them meet standards? 0.0816

(c) Graph the probability of k bad images in 200 images versus k for $0 \le k \le 10$ on a logarithmic scale for the probability.



1-41. A gambler draws cards from a well-shuffled deck. If he draws a heart he wins \$10, otherwise he loses \$2. If the game is repeated 40 times what is the probability that the net gain or loss is greater than \$10? 0.5847