Answers to Exercises in Chapter 3 - Multiple Random Variables

Mean, Standard Deviation and Expectation

3-1. Four independent random variables $X_1, X_2, X_3$ and $X_4$ each Gaussian distributed with a mean of 0 and a standard deviation of 4 are combined to form a new random variable

$$Y = X_1^2 + X_2^2 + X_3^2 + X_4^2.$$  

(a) What is the expected value of $Y$? 64  
(b) What is the variance of $Y$? 2048  
(c) What is the probability of the event $50 < Y < 80$? 0.24

3-2. Random variable $X$ has a variance of 20 and $Y$ has a variance of 5. Their correlation coefficient is 0.7.

(a) Find the variance of their sum 39  
(b) Find the variance of their difference. 11

3-3. Two independent random variables $X$ and $Y$ have variances $\sigma^2_x = 12$ and $\sigma^2_y = 18$.  

If $W$ and $Z$ are $W = -4X + 2Y$ and $Z = 3X - 6Y$.

(a) Find the variances of $W$ and $Z$. 264 and 756  
(b) Find the correlation coefficient of $W$ and $Z$. -0.672

3-4. A rural electric cooperative has a base load which is constant at 25% of its nominal power capability and three large industrial customers who operate independently and whose power demand and probability of demanding that power are listed below.

<table>
<thead>
<tr>
<th>Industrial Customer</th>
<th>Power Demand</th>
<th>Probability of Demanding Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30%</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>25%</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>35%</td>
<td>0.25</td>
</tr>
</tbody>
</table>

(a) What are the average power demanded by the system and the standard deviation of the average power demanded by the system? 52.75% 23.84%  
(b) What is the probability that the power demand on the system will exceed its nominal power capability? 0.03

Joint Probability Density

3-5. Two CV random variables $X$ and $Y$ have a joint distribution function given by
Two CV random variables $X$ and $Y$ have a joint PDF given by

$$f_{XY}(x, y) = \begin{cases} 
0, & x < 0 \text{ or } y < 0 \\
xy, & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\
x, & y \geq 1 \text{ and } 0 \leq x \leq 1 \\
y, & x \geq 1 \text{ and } 0 \leq y \leq 1 \\
1, & x > 1 \text{ and } y > 1 
\end{cases}$$

(a) Graph this distribution function. (See (b)).

(b) Graph the joint PDF.

(c) Find the probability of the event $\{ X \leq 1/2 \cap Y > 1/2 \}$. 1/4

(d) Find $E(XY)$. 1/4

3-6. Two CV random variables $X$ and $Y$ have a joint PDF given by

$$f_{XY}(x, y) = K(x+1)(y+1/2)\text{rect}(x/2)\text{rect}(y)$$

(a) Find $K$. 1

(b) Find the joint distribution function $F_{XY}(x, y)$.

$$F_{XY}(x, y) = \begin{cases} 
0, & x < -1 \text{ or } y < -1/2 \\
\frac{x^2 + x + 1/2}{2} \left(\frac{y^2 + y + 1/8}{2}\right), & -1 < x < 1 \text{ and } -1/2 < y < 1/2 \\
\frac{1}{2} \left(\frac{x^2 + x + 1}{2}\right), & -1 < x < 1 \text{ and } y > 1/2 \\
2 \left(\frac{y^2 + y + 1/8}{2}\right), & -1/2 < y < 1/2 \text{ and } x > 1 \\
1, & x > 1 \text{ and } y > 1/2 
\end{cases}$$

(c) Find the probability of the event $\{ X \leq 1/2 \cap Y > 0 \}$. 27/64

(d) Find the marginal PDF $f_x(x)$

$$f_x(x) = \frac{1}{2} (x+1) \text{rect}(x/2)$$

(e) Find $E(XY)$. 1/18

(f) Find $f_{X|Y}(x)$ and $f_{Y|X}(y)$

$$f_{X|Y}(x) = \frac{1}{2} (x+1) \text{rect}(x/2) \text{ and } f_{Y|X}(y) = 2(y+1/2) \text{rect}(y)$$

3-7. Two CV random variables $X$ and $Y$ have a joint PDF
Solutions 3-3

\[ f_{X,Y}(x,y) = \begin{cases} 
 2/3, & -2 < x < 0 \text{ and } -1/2 < y < 0 \\
 1/3, & 0 < x < 1/2 \text{ and } 0 < y < 2 \\
 0, & \text{elsewhere} 
\end{cases} \]

(a) Find \( E(XY) \). 1/4

(b) Find and graph versus \( x \) the marginal PDF \( f_X(x) \).

(c) What is the probability of the event which is the intersection of the events \( X < 1/4 \) and \( Y > 1 ? \) 1/12

3-8. For each joint PDF determine whether \( X \) and \( Y \) are uncorrelated and find their correlation \( E(XY) \).

(a) \( f_{XY}(x,y) = K(x^2 + y^2) \text{rect}(x) \text{rect}(y/2) \) \( E(XY) = 0 \)

(b) \( f_{XY}(x,y) = K(x + xy + y) \text{rect}(x - 1/2) \text{rect}(y - 1/2) \) \( E(XY) = 0.3556 \)

(c) \( f_{XY}(x,y) = K \frac{x}{y} \text{rect}(x - 1) \text{rect}(y - 1) \) \( E(XY) = 0.986 \)

3-9. The joint PDF of the random variables \( X \) and \( Y \) is given by

\[ f_{XY}(x,y) = \begin{cases} 
 1, & \text{in shaded area} \\
 0, & \text{otherwise} 
\end{cases} \]

(a) Find the marginal PDF’s of \( X \) and \( Y \). Are \( X \) and \( Y \) independent?

\[ f_X(x) = \begin{cases} 
 0, & x < -1 \\
 x + 1, & -1 \leq x < 0 \\
 1 - x, & 0 \leq x < 1 \\
 0, & x \geq 1 
\end{cases} \]

\[ f_Y(y) = \begin{cases} 
 0, & y < -1 \\
 y + 1, & -1 \leq y < 0 \\
 1 - y, & 0 \leq y < 1 \\
 0, & y \geq 1 
\end{cases} \]

\( X \) and \( Y \) are not independent.

(b) Let \( Z = X + Y \). Find \( F_Z(z) \) and \( f_Z(z) \).
\[ F_Z(z) = \frac{1}{2} \left[ (1-z^2)[u(z+1)-u(z)](z^2+1)[u(z)-u(z-1)] \right] + u(z-1) \]

\[ f_Z(z) = \begin{cases} z[u(z+1)-u(z-1)] & z \geq 0 \\ e^{-z} & z < 0 \end{cases} \]

**Linear Combinations of Random Variables and the Gaussian Distribution**

3-10. A CV random variable \( X \) has a PDF

\[ f_X(x) = \left( \frac{1}{5} \right) e^{-x/5} u(x) \]

and an independent CV random variable \( Y \) has a PDF

\[ f_Y(y) = \frac{1}{3} e^{-y/3} u(y) \]

(a) Find the probability density function of the random variable \( Z = X \cdot Y \), graph it and verify that its area is one.

\[ f_Z(z) = \begin{cases} \frac{1}{5} e^{-z} & z \geq 0 \\ \frac{1}{3} e^{-z} & z < 0 \end{cases} \]

(b) Find the probability that \(-1 < Z \leq 1\). 0.2195

3-11. \( X \) and \( Y \) are independent, identically distributed (i.i.d.) random variables with common PDF

\[ f_X(x) = e^{-x} u(x) \quad f_Y(y) = e^{-y} u(y) \]

Find the PDF of the following random variables (a) \( \min(X,Y) \), (b) \( \max(X,Y) \), (c) \( \min(X,Y) / \max(X,Y) \).

(a) \( F_Z(z) = P\left[ \min(X,Y) \leq z \right] \)

\[ f_Z(z) = 2e^{-z} u(z) \]

(b) \( F_Z(z) = P\left[ \max(X,Y) \leq z \right] \)

\[ f_Z(z) = 2e^{-z} (1-e^{-z}) u(z) \]

(c) \( Z = \min(X,Y) / \max(X,Y) \) and \( X \) and \( Y \) are never negative.

\[ f_Z(z) = \frac{2\text{rect}(z-1/2)}{(1+z)^2} \]

3-12. \( X \) and \( Y \) are independent and uniform in the interval \((0,a)\). Find the PDF of (a) \( X / Y \), (b) \( Y / (X+Y) \), (c) \( |X-Y| \).

Solutions 3-4
Solutions 3-5

(a) \( f_z(z) = \frac{1}{2} u(z) - u(z-1) + \frac{1}{2z^2} u(z-1) \)

(b) \( f_z(z) = \frac{1}{2(1-z)^2} \left[ u(z) - u \left( \frac{z-1}{2} \right) \right] + \frac{1}{2z^2} \left[ u \left( \frac{z-1}{2} \right) - u(z-1) \right] \)

(c) \( f_z(z) = \frac{2}{a} \left( 1 - \frac{z}{a} \right) \left[ u(z) - u(z-a) \right] \)

3-13. The joint PDF of \( X \) and \( Y \) is given by

\[
f_{XY}(x,y) = \begin{cases} 2(1-x), & 0 < x \leq 1, 0 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}
\]

Determine the PDF of \( Z = XY \).

\( f_z(z) = 2z - 1 - \ln(z) \)

3-14. \( X \) and \( Y \) are independent uniformly distributed random variables on \((0,1)\). Find the joint PDF of \( X + Y \) and \( X - Y \).

\[
f_{UV}(u,v) = \frac{1}{2} f_{XY} \left( \frac{u+v}{2} , \frac{u-v}{2} \right) = \begin{cases} 1, & 0 \leq u + v \leq 2, 0 \leq u - v \leq 2 \\ 0, & \text{otherwise} \end{cases}
\]

Central Limit Theorem and Gaussian Distributions

3-15. A resistor in a circuit has a Gaussian noise voltage across it with zero mean and a mean-squared value of 10^{-12} V^2. What percentage of the time is the voltage across the resistor greater than 1 \( \mu \)V? About 16%

3-16. A random variable \( X \) is Gaussian distributed and \( P[X > 2] = 0.3 \) and \( P[X > 5] = 0.1 \). What are the expected value \( E(X) \) and the variance \( \sigma_X^2 \) of \( X \)? \( \sigma_X^2 = 15.6 \), \( E(X) = -0.1 \).

3-17. A Gaussian random variable \( X \) has a mean of 3 and a variance of 16. If \( Y = \left| X \right| \) find the mean and variance of \( Y \) and graph its PDF.

\( E(Y) = 4.0531 \), \( \sigma_Y^2 = 8.572 \)

3-18. A Gaussian random variable \( X \) has a mean of -10 and a variance of 64. If \( Y = X \cdot \text{rect} \left( \frac{X}{30} \right) \), find the expected value of \( Y \) and graph its PDF. \( E(Y) = \frac{1}{\sqrt{2\pi}} \left[ -8 \times (-0.815) \right] = 7.331 = 2.6 \times 7.331 = -4.731 \)

3-19. \( X \) and \( Y \) are independent identically-distributed Gaussian random variables with zero mean and common variance \( \sigma^2 \). Find the PDF of (a) \( Z = X^2 + Y^2 \), (b) \( W = X^2 + Y^2 \) and (c) \( U = X - Y \).

Solutions 3-5
(a) \[ f_z(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} u(z) \]

(b) \[ f_w(w) = \frac{e^{-w/2\sigma^2}}{2\sigma^2} u(w) \]

(c) \[ f_u(u) = \frac{1}{(\sqrt{2\sigma})\sqrt{2\pi}} e^{-u^2/(2\sigma^2)} \]

3-20. A system consists of two cascaded subsystems 1 and 2 cascaded with another subsystem which consists of two parallel-connected systems 3 and 4. The subsystems have constant failure rates with MTTF’s of

\[ \tau_1 = 5000 \text{ hours} \quad \tau_2 = 3000 \text{ hours} \quad \tau_3 = 10000 \text{ hours} \quad \tau_4 = 1000 \text{ hours} \]

Assuming that the system only fails if both parallel-connected subsystems fail what is the MTTF of the overall system?

The overall system MTTF is 1620 hours.

General

3-21. A CV random signal \( X \) has a Rayleigh PDF and a mean value of 10 and is added to noise \( N \) that is uniformly distributed with a mean value of zero and a variance of 12. \( X \) and \( N \) are statistically independent and can be observed only as \( Y = X + N \).

(a) Find, sketch and label the conditional PDF \( f_{X|Y}(x) \) as a function of \( x \) for \( Y = 0, 6 \) and 12.

(b) If an observation yields a value of \( Y = 12 \), what is the best estimate of the true value of \( X \)? 7.98