Lattice and Lattice-Ladder Structures for IIR Systems
Lattice IIR Structure

Consider a system with finite poles and all its zeros at $z = 0$ whose transfer function is of the form

$$H(z) = \frac{1}{A_N(z)}$$

where $A_N(z) = 1 + \sum_{k=1}^{N} a_N[k] z^{-k}$ is an $N$th degree polynomial in $z$.

This system has $N$ finite poles and $N$ zeros at $z = 0$. 
Lattice IIR Structure

A Direct Form II system with $N$ finite poles and $N$ zeros at $z = 0$. 
Lattice IIR Structure

Compare the Direct Form II structures for FIR and IIR systems. If, in the FIR system, we exchange the roles of $X(z)$ and $Y(z)$, change all $b$'s to $-a$'s (with $b_0 = 1$) and let $N = M - 1$, we get the IIR system.
Modify the FIR lattice structure as illustrated below. Reverse the arrows on all the "f" signals. Reverse the lattice and apply $x[n]$ to the previous output and take $y[n]$ from the previous input. Also reverse the signs of the signals arriving from the bottom. This is now a recursive or feedback structure which can implement an IIR filter.
Take the case $N = 1$. 
$x[n] = f_1[n]$ ,  $f_0[n] = f_1[n] - K_1 g_0[n-1]$ ,  $g_1[n] = K_1 f_0[n] + g_0[n-1]$ 
and  $y[n] = f_0[n] = g_0[n] = f_1[n] - K_1 g_0[n-1]$

$z$ transforming

$$Y(z) + K_1 z^{-1} Y(z) = X(z) \Rightarrow H_1(z) = \frac{1}{1 + z^{-1} K_1} = \frac{z}{z + K_1}$$

Single pole at $z = -K_1$ and a zero at $z = 0$. 

Lattice-Ladder IIR Structure

Also, for the $N = 1$ case

$$g_1[n] = K_1 y[n] + y[n - 1]$$

$$G_1(z) = K_1 Y(z) + z^{-1} Y(z) \Rightarrow \frac{G_1(z)}{Y(z)} = K_1 + z^{-1} = K_1 \frac{z + 1/K_1}{z}$$

$$\frac{G_1(z)}{Y(z)}$$ is the transfer function of a system with a single zero at $z = -1/K_1$ and a pole at zero.
**Lattice-Ladder IIR Structure**

For the $N = 2$ case, it can be shown that

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + K_1(K_2 + 1)z^{-1} + K_2z^{-2}} = \frac{z^2}{z^2 + K_1(K_2 + 1)z + K_2}$$

and

$$\frac{G_2(z)}{Y(z)} = K_2 + K_1(K_2 + 1)z^{-1} + z^{-2} = K_2 \frac{z^2 + K_1(1 + 1/K_2)z + 1/K_2}{z^2}$$

Notice that the coefficients for the FIR and IIR systems occur in reverse order as before.
**Lattice-Ladder IIR Structure**

For any $m$,

\[ H_m(z) = \frac{Y(z)}{X(z)} = \frac{1}{A_m(z)} \quad \text{and} \quad \frac{G_m(z)}{Y(z)} = B_m(z) = z^{-m}A_m(1/z) \]

and the previous relations for FIR lattices still hold.

\[ A_0(z) = B_0(z) = 1 \quad , \quad A_m(z) = A_{m-1}(z) + K_m z^{-1}B_{m-1}(z) \]

\[ B_m(z) = z^{-m}A_m(1/z) \quad , \quad K_m = \alpha_m[m] \]

\[ A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2} \]
Lattice-Ladder IIR Structure

If we want to add finite zeros to $H_m(z)$ we can add a ladder network to the lattice. Then the transfer function will be of the form

$$H(z) = \frac{\gamma_N(0) + \gamma_N(1)z^{-1} + \cdots + \gamma_N(N)z^{-N}}{1 + \alpha_N(1)z^{-1} + \alpha_N(2)z^{-2} + \cdots + \alpha_N(N)z^{-N}} = \frac{\Gamma_N(z)}{A_N(z)}$$

$$y[n] = \sum_{m=0}^{N} v_m g_m[n] \Rightarrow Y(z) = \sum_{m=0}^{N} v_m G_m(z)$$
Lattice-Ladder Example

Synthesize the transfer function

\[ H(z) = \frac{1 - z^{-1} + 0.5z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} = \frac{z^2 - z + 0.5}{z^2 + 0.2z - 0.15} \]

using a lattice-ladder network.

\[ A_2(z) = 1 + 0.2z^{-1} - 0.15z^{-2} \]

\[ K_2 = -0.15 \text{ and } B_2(z) = -0.15 + 0.2z^{-1} + z^{-2} \]

\[ \Gamma_2(z) = \sum_{m=0}^{2} v_m B_m(z) = 1 - z^{-1} + 0.5z^{-2} \Rightarrow v_2 = 0.5 \]

\[ \Gamma_1(z) = \Gamma_2(z) - v_2 B_2(z) = 1 - z^{-1} + 0.5z^{-2} - 0.5(-0.15 + 0.2z^{-1} + z^{-2}) \]

\[ \Gamma_1(z) = 1.075 - 1.1z^{-1} \Rightarrow v_1 = -1.1 \]

\[ \Gamma_0(z) = \Gamma_1(z) - v_1 B_1(z) \]
Lattice-Ladder Example

Using $A_{m-1}(z) = \frac{A_m(z) - K_mB_m(z)}{1 - K^2_m}$

$$A_1(z) = \frac{1 + 0.2z^{-1} - 0.15z^{-2} - (-0.15)\left(-0.15 + 0.2z^{-1} + z^{-2}\right)}{1 - (-0.15)^2}$$

$$A_1(z) = \frac{0.9775 + 0.23z^{-1}}{0.9775} = 1 + 0.23529z^{-1}$$

$K_1 = 0.23529$ and $B_1(z) = 0.23529 + z^{-1}$

$$\Gamma_0(z) = 1.075 - 1.1z^{-1} - (-1.1)\left(0.23529 + z^{-1}\right) = 1.3382 \Rightarrow v_0 = 1.3382$$