Parametric Spectral Estimation
ARMA Models

The most common model used in parametric spectral estimation is the rational function model used to describe ARMA systems

\[ H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} \]

with the corresponding difference equation

\[ x[n] = -\sum_{k=1}^{p} a_k x[n-k] + \sum_{k=0}^{q} b_k w[n-k] \]
ARMA Models

The power spectral density is

\[ G_{xx}(F) = G_{ww}(F) |H(F)|^2 \]

If the excitation \( w \) is a zero-mean white noise process then

\[ G_{xx}(F) = \sigma_{ww}^2 |H(F)|^2 = \sigma_{ww}^2 \left| \frac{B(e^{j2\pi F})}{A(e^{j2\pi F})} \right|^2 \]

The most common form of model is the AR model in which \( q = 0 \) and \( b_0 = 1 \) because it usually results in the fewest parameters needed to represent a process. In that case

\[ G_{xx}(F) = \frac{\sigma_{ww}^2}{|A(e^{j2\pi F})|^2} \]
ARMA Models

The normal equations

\[ R_{xx}[l] = -\sum_{k=1}^{p} a_p[k] R_{xx}[l-k], \quad l = 1, 2, \ldots, p \]

can be use to find the coefficients \( a \).
ARMA Models

Example

Let the autocorrelation of the response of a system to an applied white noise signal with variance $\sigma_w^2 = 6$ have the following values

$$R_{xx}[0] = 4, \quad R_{xx}[1] = 2, \quad R_{xx}[2] = 3$$

Model the system producing this signal as an AR(2) system and graph the power spectral density of the signal.

The normal equations are

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \end{bmatrix} \begin{bmatrix} a_4[1] \end{bmatrix} = \begin{bmatrix} R_{xx}[1] \\ R_{xx}[2] \end{bmatrix}$$
ARMA Models

Example

\[
\begin{bmatrix}
4 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
a_4[1] \\
a_4[2]
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
3
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
a_4[1] \\
a_4[2]
\end{bmatrix}
= 
\begin{bmatrix}
-0.1667 \\
-0.667
\end{bmatrix}
\]

\[
H(z) = \frac{1}{A(z)} = \frac{1}{1 - 0.1667z^{-1} - 0.667z^{-2}}
\]

The power spectral density is

\[
G_{xx}(F) = \frac{6}{\left|1 - 0.1667e^{-j2\pi F} - 0.667e^{-j4\pi F}\right|^2}
\]

\[
G_{xx}(F) = \frac{6}{1.473 - 0.111\cos(2\pi F) - 1.333\cos(4\pi F)}
\]
ARMA Models

Example

Power Spectral Density from AR(2) Model

Power Spectral Density from Numerical Simulation
MA Models

For an MA\(\left( q \right)\) system the transfer function is

\[
H(z) = B(z) = \sum_{k=0}^{q} b_k z^{-k}
\]

and the impulse response is \(h[n] = \sum_{k=0}^{q} b_k \delta[n - k]\). So the squared magnitude of the transfer function is

\[
H(z)H(z^{-1}) = B(z)B(z^{-1}) = D(z) = \sum_{k=-q}^{q} d_k z^{-k}
\]

where \(\sum_{k=-q}^{q} d_k \delta[n - k] = h[n] * h[-n]\).
MA Models

\[
\left( h[n] \ast h[-n] \right)_{n=0} = b_0^2 + b_1^2 + \cdots + b_q^2
\]

\[
\left( h[n] \ast h[-n] \right)_{n=1} = b_0 b_1 + b_1 b_2 + \cdots + b_{q-1} b_q = \left( h[n] \ast h[-n] \right)_{n=-1}
\]

\[
\left( h[n] \ast h[-n] \right)_{n=2} = b_0 b_2 + b_1 b_3 + \cdots + b_{q-2} b_q = \left( h[n] \ast h[-n] \right)_{n=-2}
\]

\[\vdots\]

\[
\left( h[n] \ast h[-n] \right)_{n=m} = \sum_{k=0}^{q-|m|} b_k b_{k+m}, \quad |m| \leq q
\]
MA Models

The autocorrelation of $x$ is

$$R_{xx}[m] = R_{ww}[m] * h[m] * h[-m] = \sigma_w^2 h[m] * h[-m]$$

$$R_{xx}[m] = \sigma_w^2 \sum_{k=0}^{q|m|} b_k b_{k+m}, \quad |m| \le q$$

$$R_{xx}[m] = \sigma_w^2 \delta_m$$

Therefore

$$G_{xx}(F) = \sum_{m=-\infty}^{\infty} R_{xx}[m] e^{-j2\pi Fm} = \sum_{m=-q}^{q} R_{xx}[m] e^{-j2\pi Fm}$$