Web Appendix D - Derivations of Convolution Properties

D.1 Continuous-Time Convolution Properties

D.1.1 Commutativity Property

By making the change of variable, $\lambda = t - \tau$, in one form of the definition of CT convolution,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

it becomes

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t)$$

proving that convolution is commutative.

D.1.2 Associativity Property

Associativity can be proven by considering the two operations

$$\left[ x(t) * y(t) \right] * z(t) \text{ and } x(t) * \left[ y(t) * z(t) \right] .$$

Using the definition of convolution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

we get

$$\left[ x(t) * y(t) \right] * z(t) = \left[ \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \right] * z(t)$$

or

$$\left[ x(t) * y(t) \right] * z(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau_{xy})y(t-\tau_{xy})d\tau_{xy} \right] z(t-\tau_{yz})d\tau_{yz}$$

and
\[ x(t) * [y(t) * z(t)] = x(t) * \left( \int_{-\infty}^{\infty} y(\tau) z(t - \tau) d\tau \right) \]

or

\[ x(t) * [y(t) * z(t)] = \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} y(\tau_y) z(t - \tau_y - \tau_y z) d\tau \right) d\tau. \]

Then the proof consists of showing that

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_x) y(\tau_y - \tau_x) z(t - \tau_y) d\tau_x d\tau_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_x) y(\tau_y) z(t) d\tau_x d\tau_y. \]

In the right-hand \( \tau_{xy} \) integration make the change of variable \( \lambda = \tau_{xy} + \tau_{yz} \) and \( d\lambda = d\tau_{xy} \). Then

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_{xy}) y(\tau_y - \tau_{xy} - \tau_{xy} z) z(t - \tau_y) d\tau_x d\tau_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda - \tau_{xy}) y(\tau_y) z(t - \lambda) d\lambda d\tau, \]

Next, in the right-hand \( \tau_{xy} \) integration make the change of variable \( \eta = \lambda - \tau_{yz} \) and \( d\eta = d\tau_{yz} \). Then

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_{xy}) y(\tau_y - \tau_{xy}) z(t - \tau_y) d\tau_x d\tau_y = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\eta) y(\lambda - \eta) z(t - \lambda) d\lambda d\eta \] (D.1)

or

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_{xy}) y(\tau_y - \tau_{xy}) z(t - \tau_y) d\tau_x d\tau_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\eta) y(\lambda - \eta) z(t - \lambda) d\lambda d\eta \] (D.2)

Except for the names of the variables of integration, the two integrals (D.1) and (D.2) are the same, therefore the integrals are equal and the associativity of convolution is proven.

**D.1.3 Distributivity Property**

Convolution is also distributive,

\[ x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \]

\[ x(t) * [h_1(t) + h_2(t)] = \int_{-\infty}^{\infty} x(t)[h_1(t - \tau) + h_2(t - \tau)] d\tau \]
\[ x(t) \ast [h_1(t) + h_2(t)] = \int_{-\infty}^{\infty} x(t)h_1(t - \tau)d\tau + \int_{-\infty}^{\infty} x(t)h_2(t - \tau)d\tau \]

\[ x(t) \ast [h_1(t) + h_2(t)] = x(t) \ast h_1(t) + x(t) \ast h_2(t) \]

**D.1.4 Differentiation Property**

Let \( y(t) \) be the convolution of \( x(t) \) with \( h(t) \)

\[ y(t) = x(t) \ast h(t) = \int x(\tau)h(t - \tau)d\tau. \]

Taking the derivative of \( y(t) \) with respect to time,

\[ y'(t) = \int x(\tau)h'(t - \tau)d\tau = x(t) \ast h'(t) \]

and, invoking the commutativity of convolution,

\[ y'(t) = x'(t) \ast h(t). \]

**D.1.5 Area Property**

Let \( y(t) \) be the convolution of \( x(t) \) with \( h(t) \)

\[ y(t) = x(t) \ast h(t) = \int x(\tau)h(t - \tau)d\tau. \]

The area under \( y(t) \) is

\[ \int_{-\infty}^{\infty} y(t)dt = \int_{-\infty}^{\infty} \int x(\tau)h(t - \tau)d\tau dt \]

or, exchanging the order of integration,

\[ \int_{-\infty}^{\infty} y(t)dt = \int_{-\infty}^{\infty} x(\tau)d\tau \int_{-\infty}^{\infty} h(t - \tau)dt \]

proving that the area of \( y \) is the product of the areas of \( x \) and \( h \).
D.1.6 Scaling Property

Let \( y(t) = x(t) * h(t) \) and \( z(t) = x(at) * h(at) \), \( a > 0 \). Then

\[
z(t) = \int_{-\infty}^{\infty} x(a\tau)h(a(t-\tau))d\tau.
\]

Making the change of variable, \( \lambda = at \Rightarrow d\lambda = d\tau / a \), for \( a > 0 \) we get

\[
z(t) = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)h(at - \lambda)d\lambda.
\]

Since

\[
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
\]

it follows that \( z(t) = (1/a)y(at) \) and \( (1/a)y(at) = x(at) * h(at) \). If we do a similar proof for \( a < 0 \) we get \(- (1/a)y(at) = x(at) * h(at)\). Therefore, in general, if \( y(t) = x(t) * h(t) \) then

\[
y(at) = |a|x(at) * h(at).
\]

D.2 Discrete-Time Convolution Properties

D.2.1 Commutativity Property

The commutativity of DT convolution can be proven by starting with the definition of convolution

\[
x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]
\]

and letting \( q = n - k \). Then we have

\[
x[n] * h[n] = \sum_{q=-\infty}^{\infty} x[n-q]h[q] = \sum_{q=-\infty}^{\infty} h[q]x[n-q] = h[n] * x[n]
\]

D.2.2 Associativity Property
If we convolve \( g[n] = x[n] * y[n] \) with \( z[n] \) we get

\[
g[n] * z[n] = \left( \sum_{k=-\infty}^{\infty} x[k] y[n-k] \right) * z[n]
\]

or

\[
g[n] * z[n] = \sum_{q=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k] y[q-k] \right) z[n-q]
\]

Exchanging the order of summation,

\[
\left( \sum_{k=-\infty}^{\infty} x[k] \right) \sum_{q=-\infty}^{\infty} y[q-k] z[n-q] = \sum_{k=-\infty}^{\infty} x[k] \sum_{q=-\infty}^{\infty} y[q-k] z[n-q]
\]

Let \( n - q = m \) and let \( h[n] = y[n] * z[n] \). Then

\[
\left( \sum_{k=-\infty}^{\infty} x[k] \right) \sum_{q=-\infty}^{\infty} y[q-k] z[n-q] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \left( \sum_{k=-\infty}^{\infty} x[k] \right) \left( \sum_{k=-\infty}^{\infty} y[k] z[n-k] \right)
\]

D.2.3 Distributivity Property

If we convolve \( x[n] \) with the sum of \( y[n] \) and \( z[n] \) we get

\[
x[n] * (y[n] + z[n]) = \sum_{k=-\infty}^{\infty} x[k] (y[n-k] + z[n-k])
\]

or

\[
x[n] * (y[n] + z[n]) = \sum_{k=-\infty}^{\infty} x[k] y[n-k] + \sum_{k=-\infty}^{\infty} x[k] z[n-k].
\]

Therefore

\[
x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n].
\]
D.2.4 Differencing Property

Let \( y[n] = x[n] * h[n] \). Using the time-shifting property

\[
y[n-n_0] = x[n] * h[n-n_0] = x[n-n_0] * h[n]
\]

the first backward difference of their convolution sum is

\[
\]

or

\[
y[n] - y[n-1] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] - \sum_{m=-\infty}^{\infty} x[m] h[n-m-1].
\]

Combining summations,

\[
y[n] - y[n-1] = \sum_{m=-\infty}^{\infty} x[m] (h[n-m] - h[n-m-1])
\]

or

\[
y[n] - y[n-1] = x[n] (h[n] - h[n-1])
\]

D.2.5 Sum Property

Let \( y[n] = x[n] * h[n] \) and let the sum of all the impulses in the functions \( y, x, \) and \( h \) be \( S_y, S_x \) and \( S_h \), respectively. Then

\[
y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m]
\]

and

\[
S_y = \sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[n-m].
\]

Interchanging the order of summation,

\[
S_y = \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m] = S_x S_h.
\]