Web Appendix F - Derivations of the Properties of the Continuous-Time Fourier Series

F.1 Numerical Computation of the CTFS

The harmonic function of a periodic signal with period $T_F$ is

$$X[k] = \frac{1}{T_F} \int_{T_F} x(t) e^{-j2\pi k t/T_F} dt.$$ 

Since the starting point of the integral is arbitrary, for convenience set it to $t = 0$

$$X[k] = \frac{1}{T_F} \int_0^{T_F} x(t) e^{-j2\pi k t/T_F} dt.$$ 

Suppose we don’t know the function $x(t)$ but we have a set of $N_F$ samples over one period starting at $t = 0$, the time between samples is $T_s = T_F / N_F$. Then we can approximate the integral by the sum of several integrals, each covering a time of length $T_s$

$$X[k] = \frac{1}{T_F} \int_0^{T_F} x(t) e^{-j2\pi k t/T_F} dt.$$ 

(Figure F-1).
Figure F-1  Sampling the arbitrary periodic signal to estimate its CTFS harmonic function

(In Figure F-1, the samples extend over one fundamental period but they could extend over any period and the analysis would still be correct.) If the samples are close enough together \(x(t)\) does not change much between samples and the integral (F.1) becomes a good approximation. We can now complete the integration.

\[
X[k] = \frac{1}{T_F} \sum_{n=0}^{N_s-1} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j2\pi kf_F t} dt = \frac{1}{T_F} \sum_{n=0}^{N_s-1} x(nT_s) \left[ \frac{e^{-j2\pi k f_F (n+1)T_s}}{-j2\pi k f_F} \right]_{nT_s}^{(n+1)T_s}
\]

\[
X[k] = \frac{1}{T_F} \sum_{n=0}^{N_s-1} x(nT_s) \left[ e^{-j2\pi k f_F nT_s} - e^{-j2\pi k f_F (n+1)T_s} \right] \frac{1}{j2\pi k f_F} \sum_{n=0}^{N_s-1} x(nT_s) e^{-j2\pi k f_F nT_s}
\]

Using \(T_s = T_F / N_F\),

\[
X[k] = \frac{1}{j2\pi k} \sum_{n=0}^{N_s-1} x(nT_s) e^{-j2\pi k f_F nT_s} = e^{-j\pi k N_f} \frac{e^{j\pi k N_f} - e^{-j\pi k N_f}}{j2\pi k} \sum_{n=0}^{N_s-1} x(nT_s) e^{-j2\pi k f_F nT_s}
\]

\[
X[k] = e^{-j\pi k N_f} \frac{\sin(\pi k / N_F)}{N_F} \sum_{n=0}^{N_s-1} x(nT_s) e^{-j2\pi k f_F nT_s}
\]

For harmonic numbers \(|k| << N_F\) we can further approximate the harmonic function as

\[
X[k] \approx \frac{1}{N_F} \sum_{n=0}^{N_s-1} x(nT_s) e^{-j2\pi k f_F nT_s}.
\]

**F.2 Linearity**

Let \(z(t) = \alpha x(t) + \beta y(t)\). If \(T_F = mT_{0x} = qT_{0y}\), where \(m\) and \(q\) are integers, then the CTFS harmonic function of \(z(t)\) is

\[
Z[k] = \frac{1}{T_F} \int_{T_F} z(t) e^{-j2\pi kf_F t} dt = \frac{1}{T_F} \int_{T_F} [\alpha x(t) + \beta y(t)] e^{-j2\pi kf_F t} dt
\]
and
\[ z(t) = \sum_{n=-\infty}^{\infty} Z[k] e^{j2\pi n f_s t} = \sum_{n=-\infty}^{\infty} \left( \alpha X[k] + \beta Y[k] \right) e^{j2\pi n f_s t} \]
and
\[ Z[k] = \alpha X[k] + \beta Y[k] . \]
So the linearity property is
\[ T_F = m T_{0_x} = q T_{0_y} \]
\[ \alpha x(t) + \beta y(t) \xrightarrow{F.S.} \alpha X[k] + \beta Y[k] . \]

F.3 Time Shifting

Let \( z(t) = x(t - t_0) \) and let \( T_F = m T_{0_x} = m T_{0_z} \), where \( m \) is an integer. Then
\[ Z[k] = \frac{1}{T_F} \int_{T_F} z(t) e^{-jk\omega_0 t} dt = \frac{1}{T_F} \int_{T_F} x(t - t_0) e^{-jk\omega_0 t} dt . \]
Making the change of variable \( \tau = t - t_0 \Rightarrow d\tau = dt \),
\[ Z[k] = \frac{1}{T_F} \int_{T_F} x(\tau) e^{-jk\omega_0 (\tau + t_0)} d\tau = e^{-jk\omega_0 t_0} \frac{1}{T_F} \int_{T_F} x(\tau) e^{-jk\omega_0 \tau} d\tau \]
\[ = X[k] . \]
So the time-shifting property is
\[ T_F = m T_0 \]
\[ x(t - t_0) \xrightarrow{F.S.} e^{-j2\pi f_s t_0} X[k] . \]
\[ x(t - t_0) \xrightarrow{F.S.} e^{-jk\omega_0 t_0} X[k] . \]
F.4 Frequency Shifting

Let \( z(t) = e^{j2\pi k_0 f_F t} x(t) \), with \( k_0 \) being an integer and \( T_F = m T_{0x} = m T_{0z} \), where \( m \) is an integer. Then

\[
Z[k] = \frac{1}{T_F} \int_{T_F} z(t) e^{-j2\pi k_0 f_F t} dt = \frac{1}{T_F} \int_{T_F} e^{j2\pi k_0 f_F t} x(t) e^{-j2\pi k_0 f_F t} dt
\]

\[
Z[k] = \frac{1}{T_F} \int_{T_F} x(t) e^{-j2\pi (k-k_0) f_F t} dt = X[k-k_0].
\]

So the frequency shifting property is

\[
T_F = m T_0
\]

\[
e^{j2\pi k_0 f_F t} x(t) \xrightarrow{FS} X[k-k_0].
\]

\[
e^{j2\pi k_0 f_F t} x(t) \xrightarrow{FS} X[k-k_0].
\]

F.5 Time Reversal

Let \( z(t) = x(-t) \) and let \( T_F = m T_{0x} = m T_{0z} \), where \( m \) is an integer. If

\[
x(t) = \sum_{k=\infty}^{\infty} X[k] e^{j2\pi k f_F t}
\]

then

\[
x(-t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F (-t)} = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi (-k) f_F t}.
\]

Let \( q = -k \), then

\[
x(-t) = \sum_{q=-\infty}^{\infty} X[-q] e^{j2\pi q f_F t}
\]

and, since changing the order of summation does not change the sum,

\[
x(-t) = \sum_{q=-\infty}^{\infty} X[-q] e^{j2\pi q f_F t}.
\]

Therefore, since

\[
z(t) = \sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi k f_F t}
\]

we can say that
and \( Z[k] = X[-k] \). So the time reversal property is

\[
T_F = mT_0
\]

\[
x(-t) \xrightarrow{F_S} X[-k].
\]

**F.6 Time Scaling**

Let \( z(t) = x(at) \), \( a > 0 \) and let \( T_F = mT_{0x} \), \( m \) an integer. (Figure F-2).

The first thing to realize is that if \( x(t) \) is periodic with fundamental period \( T_{0x} \) that \( z(t) \) is periodic with fundamental period \( T_{oz} = T_{0x} / a \) and fundamental frequency \( af_{0x} \).

Case 1. \( z(t) \) represented by a CTFS over a period of \( z(t), T_F / a \).

The CTFS harmonic function will be

\[
Z[k] = \frac{a}{T_F} \int_{t_0}^{t_0 + T_F / a} z(t) e^{-j2\pi kaf_F t} dt = \frac{a}{T_F} \int_{t_0}^{t_0 + T_F / a} x(at) e^{-j2\pi kaf_F t} dt.
\]

We can make the change of variable \( \tau = at \Rightarrow d\tau = adt \) yielding
\[
Z[k] = \frac{a}{T_F a} \int_{at_0}^{at_0+T_F} x(\tau) e^{-j2\pi kf_F \tau} d\tau = \frac{1}{T_F} \int_{at_0}^{at_0+T_F} x(\tau) e^{-j2\pi kf_F \tau} d\tau.
\]

Since the starting point \( t_0 \) is arbitrary

\[
Z[k] = \frac{1}{T_F} \int_{T_F} x(\tau) e^{-j2\pi kf_F \tau} d\tau = X[k]
\]

and the CTFS harmonic function describing \( z(t) \) over the period \( T_F / a \) is the same as the CTFS harmonic function describing \( x(t) \) over the period \( T_F \).

\[
z(t) = x(at)
\]

\[
T_F = mT_0 \iff T_F / a = mT_0 / a
\]

\[
Z[k] = X[k]
\]

Even though the CTFS harmonic functions of \( x(t) \) and \( z(t) \) are the same, the CTFS representations themselves are not because the fundamental frequencies are different. The representations are

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kf_F t} \text{ and } z(t) = x(at) = \sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi kf_F t}.
\]

Case 2. \( z(t) \) represented by a CTFS over a period of \( x(t), T_F \)

The CTFS harmonic function will be

\[
Z[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} z(t) e^{-j2\pi kf_F t} dt = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(at) e^{-j2\pi kf_F t} dt.
\]

Let \( \tau = at \iff d\tau = adt \). Then

\[
Z[k] = \frac{1}{aT_F} \int_{at_0}^{at_0+aT_F} x(\tau) e^{-j2\pi kf_F \tau} d\tau.
\]

If \( a \) is not an integer, the relationship between the two harmonic functions \( Z[k] \) and \( X[k] \) cannot be simplified further.
Let \( a \) be a non-zero integer. The signal \( x(t) \) is made up of frequency components at integer multiples of \( f_F \). Therefore for ratios \( k/a \) that are not integers, \( x(t) \) and 
\[
e^{-j2\pi f_F t/a}
\] are orthogonal on the interval \( at_0 < t < at_0 + aT_F \) and \( Z[k] = 0 \). For ratios \( k/a \) that are integers, the integral over \( a \) periods is \( a \) times the integral over one period and

\[
Z[k] = a \left( \frac{1}{aT_F} \int_{at_0}^{at_0 + aT_F} x(\tau) e^{-j2\pi (k/a)f_F \tau} d\tau \right) = X[k/a], \ k/a \text{ an integer}.
\]

So the time scaling property, for this kind of time scaling, is

\[
T_F = mT_0, \ z(\tau) = x(at), \ a \text{ a non-zero integer}
\]

\[
Z[k] = \begin{cases} 
X[k/a], & k/a \text{ an integer} \\
0, & \text{otherwise}
\end{cases}
\]

**F.7 Change of Period**

If the CTFS harmonic function of \( x(t) \) over any period \( T_F \) is \( X[k] \), we can find the CTFS harmonic function \( X_q[k] \) of \( x(t) \) over a time \( qT_F \) where \( q \) is a positive integer. The new fundamental CTFS frequency is then \( f_F / q \) and

\[
X_q[k] = \frac{1}{qT_F} \int_{qT_F}^{qT_F} x(t) e^{-j2\pi (f_F/q)t} dt
\]

This is exactly the same as the result for time scaling by a positive integer in the previous section and the result is

\[
T_F \rightarrow qT_F \Rightarrow X_q[k] = \begin{cases} 
X[k/q], & k/q \text{ an integer} \\
0, & \text{otherwise}
\end{cases}
\]

**F.8 Time Differentiation**

Let \( z(t) = \frac{d}{dt} x(t) \) and let \( T_F = mT_0 = mT_z \), where \( m \) is an integer. Then we can represent \( z(t) \) by
\[ z(t) = \frac{d}{dt} \left( x(t) \right) = \frac{d}{dt} \left( \sum_{n=-\infty}^{\infty} X[k] e^{j\omega_n t} \right) = \sum_{n=-\infty}^{\infty} jk\omega_F X[k] e^{j\omega_n t} . \]

Then, if
\[ z(t) = \sum_{k=-\infty}^{\infty} Z[k] e^{j\omega_n t} \]
it follows that
\[ \sum_{k=-\infty}^{\infty} Z[k] e^{j\omega_n t} = jk\omega_F \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_n t} , \]
\[ Z[k] = jk\omega_F X[k] . \]

So the differentiation property is
\[ T_F = mT_0 \]
\[ \frac{d}{dt} \left( x(t) \right) \xrightarrow{F.S.} j2\pi k f_F X[k] , \quad \frac{d}{dt} \left( x(t) \right) \xrightarrow{F.S.} jk\omega_F X[k] . \]

**F.9 Time Integration**

Let \( z(t) = \int_{-\infty}^{t} x(\tau) d\tau \) and let \( T_F = mT_{0x} \), where \( m \) is an integer. We must consider two cases separately, \( X[0] = 0 \) and \( X[0] \neq 0 \). If \( X[0] \neq 0 \) then, even though \( x(t) \) is periodic, \( z(t) \) is not and we cannot represent it exactly for all time with a CTFS (Figure F-3).
Figure F-3  Effect of a non-zero average value on the integral of a periodic function

If \( X[0] = 0 \) then we can represent \( z(t) \) by

\[
z(t) = \int_{-\infty}^{t} x(\tau) d\tau
\]

and its CTFS harmonic function is

\[
Z[k] = \frac{1}{T_F} \int_{T_F} \left[ \int_{-\infty}^{t} x(\tau) d\tau \right] e^{-jk\omega_F t} dt.
\]

For \( k = 0 \),

\[
Z[0] = \frac{1}{T_F} \int_{T_F} \left[ \int_{-\infty}^{t} x(\tau) d\tau \right] dt
\]

which is the average value of the integral of \( x(t) \) over one fundamental period. We know that the average value of \( x(t) \) is zero but, without some other information, we don’t know what the average value of the integral of \( x(t) \) is and cannot determine the value of \( Z[0] \). However we can determine all the other values of \( Z[k] \).

\[
z(t) = \int_{-\infty}^{t} x(\tau) d\tau = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_F \tau} d\tau = \sum_{k=-\infty}^{\infty} X[k] \int_{-\infty}^{t} e^{jk\omega_F \tau} d\tau
\]

\[
Z[k] = \frac{1}{T_F} \int_{T_F} \left[ \sum_{q=-\infty}^{\infty} X[q] \int_{-\infty}^{t} e^{jq\omega_F \tau} d\tau \right] e^{-jk\omega_F t} dt.
\]
In (F.2) \( k \) has been replaced by \( q \) to avoid confusion with \( k \) which is independent of \( q \). To finish the integration we must evaluate the integral \( \int_{-\infty}^{*} e^{j\omega_F t} d\tau \) which is

\[
\left[ \frac{e^{j\omega_F t}}{j\omega_F} \right]_{-\infty}^{*}.
\]

Evaluation at the upper limit is not a problem but the lower limit does present a problem. Since the complex sinusoid \( e^{j\omega_F t} \) is periodic it is impossible to define what its value is at the lower limit of negative infinity because negative infinity is a limit, not a number. The magnitude of \( e^{j\omega_F t} \) is one but its phase could be any value in a range of \( 2\pi \) radians. But, it turns out that this won’t matter. Call the indeterminate value at the lower limit \( C \).

Then

\[
\left[ \frac{e^{j\omega_F t}}{j\omega_F} \right]_{-\infty}^{*} = e^{j\omega_F t} - C
\]

and

\[
Z[k] = \frac{1}{T_F} \int_{T_F} \sum_{q=-\infty}^{\infty} X[q] \left[ e^{j\omega_F t} - C \right] e^{-j\omega_F t} dt
\]

\[
= \sum_{q=-\infty}^{\infty} X[q] \int_{T_F} \left[ e^{j(q-k)\omega_F t} - Ce^{-j\omega_F t} \right] dt
\]

For each \( k \neq 0 \) the computation involves a summation of terms involving \( q \). But all of those terms are zero unless \( q = k \) because the integration is over an integer number of periods of a complex sinusoid. So, in the end,

\[
Z[k] = \frac{X[k]}{T_F} \int_{T_F} \left[ \frac{1}{jk\omega_F} \right] dt = \frac{X[k]}{jk\omega_F T_F} T_F = \frac{X[k]}{jk\omega_F}, \quad k \neq 0
\]

and the integration property is

\[
T_F = mT_0
\]

\[
\int_{-\infty}^{*} x(\tau) d\tau \xleftarrow{\text{FS}} \int_{-\infty}^{*} x(\tau) d\tau \xleftarrow{\text{FS}} X[k], \quad k \neq 0, \quad \text{if} \quad X[0] = 0.
\]
F.10 Multiplication-Convolution Duality

Let \( z(t) = x(t) y(t) \) and let \( T_F = mT_{0x} = qT_{0y} \), where \( m \) and \( q \) are integers. Then

\[
Z[k] = \frac{1}{T_F} \int_{T_F} z(t) e^{-j 2\pi kf_F} dt = \frac{1}{T_F} \int_{T_F} x(t) y(t) e^{-j 2\pi kf_F} dt.
\]

Then, using

\[
y(t) = \sum_{k=-\infty}^{\infty} Y[k] e^{j 2\pi kf_F} = \sum_{q=-\infty}^{\infty} Y[q] e^{j 2\pi qf_F}
\]

we get

\[
Z[k] = \frac{1}{T_F} \int_{T_F} x(t) \left( \sum_{q=-\infty}^{\infty} Y[q] e^{j 2\pi qf_F} \right) e^{-j 2\pi kf_F} dt.
\]

Reversing the order of integration and summation,

\[
Z[k] = \frac{1}{T_F} \sum_{q=-\infty}^{\infty} Y[q] \int_{T_F} x(t) e^{j 2\pi qf_F} e^{-j 2\pi kf_F} dt
\]

or

\[
Z[k] = \sum_{q=-\infty}^{\infty} Y[q] \frac{1}{T_F} \int_{T_F} x(t) e^{-j 2\pi (k-q)f_F} dt.
\]

Then

\[
Z[k] = \sum_{q=-\infty}^{\infty} Y[q] X[k-q]
\]

and the multiplication property is

\[
T_F = mT_{0x} = qT_{0y}
\]

\[
x(t) y(t) \xrightarrow{FS} \sum_{q=-\infty}^{\infty} Y[q] X[k-q] = X[k] * Y[k].
\]

This result \( \sum_{q=-\infty}^{\infty} Y[q] X[k-q] \) is a convolution sum. So the product of CT signals corresponds to the convolution sum of their CTFS harmonic functions.

Now let \( Z[k] = X[k] Y[k] \) and let \( T_F = mT_{0x} = qT_{0y} \), where \( m \) and \( q \) are integers. Then
\[ z(t) = \sum_{k=-\infty}^{\infty} X[k] Y[k] e^{jk\omega_F t} \]

\[ z(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_F} \int_{T_F} x(\tau) e^{-jk\omega_F \tau} d\tau \ Y[k] e^{jk\omega_F t} = \frac{1}{T_F} \int_{T_F} x(\tau) d\tau \sum_{k=-\infty}^{\infty} Y[k] e^{jk\omega_F (t-\tau)} \]

or

\[ z(t) = \frac{1}{T_F} \int_{T_F} x(\tau) y(t-\tau) d\tau . \]

This integral looks just like a convolution integral except that it covers the range \( t_0 \leq \tau < t_0 + T_F \) instead of \( -\infty < \tau < \infty \). This integral operation is called *periodic convolution* and is indicated by the notation

\[ x(t) \otimes y(t) = \int_{T_F} x(\tau) y(t-\tau) d\tau . \]

Therefore

\[ z(t) = (1/T_F) x(t) \otimes y(t) . \]

Since \( x(t) \) is periodic it can be expressed as the periodic extension of an aperiodic function \( x_{ap}(t) \)

\[ x(t) = \sum_{q=-\infty}^{\infty} x_{ap}(t-qT_F) = x_{ap}(t) \ast \delta_{T_F}(t) . \]

(The function, \( x_{ap}(t) \), is not unique. It can be any function which satisfies this equation.) Then

\[ x(t) \otimes y(t) = \int_{T_F} \left[ \sum_{q=-\infty}^{\infty} x_{ap}(\tau-qT_F) \right] y(t-\tau) d\tau \]

\[ x(t) \otimes y(t) = \sum_{q=-\infty}^{\infty} \int_{0}^{T_F} x_{ap}(\tau-qT_F) y(t-\tau) d\tau . \]

Let \( \lambda = \tau - qT_F \). Then \( d\lambda = d\tau \) and

\[ x(t) \otimes y(t) = \sum_{q=-\infty}^{\infty} \int_{b_q}^{b_q+qT_F} x_{ap}(\lambda) y(t-(\lambda + qT_F)) d\lambda . \]

Since \( y(t) \) is periodic, with period \( T_F \),

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\[
y(t - (\lambda + qT_F)) = y(t - qT_F - \lambda) = y(t - \lambda)
\]
and the summation of integrals \(\sum_{q=-\infty}^{\infty} \int_{t_0+qT_F}^{t_0+(q+1)T_F} \) is equivalent to the single integral over infinite limits \(\int_{-\infty}^{\infty} \) we conclude that
\[
x(t) \ast y(t) = \int_{-\infty}^{\infty} x_\text{ap}(\lambda)y(t-\lambda)\,d\lambda = x_\text{ap}(t) \ast y(t).
\]
So the periodic convolution of two functions \(x(t)\) and \(y(t)\) each with period \(T_F\) can be expressed as an aperiodic convolution of \(y(t)\) with a function \(x_\text{ap}(t)\) which, when periodically repeated with the same period \(T_F\) equals \(x(t)\). The periodic convolution of two periodic functions corresponds to the product of their CTFS harmonic function representations and the period \(T_F\) and the convolution property is
\[
T_F = mT_{0x} = qT_{0y}.
\]
\[
x(t) \ast y(t) \overset{FS}{\Longleftarrow} T_F X[k]Y[k] .
\]

F.11 Conjugation

Let \(z(t) = x^*(t)\) and let \(T_F = mT_{0x} = mT_{0z}\), where \(m\) is an integer. Then
\[
\sum_{k=-\infty}^{\infty} Z[k]e^{j2\pi k f_F t} = \left(\sum_{k=-\infty}^{\infty} X[k]e^{j2\pi k f_F t}\right)^* = \sum_{k=-\infty}^{\infty} X^*[k]e^{-j2\pi k f_F t} = \sum_{k=-\infty}^{\infty} X^*[\bar{k}]e^{j2\pi k f_F t},
\]
and, since changing the order of summation does not change the sum,
\[
\sum_{k=-\infty}^{\infty} Z[k]e^{j2\pi k f_F t} = \sum_{k=-\infty}^{\infty} X^*[\bar{k}]e^{j2\pi k f_F t},
\]
\(Z[k] = X^*[\bar{k}]\) and the conjugation property is
\[
T_F = mT_0, \quad x^*(t) \overset{FS}{\Longleftarrow} X^*[\bar{k}] .
\]
F.12 Parseval's Theorem

The signal energy in any period \( T_F = mT_0 \), where \( m \) is an integer, of any periodic signal \( x(t) \) is

\[
E_{x,T_F} = \int_{T_F} |x(t)|^2 \, dt = \int_{T_F} \left| \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kf_F t} \right|^2 \, dt
\]

\[
E_{x,T_F} = \int_{T_F} \left( \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kf_F t} \right)^* \left( \sum_{q=-\infty}^{\infty} X[q] e^{j2\pi qf_F t} \right) \, dt
\]

\[
E_{x,T_F} = \int_{T_F} \left( \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q)f_F t} \right) \, dt
\]

\[
E_{x,T_F} = \int_{T_F} \left( \sum_{k=-\infty}^{\infty} X[k] X^*[k] + \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q)f_F t} \right) \, dt
\]

\[
E_{x,T_F} = \int_{T_F} \sum_{k=-\infty}^{\infty} |X[k]|^2 \, dt + \int_{T_F} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q)f_F t} \, dt
\]

\[
E_{x,T_F} = T_F \sum_{k=-\infty}^{\infty} |X[k]|^2
\]

Therefore, for any periodic signal \( x(t) \)

\[
T_F = mT_0
\]

\[
\frac{1}{T_F} \int_{T_F} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2.
\]