Web Appendix G - Derivations of the Properties of the Discrete-Time Fourier Series

G.1 The Trigonometric Discrete-Time Fourier Series

Assuming a DTFS harmonic function has been found, we can say that

\[ x[n] = x_F[n], \quad n_0 < n < n_0 + N_F \]

where

\[ x_F[n] = \sum_{k=\{N_F\}} X[k] e^{j2\pi kn/N_F}. \]  \hspace{1cm} (G.1)

It is useful to explore the characteristics of the complex conjugate of \( x_F[n] \). If we conjugate both sides of (G.1) we get

\[ x_F^*[n] = \sum_{k=\{N_F\}} X^*[k] e^{-j2\pi kn/N_F}. \]

Since any range of \( k \) exactly \( N_F \) in length will work we can replace \( k \) with \(-k\) and still have an equality

\[ x_F^*[n] = \sum_{k=\{N_F\}} X^*[-k] e^{j2\pi kn/N_F}. \]

In words, this says that to find the DTFS harmonic function \( X[k] \) for the complex conjugate of a signal, conjugate it and change the sign of \( k \). The transformation is \( X[k] \rightarrow X^*[-k] \) and then for any \( x[n] \), \( x^*[n] \leftarrow \) F.S. \( X^*[-k] \). In the very important special case in which \( x[n] \) is a real-valued function, \( x[n] = x^*[n] \) and therefore \( x_F[n] = x_F^*[n] \). That means that the two representations,

\[ x_F[n] = \sum_{k=\{N_F\}} X[k] e^{j2\pi kn/N_F} \quad \text{and} \quad x_F^*[n] = \sum_{k=\{N_F\}} X^*[-k] e^{j2\pi kn/N_F} \]
must be equal and therefore that \( X[k] = X^*[−k] \), implying that, for real-valued signals and for any \( k \), \( X[k] \) and \( X[−k] \) are complex conjugates.

Any set of consecutive harmonics exactly \( N_F \) in length is sufficient to represent a signal over the time range \( n_0 \leq n < n_0 + N_F \). The time range \( N_F \) is either an even integer or an odd integer. For \( N_F \) even, consider the harmonics \(-N_F/2 \leq k < N_F/2\). All the harmonics except \( k = 0 \) and \( k = −N_F/2 \) occur in complex conjugate pairs \( \pm 1, \pm 2, \ldots, \pm (N_F/2−1) \). The \( k = 0 \) harmonic is

\[
X[0] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F−1} x[n]
\]

and is therefore guaranteed to be a real number (if \( x[n] \) is real). The \( k = −N_F/2 \) harmonic is

\[
X[−N_F/2] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F−1} x[n]e^{j\pi n} = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F−1} x[n]\cos(\pi n)
\]

\[
= \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F−1} x[n](-1)^n
\]

which is also guaranteed real. Therefore, we can write the DTFS representation of the signal, \( x_F[n] = \sum_{k=−N_F/2}^{N_F/2−1} X[k]e^{j2\pi kn/N_F} \), as

\[
x_F[n] = X[0] + X[−N_F/2]e^{j\pi n}\cos(\pi n) + \sum_{k=1}^{N_F/2−1} \left[ X[k]e^{j2\pi kn/N_F} + X^*[k]e^{−j2\pi kn/N_F} \right]
\]

or

\[
x_F[n] = X[0] + (−1)^n X[N_F/2] + \sum_{k=1}^{N_F/2−1} \left[ \text{Re}(X[k])e^{j2\pi kn/N_F} + \text{Re}(X^*[k])e^{−j2\pi kn/N_F} + \text{Im}(X[k])e^{j2\pi kn/N_F} + \text{Im}(X^*[k])e^{−j2\pi kn/N_F} \right]
\]

\[
x_F[n] = X[0] + (−1)^n X[N_F/2] + \sum_{k=1}^{N_F/2−1} \left[ 2\text{Re}(X[k])\cos(2\pi kn/N_F) \right. + \left. 2\text{Im}(X[k])\sin(2\pi kn/N_F) \right]
\]
\[ x_F[n] = X[0] + (-1)^n X[N_F / 2] + \sum_{k=1}^{N_F/2-1} \left[ (X[k] + X^*[k]) \cos\left(\frac{2\pi kn}{N_F}\right) + j(X[k] - X^*[k]) \sin\left(\frac{2\pi kn}{N_F}\right) \right] \]

\[ x_F[n] = X[0] + (-1)^n X[N_F / 2] + \sum_{k=1}^{N_F/2-1} \left[ X_c[k] \cos\left(\frac{2\pi kn}{N_F}\right) + X_s[k] \sin\left(\frac{2\pi kn}{N_F}\right) \right] \quad (G.2) \]

where \( X_c[k] = X[k] + X^*[k] \) and \( X_s[k] = j(X[k] - X^*[k]) \), \( 0 < k < N_F / 2 \).

Therefore, using \( X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] e^{-j2\pi kn/N_F} \),

\[ X_c[k] = \frac{2}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] \cos\left(\frac{2\pi kn}{N_F}\right) \] and \( X_s[k] = \frac{2}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] \sin\left(\frac{2\pi kn}{N_F}\right) \)

For \( N_F \) odd the development is similar and the result is

\[ x_F[n] = X[0] + \sum_{k=1}^{(N_F-1)/2} \left[ X_c[k] \cos\left(\frac{2\pi kn}{N_F}\right) + X_s[k] \sin\left(\frac{2\pi kn}{N_F}\right) \right] . \quad (G.3) \]

Equations (G.2) and (G.3) are each a representation of the real-valued signal \( x[n] \) in terms of a linear combination of a real constant and real-valued cosines and sines. These are known as the trigonometric forms of the CTFS harmonic function for real-valued signals. The relationships between the complex and trigonometric harmonic functions are

\[ X_c[0] = X[0] , \quad X_s[0] = 0 \]

\[ X_c[k] = \begin{cases} X[k] + X^*[k] , & 0 < k < N_F / 2 \\ X[k] , & k = N_F / 2 \end{cases} \quad (G.4) \]

\[ X_s[k] = \begin{cases} j(X[k] - X^*[k]) , & 0 < k < N_F / 2 \\ 0 , & k = N_F / 2 \end{cases} \]

and

\[ \]
The complex and trigonometric forms of the DTFS are closely related because of Euler’s identity \( e^{jx} = \cos(x) + j\sin(x) \) which indicates that when we find a complex sinusoid in a DTFS representation of a signal we are, by implication, simultaneously finding a cosine and a sine.

G.2 Properties

Linearity

Let \( z[n] = \alpha x[n] + \beta y[n] \). Then

\[
z[n] = \sum_{k=\langle N_0 \rangle} X[k] e^{j2\pi kF_0 n} + \sum_{k=\langle N_0 \rangle} Y[k] e^{j2\pi kF_0 n} = \sum_{k=\langle N_0 \rangle} \left( \alpha X[k] + \beta Y[k] \right) e^{j2\pi kF_0 n}
\]

But \( z[n] \) also has a DTFS representation,

\[
z[n] = \sum_{k=\langle N_0 \rangle} Z[k] e^{j2\pi kF_0 n}.
\]

Therefore, we can conclude that

\[
Z[k] = \alpha X[k] + \beta Y[k]
\]

and

\[
\alpha x[n] + \beta y[n] \leftrightarrow \alpha X[k] + \beta Y[k].
\]

Time Shifting

Let \( z[n] = x[n-n_0] \). Then

\[
Z[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} z[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n-n_0] e^{-j2\pi kF_0 n}
\]
Now let $q = n - n_0$ in the $x$ summation. Then, since $n$ covers a range of $N_0$, $q$ does also and

$$Z[k] = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} z[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0} \sum_{q = \langle N_0 \rangle} x[q] e^{-j2\pi kF_0 (q + n_0)}$$

$$Z[k] = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} z[n] e^{-j2\pi kF_0 n} = e^{-j2\pi kF_0 n_0} \frac{1}{N_0} \sum_{q = \langle N_0 \rangle} x[q] e^{-j2\pi kF_0 q}$$

and

$$Z[k] = e^{-j2\pi kF_0 n_0} X[k]$$

Frequency Shifting

Let $z[n] = e^{j2\pi k_0 F_0 n} x[n]$, $k_0$ an integer. Then

$$Z[k] = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} z[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} e^{j2\pi k_0 F_0 n} x[n] e^{-j2\pi kF_0 n}$$

$$Z[k] = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} x[n] e^{-j2\pi (k-k_0) F_0 n} = x[k-k_0]$$

$$Z[k] = X[k-k_0]$$

$$e^{j2\pi k_0 F_0 n} x[n] \xrightarrow{FS} X[k-k_0]$$

Conjugation

Let $z[n] = x^*[n]$. Then

$$Z[k] = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} z[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0} \sum_{n = \langle N_0 \rangle} x^*[n] e^{-j2\pi kF_0 n}$$

Conjugating both sides,
\[ Z^*[k] = \frac{1}{N_0} \sum_{n=N_0}^{N_0} x[n] e^{j2\pi F_0 n} = \frac{1}{N_0} \sum_{n=N_0}^{N_0} x[n] e^{-j2\pi (-kF_0)n} \]

\[ Z^*[k] = X[-k] \]

or

\[ Z[k] = X^*[-k] \]

and

\[ x^*[n] \xrightarrow{F S} X^*[-k] \]

**Time Reversal**

Let \( z[n] = x[-n] \). Then

\[ Z[k] = \frac{1}{N_0} \sum_{n=N_0}^{N_0} z[n] e^{-j2\pi F_0 n} = \frac{1}{N_0} \sum_{n=N_0}^{N_0} x[-n] e^{-j2\pi kF_0 n} \]

Let \( m = -n \). Then if \( n \) covers a range of \( N_0 \), so does \( m \) and

\[ Z[k] = \frac{1}{N_0} \sum_{m=N_0}^{N_0} x[-m] e^{-j2\pi F_0 (-m)} = \frac{1}{N_0} \sum_{m=N_0}^{N_0} x[-m] e^{-j2\pi (-kF_0)m} \]

\[ Z[k] = X[-k] \]

and

\[ x[-n] \xrightarrow{F S} X[-k] \]

**Time Scaling**

Let \( z[n] = x[an] \), \( a > 0 \). If \( a \) is not an integer then some values of \( z[n] \) will be undefined and a DTFS cannot be found for it. If \( a \) is an integer, then \( z[n] \) is a decimated version of \( x[n] \) and some of the values of \( x[n] \) do not appear in \( z[n] \). In that case, there cannot be a unique relationship between the harmonic functions of \( x[n] \) and \( z[n] \) through the transformation, \( n \to an \) (Figure G-1).
However there is an operation for which the relationship between $x[n]$ and $z[n]$ is unique. Let $m$ be a positive integer and let

$$z[n] = \begin{cases} x[n/m] , & n/m \text{ an integer} \\ 0 , & \text{otherwise} \end{cases}$$

That is, $z[n]$ is a time-expanded version of $x[n]$ formed by placing $m-1$ zeros between adjacent values of $x[n]$ (Figure G-2).

Figure G-1  Two different signals decimated to yield the same signal

Figure G-2  A DT function and an expanded version formed by inserting zeros between values
If the fundamental period of $x[n]$ is $N_{0x}$, the fundamental period of $z[n]$ is $N_{0z} = mN_{0x}$. Then the DTFS harmonic function for $z[n]$ with a representation time of $N_F = qN_{0z}$, where $q$ is an integer is

$$Z[k] = \frac{1}{N_F} \sum_{n=[N_F]} z[n] e^{-j2\pi nk/N_F}.$$  

Since all the values of $z$ are zero when $n/m$ is not an integer,

$$Z[k] = \frac{1}{N_F} \sum_{n=[N_F]} z[n] e^{-j2\pi nk/N_F}.$$  

Let $p = n/m$, when $n/m$ is an integer. Then

$$Z[k] = \frac{1}{N_F} \sum_{p=[N_F/m]} z[mp] e^{-j2\pi kp/N_F}$$  

and $z[mp] = x[p]$. Therefore, since $N_F/m = qN_{0z}/m = qmN_{0x}/m = qN_{0x},$

$$Z[k] = \frac{1}{N_F} \sum_{p=[qN_{0x}]} x[p] e^{-j2\pi kp/qN_{0x}} = \frac{1}{m} X[k]$$

where $X[k]$ is the harmonic function for $x[n]$ using a representation time of $qN_{0x}$. So the time-scaling property is

$$z[n] = \begin{cases} x[n/m], & n/m \text{ an integer} \\ 0, & \text{otherwise} \end{cases}.$$  

$N_F \rightarrow mN_F$, $Z[k] = (1/m)X[k]$

Change of Period

If we know that the DTFS harmonic function of $x[n]$ over the representation time $N_F = mN_{0x}$, where $m$ is an integer, is $X[k]$ we can find the harmonic function of $x[n]$ over the representation time $qN_F$, which is $X[qk]$, with $q$ being a positive integer. It is
\[ X_q[k] = \frac{1}{qN_F} \sum_{n=\{qN_F\}} x[n] e^{-j2\pi nk / qN_F}. \]

The DT function \( x[n] \) has a period \( N_F \) and therefore is represented by DT sinusoids at integer multiples of \( 1 / N_F \). The DT function \( e^{-j2\pi nk / qN_F} \) has a fundamental period \( qN_F \) and fundamental frequency \( 1 / qN_F \). Therefore, on the DT interval \( n_0 \leq n < n_0 + qN_F \) the two DT functions \( x[n] \) and \( e^{-j2\pi nk / qN_F} \) are orthogonal unless \( k / q \) is an integer. Therefore, for \( k / q \) not an integer, \( X_q[k] = 0 \). For \( k / q \) an integer, the summation over \( qN_F \) is equivalent to \( q \) summations over \( N_F \) and

\[ X_q[k] = q \left( \frac{1}{qN_F} \sum_{n=\{qN_F\}} x[n] e^{-j2\pi nk / qN_F} \right) = \frac{1}{N_F} \sum_{n=\{N_F\}} x[n] e^{-j2\pi nk / qN_F} = X[k/q]. \]

Summarizing,

\[ N_F \rightarrow qN_F, \quad q \text{ a positive integer} \]

\[ X_q[k] = \begin{cases} X[k/q], & k/q \text{ an integer} \\ 0, & \text{otherwise} \end{cases} \]

(Figure G-3).

Figure G-3  DT signal and the magnitude of its DTFS harmonic function with \( N_F = N_0 \) and with \( N_F = 2N_0 \)
Multiplication-Convolution Duality

Let \( z[n] = x[n]y[n] \) and let \( N_F = mN_{x0} = qN_{y0} \), where \( m \) and \( q \) are integers. Then

\[
Z[k] = \frac{1}{N_F} \sum_{n=\{N_F\}} z[n]e^{-j2\pi kn/N_F} = \frac{1}{N_F} \sum_{n=\{N_F\}} x[n]y[n]e^{-j2\pi kn/N_F},
\]

and, using

\[
y[n] = \sum_{p=\{N_F\}} Y[p]e^{j2\pi pn/N_F},
\]

\[
Z[k] = \frac{1}{N_F} \sum_{n=\{N_F\}} x[n] \sum_{p=\{N_F\}} Y[p]e^{j2\pi pn/N_F} e^{-j2\pi kn/N_F},
\]

\[
Z[k] = \frac{1}{N_F} \sum_{n=\{N_F\}} x[n] \sum_{p=\{N_F\}} Y[p]e^{-j2\pi (k-p)n/N_F},
\]

\[
Z[k] = \sum_{p=\{N_F\}} Y[p] \frac{1}{N_F} \sum_{n=\{N_F\}} x[n]e^{-j2\pi (k-p)n/N_F} e^{X[k-p]} = X[k-p]
\]

\[
Z[k] = \sum_{p=\{N_F\}} Y[p]X[k-p]
\]

This result looks just like a convolution sum except that \( q \) extends over a finite range instead of an infinite one. This is a periodic convolution sum which is indicated by the notation

\[
Z[k] = Y[k] \circledast X[k].
\]

Therefore

\[
N_F = mN_{x0} = qN_{y0}
\]

\[
x[n]y[n] \xrightarrow{\text{FS}} Y[k] \circledast X[k] = \sum_{p=\{N_F\}} Y[p]X[k-p]. \quad (G.6)
\]

Multiplication of two DT signals corresponds to the convolution of their DTFS harmonic functions but the convolution is now a periodic convolution.
Now let $\mathbf{Z}[k] = \mathbf{Y}[k] \mathbf{X}[k]$ and let $N_F = mN_{x_0} = qN_{y_0}$, where $m$ and $q$ are integers. Then

$$z[n] = \sum_{k=\langle N_F \rangle} X[k] Y[k] e^{j2\pi n/N_F}$$

$$z[n] = \sum_{k=\langle N_F \rangle} 1/N_F \sum_{p=\langle N_F \rangle} X[p] e^{-j2\pi kp/N_F} Y[k] e^{j2\pi n/N_F}$$

$$z[n] = \frac{1}{N_F} \sum_{p=\langle N_F \rangle} X[p] \sum_{k=\langle N_F \rangle} Y[k] e^{j2\pi (n-p)/N_F}$$

$$z[n] = \frac{1}{N_F} \sum_{p=\langle N_F \rangle} X[p] y[n-p]$$

or

$$N_F = mN_{x_0} = qN_{y_0}$$

$$x[n] \circledast y[n] \xrightarrow{FS} N_F Y[k] X[k].$$

Multiplication in either domain corresponds to a periodic convolution sum in the other domain (except for a scale factor of $N_F$ in the case of discrete-time periodic convolution).

First Backward Difference

Let $z[n] = x[n] - x[n-1]$ and let $N_F = mN_{x_0} = mN_{x_0}$, where $m$ is an integer. Then using the time-shifting property,

$$x[n-1] \xrightarrow{FS} X[k] e^{-j2\pi k/N_0}$$

and invoking the linearity property,

$$x[n] + x[n-1] \xrightarrow{FS} X[k] + X[k] e^{-j2\pi k/N_0}$$

or

$$N_F = mN_0$$

$$x[n] - x[n-1] \xrightarrow{FS} (1 - e^{-j2\pi k/N_0}) X[k].$$
Accumulation

Let \( z[n] = \sum_{m=-\infty}^{n} x[m] \). It is important for this property to consider the effect of the average value of \( x[n] \). We can write the signal \( x[n] \) as

\[
x[n] = x_0[n] + X[0]
\]

where \( x_0[n] \) is a signal with an average value of zero and \( X[0] \) is the average value of \( x[n] \). Then

\[
z[n] = \sum_{m=-\infty}^{n} x_0[m] + \sum_{m=-\infty}^{n} X[0].
\]

Since \( X[0] \) is a constant, \( \sum_{m=-\infty}^{n} X[0] \) increases or decreases linearly with \( n \), unless \( X[0] = 0 \). Therefore, if \( X[0] \neq 0 \), \( z[n] \) is not periodic and we cannot find its DTFS. If the average value of \( x[n] \) is zero, \( z[n] \) is periodic and we can find a DTFS for it. Since accumulation is the inverse of the first backward difference,

\[
\text{if } z[n] = \sum_{m=-\infty}^{n} x[m] \text{ then } x[n] = z[n] - z[n-1].
\]

But remember, multiple signals can have the same backward difference. For example we just showed that \( x[n] = z[n] - z[n-1] \) where \( z[n] = \sum_{m=-\infty}^{n} x[m] \). But if we redefine \( z[n] \) as \( C + \sum_{m=-\infty}^{n} x[m] \) where \( C \) is any constant we can still say that \( x[n] = z[n] - z[n-1] \). So, in finding the DTFS of the accumulation of a signal we can find it exactly except for the effect of the constant. The constant only affects \( Z[0] \). So we can relate the harmonic functions of \( x[n] \) and \( z[n] \) except for the \( k = 0 \) values. The first backward difference property proved that \( X[k] = (1 - e^{-j2\pi k/N_T}) Z[k] \). If follows that

\[
Z[k] = \frac{X[k]}{1 - e^{-j2\pi k/N_T}}, \quad k \neq 0, \text{ if } X[0] = 0
\]

and
Parseval's Theorem

The total signal energy of a periodic signal $x[n]$ is infinite (unless it is the trivial signal $x[n] = 0$). The signal energy over one period $N_F = mN_0$ is defined as

$$E_{x,N_F} = \sum_{n=0}^{N_F} \left| x[n] \right|^2 = \sum_{k=0}^{N_F} X[k] e^{j2\pi kn/N_F} \left| X[k] e^{j2\pi kn/N_F} \right|^2$$

where

$$E_{x,N_F} = \sum_{n=0}^{N_0} \left( \sum_{k=0}^{N_F} X[k] e^{j2\pi kn/N_F} \sum_{q=0}^{N_F} X[q] e^{-j2\pi qn/N_F} \right)^*$$

$$E_{x,N_F} = \sum_{n=0}^{N_F} \left( \sum_{k=0}^{N_F} \left| X[k] \right|^2 + \sum_{k=0}^{N_F} \sum_{q=0}^{N_F} X[k] e^{j2\pi kn/N_F} X^*[q] e^{-j2\pi qn/N_F} \right)$$

$$E_{x,N_F} = N_F \sum_{k=0}^{N_F} \left| X[k] \right|^2$$

Then

$$N_F = mN_0$$

$$\frac{1}{N_F} \sum_{n=0}^{N_F} \left| x[n] \right|^2 = \sum_{k=0}^{N_F} \left| X[k] \right|^2$$

which, in words, says that the average signal power of the signal is equal to the sum of the average signal powers in its DTFS harmonics.