A Complete Solution to the Harmonic Elimination Problem

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Abstract— **The problem of eliminating harmonics in a switching converter is considered. That is, given a desired fundamental output voltage, the problem is to find the switching times (angles) that produce the fundamental while not generating specifically chosen harmonics. In contrast to the well known work of Patel and Hoft [1][2] and others, here *all* possible solutions to the problem are found. This is done by first converting the *transcendental* equations that specify the harmonic elimination problem into an equivalent set of *polynomial* equations. Then, using the mathematical theory of resultants, all solutions to this equivalent problem can be found. In particular, it is shown that there are new solutions that have not been previously reported in the literature. The complete solutions for both unipolar and bipolar switching patterns to eliminate the 5^{th} and 7^{th} harmonics are given. Finally, the unipolar case is again considered where the $5^{th}, 7^{th}, 11^{th}$, and 13^{th} harmonics are eliminated along with corroborative experimental results.**

Keywords— **Harmonic Elimination, Unipolar, Bipolar, Programmed PWM, Resultants.**

I. Introduction

The problem of eliminating harmonics in switching converters has been the focus of research for many years. If the switching losses in an inverter are not a concern (i.e., switching on the order of a few kHz is acceptable), then the sine-triangle PWM method and its variants are very effective for controlling the inverter [3]. This is because the generated harmonics are beyond the bandwidth of the system being actuated and therefore these harmonics do not dissipate power. On the other hand, for systems where high switching efficiency is of utmost importance, it is desirable to keep the switching frequency much lower. In this case, another approach is to choose the switching times (angles) such that a desired fundamental output is generated and specifically chosen harmonics of the fundamental are suppressed [1][2]. This is referred to as *harmonic elimination* [1][2] or *programmed harmonic elimination* [3] as the switching angles are chosen (programmed) to eliminate specific harmonics.

In this work, it is shown how the complete solution (i.e., *all* possible solutions) to the problem considered in [1][2] is obtained. Specifically, in [1][2] the authors formulated the harmonic elimination problem as a set of transcendental equations that must be solved to determine the times (angles) in an electrical cycle for turning the switches on and off in a full bridge inverter so as to produce a desired fundamental amplitude while eliminating, for example, the 5^{th} and 7^{th} harmonics. The authors in [1][2] then solved the transcendental equations using *iterative numerical techniques* to compute the switching angles (See Figure 8-34 of [3] for a plot of these angles as a percent of the fundamental or Figure 2 below.). Here a method is presented that not only obtains these solutions, but also another (different) set of the switching angles, and this other set of switching angles actually generates a smaller harmonic distortion due to the 11^{th} and 13^{th} harmonics. The unipolar PWM scheme is also considered.

The paper is organized as follows: In section II, the solution method is illustrated for the bipolar case with the problem formulated as achieving the fundamental while not generating the 5^{th} and 7^{th} harmonics. In section III, it is then shown how the method can be used in the case of a unipolar PWM switching scheme, again formulating the problem so as to achieve the fundamental while not generating the 5^{th} and 7^{th} harmonics. Section IV then formulates and solves the unipolar case using five switching angles in which the fundamental is achieved and the $5^{th}, 7^{th}, 11^{th}$ and 13^{th} are not generated. Experimental results are presented in Section V, and a summary of the results in presented in Section VI.

II. Bipolar Case

In this work, a standard H-bridge is used wherein choosing the switching angles $\theta_1, \theta_2, \theta_3$ for the bipolar case results in an output waveform of the form shown in Figure 1. (In this figure, the angle θ_1 corresponds to the time $\theta_1(T/2\pi)$, etc and 2π corresponds to the fundamental period T.) The Fourier series expansion of this output voltage waveform is

$$V(\omega t) = \frac{4V_{dc}}{\pi}\left\{\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin(n\omega t)}{n}\times \right. \tag{1}$$
$$\left.\Big(1 - 2\cos(n\theta_1) + 2\cos(n\theta_2) - 2\cos(n\theta_3)\Big)\right\}.$$

Given a desired fundamental voltage V_1, the problem here

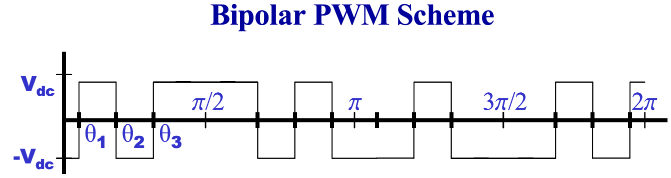

Fig. 1. Bipolar Switching Scheme

is to determine the switching angles $\theta_1, \theta_2, \theta_3$ so that

$$\begin{aligned} 1-2\cos(\theta_1)+2\cos(\theta_2)-2\cos(\theta_3) &= -m \\ 1-2\cos(5\theta_1)+2\cos(5\theta_2)-2\cos(5\theta_3) &= 0 \\ 1-2\cos(7\theta_1)+2\cos(7\theta_2)-2\cos(7\theta_3) &= 0 \end{aligned} \quad (2)$$

where $m \triangleq V_1/\left(\frac{4V_{dc}}{\pi}\right)$. This is a system of 3 transcendental equations in the unknowns $\theta_1, \theta_2, \theta_3$. One approach to solving this set of nonlinear transcendental equations (2) is to use an iterative technique such as the Newton-Raphson method [1][2]. Such a method results in the solution in Figure 8-34 of [3] (or Figure 2). Here, a methodology for finding *all* the solutions to (2) is presented, and our method not only gives the solutions reported in [3][1][2], but also a new set of solutions which are found to generate a lower harmonic distortion due to the 11^{th} and 13^{th} harmonics (see Figure 3).

To use the method, the conditions (2) are first converted to an equivalent polynomial system. Specifically, one defines $x_1 = \cos(\theta_1), x_2 = \cos(\theta_2), x_3 = \cos(\theta_3)$ and uses the trigonometric identities

$$\begin{aligned} \cos(5\theta) &= 5\cos(\theta) - 20\cos^3(\theta) + 16\cos^5(\theta) \\ \cos(7\theta) &= -7\cos(\theta) + 56\cos^3(\theta) - 112\cos^5(\theta) + 64\cos^7(\theta) \end{aligned}$$

to transform the conditions (2) into the equivalent conditions

$$\begin{aligned} p_1(x) &\triangleq 1+m-2x_1+2x_2-2x_3 = 0 \\ p_5(x) &\triangleq 1+2\sum_{i=1}^{3}(-1)^i\left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \quad (3) \\ p_7(x) &\triangleq 1+2\sum_{i=1}^{3}(-1)^i\left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \end{aligned}$$

where $x = (x_1, x_2, x_3)$ and $m \triangleq V_1/(4V_{dc}/\pi)$. Equation (3) is a set of three *polynomial* equations in the three unknowns x_1, x_2, x_3. Further, the solutions must satisfy $0 \le x_3 < x_2 < x_1 \le 1$.

A. Elimination Using Resultants

In order to explain how one computes the zero sets of polynomial systems, a brief discussion of the procedure of solving such systems is now given. A systematic procedure to do this is known as *elimination theory* and uses the notion of *resultants* [4][5][6][7]. Briefly, one considers $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. Then, for example, letting $a(x_1, x_2)$ and $b(x_1, x_2)$ have degrees 3 and 2, respectively in x_2, they may be written in the form

$$\begin{aligned} a(x_1,x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1,x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1). \end{aligned}$$

The $n \times n$ *Sylvester* matrix, where $n = \deg_{x_2}\{a(x_1,x_2)\} + \deg_{x_2}\{b(x_1,x_2)\} = 3+2 = 5$, is defined by

$$S_{a,b}(x_1) = \begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & 0 & b_2(x_1) \end{bmatrix}.$$

The *resultant* polynomial is then defined by

$$r(x_1) = \text{Res}\Big(a(x_1,x_2), b(x_1,x_2), x_2\Big) \triangleq \det S_{a,b}(x_1) \quad (4)$$

and is the result of solving $a(x_1,x_2) = 0$ and $b(x_1,x_2) = 0$ simultaneously for x_1, i.e., eliminating x_2. See the Appendix for a brief explanation of this fact.

B. Solving the Bipolar Equations

Following the procedure just outlined (see also [8]), the resultant methodology is used to solve for *all* possible switching angles. That is, $x_3 = m - (x_1 + x_2)$ is used to eliminate x_3 from p_5 and p_7 in (3) to get, the two polynomials equations $p_5(x_1,x_2) = 0, p_7(x_1,x_2) = 0$ in two unknowns which must be solved simultaneously. This is reduced to one polynomial in one unknown by computing the resultant polynomial $r_{p_5,p_7}(x_1)$ of the polynomial pair $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ (See [5][6] for background on resultants) to get

$$r_{p_5,p_7}(x_1) = 16777216m^2(1+m-2x_1)^4 r_{bi}^2(x_1)$$

where $r_{bi}(x_1)$ is a polynomial of 9^{th} degree (see the Appendix). As the parameter m is incremented in steps of 0.01, the roots of $r_{bi}(x_1)$ are found and used to back solve for x_2 and x_1. The set of all three tuples $(x_{3\ell}, x_{2\ell}, x_{1\ell})$ which satisfy $0 \le x_{3\ell} < x_{2\ell} < x_{1\ell} \le 1$ then give

$$\Big\{(\theta_{1l}, \theta_{2l}, \theta_{3l})\Big\} = \Big\{\left(\cos^{-1}(x_{1l}), \cos^{-1}(x_{2l}), \cos^{-1}(x_{3l})\right)\Big\}$$

as the set of all possible solutions to (2) for the particular value of m. This computation was done as m was incremented between 0 and 1 resulting in the switching angles versus m as given in Figure 2. As the figure shows, only at high values of m (> 0.91) do the two sets of solutions merge into one. To compare the two sets of solutions, the normalized magnitude of their 11^{th} and 13^{th} harmonics (i.e., $\sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}$ where a_k is the k^{th} harmonic) is

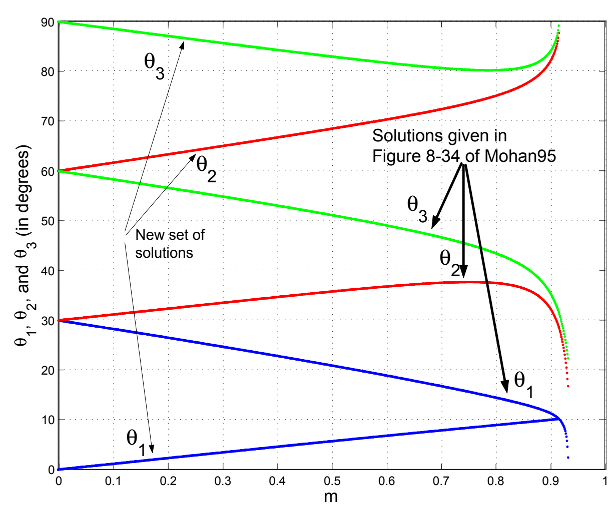

Fig. 2. Bipolar switching angles vs. m.

plotted in Figure 3. As this figure shows, the new set of solutions generates less harmonic distortion due to the 11^{th} and 13^{th} harmonics.

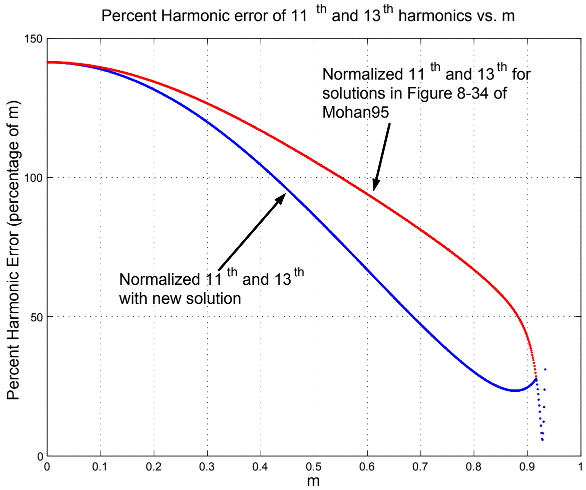

Fig. 3. Normalized error $\sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}$ for Bipolar PWM due to the 11^{th} and 13^{th} harmonics.

III. Unipolar Case

The Fourier expansion of the unipolar waveform as given in Figure 4 is

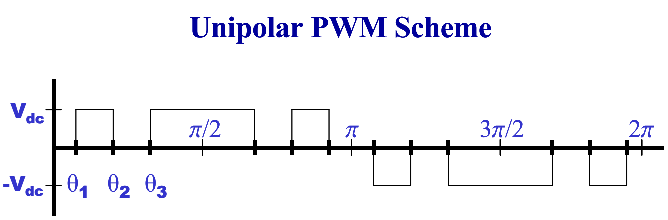

Fig. 4. Unipolar Switching Scheme

$$V(\omega t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{dc}}{n\pi} \Big(\cos(n\theta_1) - \cos(n\theta_2) + \cos(n\theta_3) \Big) \sin(n\omega t). \quad (5)$$

The problem is to determine the switching angles $\theta_1, \theta_2, \theta_3$ such that $(m \triangleq V_1/(4V_{dc}/\pi))$

$$\begin{aligned} \cos(\theta_1) - \cos(\theta_2) + \cos(\theta_3) &= m \\ \cos(5\theta_1) - \cos(5\theta_2) + \cos(5\theta_3) &= 0 \\ \cos(7\theta_1) - \cos(7\theta_2) + \cos(7\theta_3) &= 0. \end{aligned} \quad (6)$$

Converting (6) to polynomial equations

$$\begin{aligned} p_1(x) &\triangleq x_1 - x_2 + x_3 - m = 0 \\ p_5(x) &\triangleq \sum_{i=1}^{3} (-1)^{i-1} \left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ p_7(x) &\triangleq \sum_{i=1}^{3} (-1)^{i-1} \left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \end{aligned} \quad (7)$$

as in the bipolar example, the resultant methodology as presented in [8] was again used to solve for *all* possible switching angles. That is, $x_3 = m - (x_1 + x_2)$ is used to eliminate x_3 from p_5 and p_7 in (7) to get the pair of polynomial equations $p_5(x_1, x_2) = 0, p_7(x_1, x_2) = 0$ that must be solved simultaneously. As in the bipolar case, this is done by computing resultant polynomial $r_{p_5,p_7}(x_1)$ of the pair $\{p_5(x_1, x_2), p_7(x_1, x_2)\}$ to get

$$r_{p_5,p_7}(x_1) = 16777216m^4(m - x_1)^4 r_{uni}^2(x_1)$$

where $r_{uni}(x_1)$ is a polynomial of 9^{th} degree (see the Appendix).

As the parameter m is incremented in steps of 0.01, the roots of $r_{uni}(x_1)$ are found and used to back solve for x_2 and x_1. The set of all three tuples $(x_{3\ell}, x_{2\ell}, x_{1\ell})$ which satisfy $0 \le x_{3\ell} < x_{2\ell} < x_{1\ell} \le 1$ then give

$$\Big\{(\theta_{1l}, \theta_{2l}, \theta_{3l})\Big\} = \Big\{\big(\cos^{-1}(x_{1l}), \cos^{-1}(x_{2l}), \cos^{-1}(x_{3l})\big)\Big\}$$

as the set of all possible solutions to (6) for the particular value of m. The parameter m is then varied between 0 and 1 and these switching angles are plotted versus m in Figure 5. Figure 6 is a plot of magnitude of the distortion $\left(\text{i.e., } \sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}\right)$ due to the 11^{th} and 13^{th} harmonics. As seen in the figure, there are two sets of solutions for $m \in [0.5, 9.1]$ and that the two sets of solutions produce approximately the same distortion.

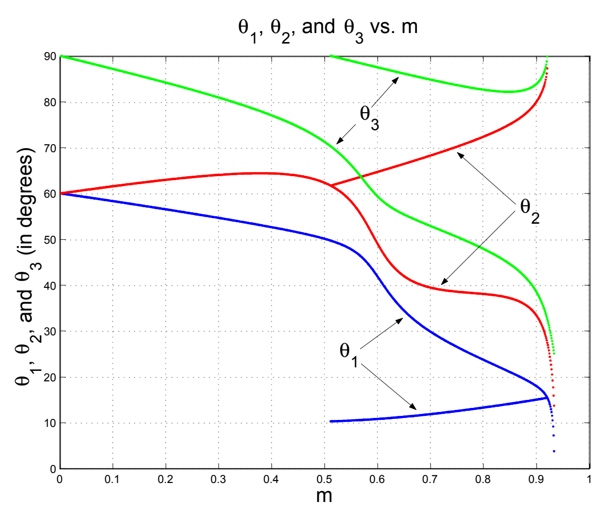

Fig. 5. Unipolar switching angles vs. m.

IV. Unipolar PWM with 5 Switching Angles

In the bipolar scheme, the RMS voltage $\frac{1}{2\pi}\int_0^{2\pi} \sqrt{V^2(\omega t)^2} d\omega = V_{dc}$ is constant because $V(\omega t) = \pm V_{dc}$ and therefore the THD is constant and is only being shifted in the frequency spectrum. However, the unipolar PWM scheme can also produce zero voltage and therefore inherently has lower harmonic content than the bipolar scheme. Consequently, this scheme is now considered for the case where five switching angles are used. The Fourier expansion of the unipolar waveform is still given by equation (5), and the problem

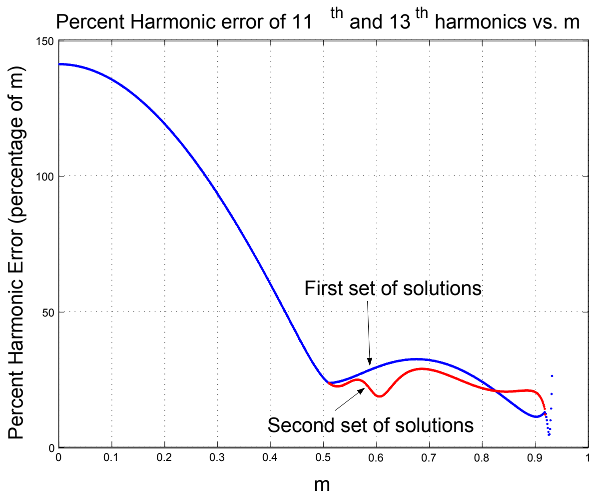

Fig. 6. Normalized error $\sqrt{(a_{11}/a_1)^2+(a_{13}/a_1)^2}$ for Unipolar PWM due to the 11^{th} and 13^{th} harmonics.

is to determine the switching angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ such that

$$\begin{aligned}
\cos(\theta_1)-\cos(\theta_2)+\cos(\theta_3)-\cos(\theta_4)+\cos(\theta_5) &= m\\
\cos(5\theta_1)-\cos(5\theta_2)+\cos(5\theta_3)-\cos(5\theta_4)+\cos(5\theta_5) &= 0\\
\cos(7\theta_1)-\cos(7\theta_2)+\cos(7\theta_3)-\cos(7\theta_4)+\cos(7\theta_5) &= 0\\
\cos(11\theta_1)-\cos(11\theta_2)+\cos(11\theta_3)-\cos(11\theta_4)+\cos(11\theta_5) &= 0\\
\cos(13\theta_1)-\cos(13\theta_2)+\cos(13\theta_3)-\cos(13\theta_4)+\cos(13\theta_5) &= 0.
\end{aligned} \quad (8)$$

where $m \triangleq V_1/(4V_{dc}/\pi)$ is the modulation index and the angles must satisfy $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4 \le \theta_5$ (see Figure 9 for a typical waveform). Let $\theta_i' = \theta_i$ if the coefficient of $\cos(n\theta_i)$ is $+1$ and $\theta_i' = \pi - \theta_i$ if it is -1 ($\cos(n\theta_i') = -\cos(n\theta_i)$ for n odd) and letting $x_1 = \cos(\theta_1'), x_2 = \cos(\theta_2'), x_3 = \cos(\theta_3'), x_4 = \cos(\theta_4'), x_5 = \cos(\theta_5')$ the conditions become

$$\begin{aligned}
p_1(x) &\triangleq x_1+x_2+x_3+x_4+x_5-m=0\\
p_5(x) &\triangleq \sum_{i=1}^{5}\left(5x_i-20x_i^3+16x_i^5\right)=0\\
p_7(x) &\triangleq \sum_{i=1}^{5}\left(-7x_i+56x_i^3-112x_i^5+64x_i^7\right)=0\\
p_{11}(x) &\triangleq \sum_{i=1}^{5}\Big(-11x_i+220x_i^3-1232x_i^5+2816x_i^7\\
&\quad -2816x_i^9+1024x_i^{11}\Big)=0 \quad (9)\\
p_{13}(x) &\triangleq \sum_{i=1}^{5}\Big(13x_i-364x_i^3+2912x_i^5-9984x_i^7\\
&\quad +16640x_i^9-13312x_i^{11}+4096x_i^{13}\Big)=0.
\end{aligned}$$

where $0 \le x_5 \le -x_4 \le x_3 \le -x_2 \le x_1 \le 1$.

Remark It is interesting to note that the set of polynomials in (9) are the same equations as that of a multilevel inverter with five dc sources [8]. The difference between the two solutions is in the region where the x_i must lie. In the multilevel case, the conditions are $0 \le x_5 \le x_4 \le x_3 \le x_2 \le x_1 \le 1$.

Following a procedure similar to that given in Sections II and III, one systematically solves these equations by elimination theory. This was done, and the *complete* set of switching angle solutions are plotted versus m in Figure 7. Note that for $0 \le m \le 0.48$ there are two sets of solutions, for $0.40 \le m \le 0.53$ there is only one solution set, for $0.53 \le m \le 0.78$ there are three sets of solutions, and finally, for $0.78 \le m \le 0.91$, there are again two sets of solutions.

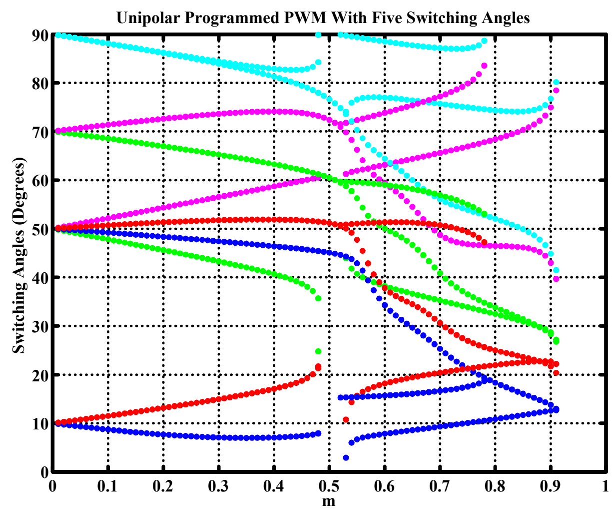

Fig. 7. Unipolar switching angles vs. m with 5 switching angles.

The corresponding total harmonic distortion (THD) was computed out to the 31^{st} according to

$$THD = \sqrt{\frac{V_5^2+V_7^2+V_{11}^2+V_{13}^2+V_{17}^2+\cdots+V_{31}^2}{V_1^2}}$$

and is plotted versus m in Figure 8 for each of the solution sets shown in Figure 7. As this figure shows, one can chose a particular solution for the switching angles such that the THD is 32% or less for $0.55 \le m \le 0.9$. It is important to point out that if one had used an iterative method such as Newton-Raphson, then the third solution set that exists for $0.53 \le m \le 0.78$ would not have been found, and this is the solution set that results in the lowest THD for this range of modulation indices. The reason the Newton-Raphson method would not have found this solution set is simply due to the way it is implemented. One starts with an initial guess for the angles at $m = 0$. Then this solution is used as the initial guess for the solution when m is incremented by Δm to its next value and so on. At $m = 0$, the only possible solutions are $\theta_1 = 50°, \theta_2 = 50°, \theta_3 = 70°, \theta_4 = 70°, \theta_5 = 90°$ or $\theta_1 = 10°, \theta_2 = 10°, \theta_3 = 50°, \theta_4 = 70°, \theta_5 = 90°$. As Figure 7 shows, if the first solution set is used as the starting point in the Newton-Raphson scheme for $m = 0$, then as m is incremented, one would obtain a set of solutions valid for $0 \le m \le 0.91$. If the second set of solutions is

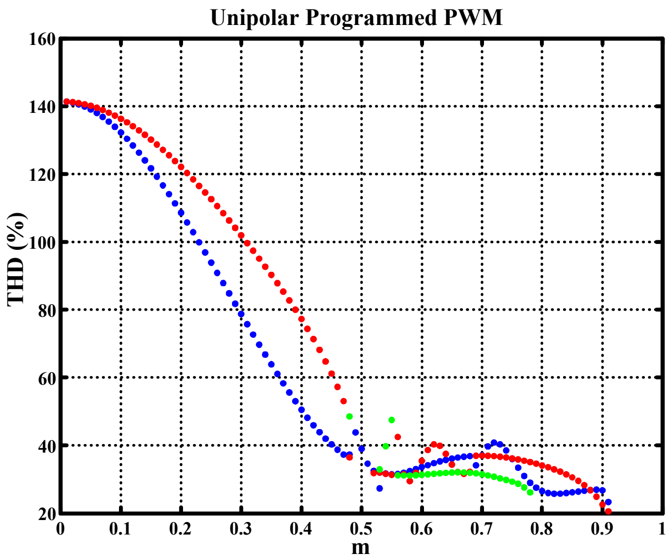

Fig. 8. THD vs. m for each set of switching angles.

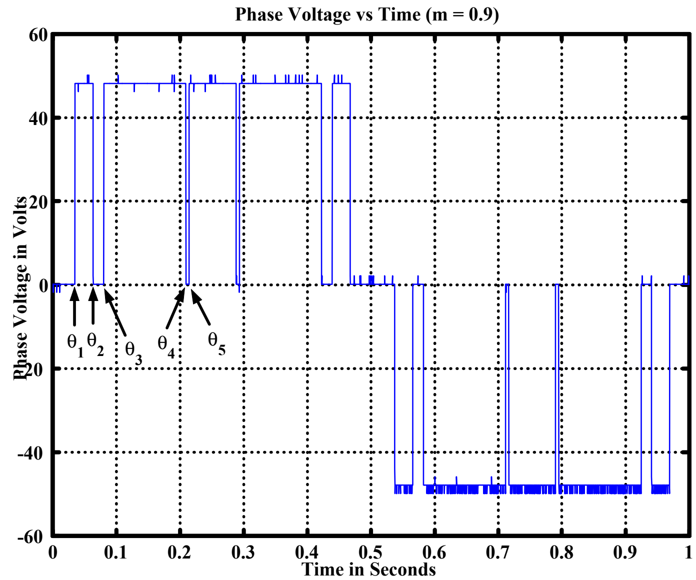

Fig. 9. Output waveform from a unipolar PWM scheme with $m = 0.9$.

used as the starting point, then a set of solutions valid for $0 \leq m \leq 0.48$ would be obtained. Neither of these sets results in the minimum THD for $0.53 \leq m \leq 0.78$. *Consequently, the method proposed here that finds the complete solution set allows one to be sure that the solution with the lowest THD is used.*

V. Experimental Results

A prototype inverter has been built using 100 V, 70 A MOSFETs as the switching devices [8][9]. In the experimental study here, this prototype system was configured to have a nominal DC link of 48 Volts. This inverter was then used to perform experiments to validate the predicted results, that is, the elimination of the $5^{th}, 7^{th}, 11^{th}$, and 13^{th} harmonics in the output of a three phase inverter. In this work, a real-time computing platform [10] was used to interface the computer (which determines the switching times as logic signals) to the gate driver board of the inverter. The switching algorithm is implemented as a lookup table in Simulink which is then converted to C code. The software provides icons to interface the Simulink model to the digital I/O board and converts the C code into executables. The time resolution (the precision for the time at which a switch is turned on or off) was chosen to be no larger than 1/1000 of an electrical cycle. Experiments were performed for several values of the parameter m, and as seen in the following figures, each was consistent with predicted results given in Figures 7 and 8.

Figure 9 is the output waveform from the inverter using the unipolar PWM scheme with a modulation index of $m = 0.9$ while Figure 10 is its corresponding Fast Fourier Transform (FFT). As predicted, the $5^{th}, 7^{th}, 11^{th}$ and 13^{th} harmonics do not appear in the waveform.

Figure 11 is the output waveform from the inverter using the unipolar PWM scheme with a modulation index of $m = 0.7$ with Figure 12 being its corresponding FFT. Again,

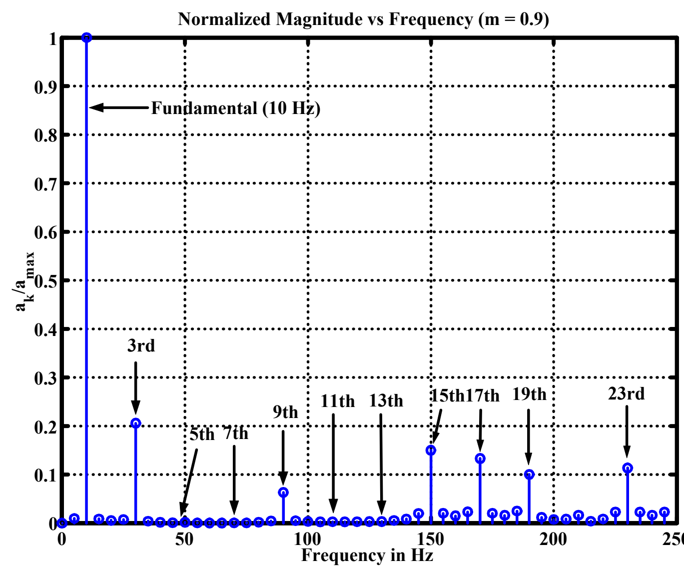

Fig. 10. Normalized FFT of the waveform in Figure 9.

as predicted, the $5^{th}, 7^{th}, 11^{th}$ and 13^{th} harmonics do not appear in the waveform.

Figure 13 is the output waveform from the inverter using the unipolar PWM scheme with a modulation index of $m = 0.55$ and Figure 14 is its corresponding FFT. Again, as predicted, the $5^{th}, 7^{th}, 11^{th}$ and 13^{th} harmonics do not appear in the waveform.

Figure 15 is the output waveform from the inverter using the unipolar PWM scheme with a modulation index of $m = 0.35$ and Figure 16 is its corresponding FFT. In this case, the $5^{th}, 7^{th}$ harmonics are zero, but there is a few percent harmonic content in the 11^{th} and 13^{th}. However, at this modulation index, the THD due to the remaining harmonics is over 60% as shown in Figure 8.

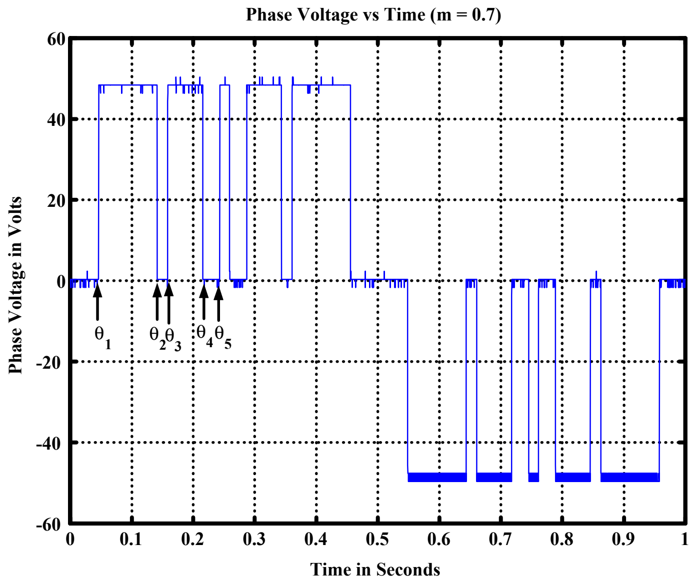

Fig. 11. Output waveform from a unipolar PWM scheme with $m = 0.7$.

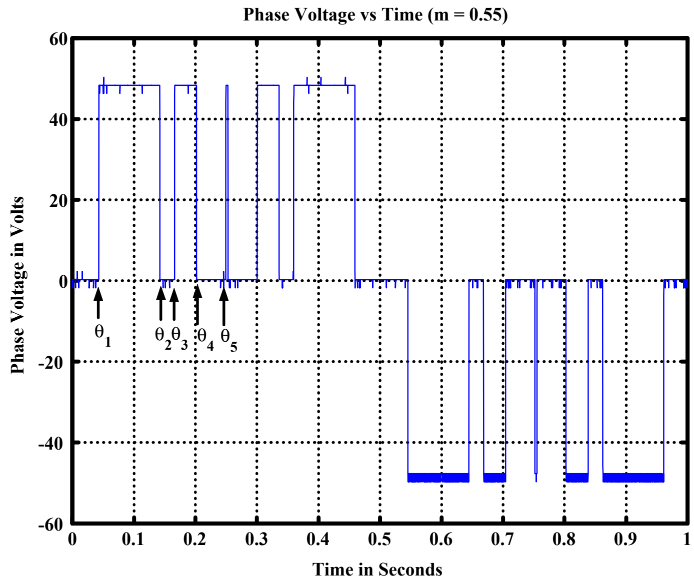

Fig. 13. Output waveform from a unipolar PWM scheme with $m = 0.55$.

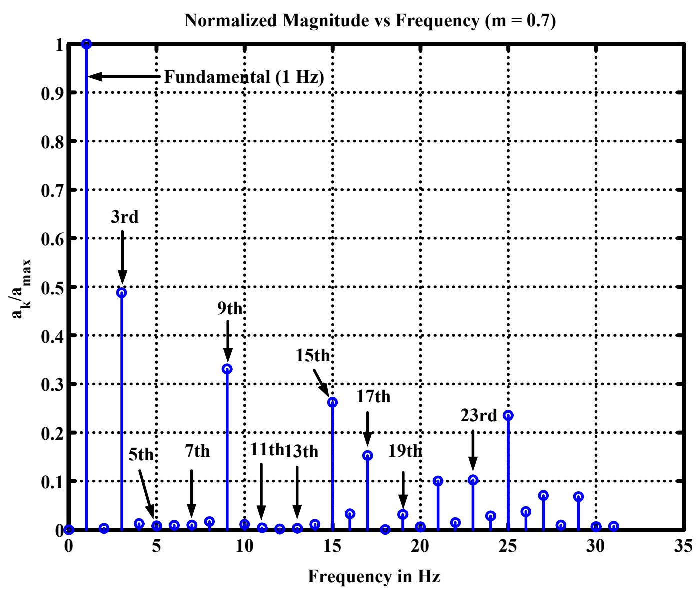

Fig. 12. Normalized FFT of the waveform in Figure 11.

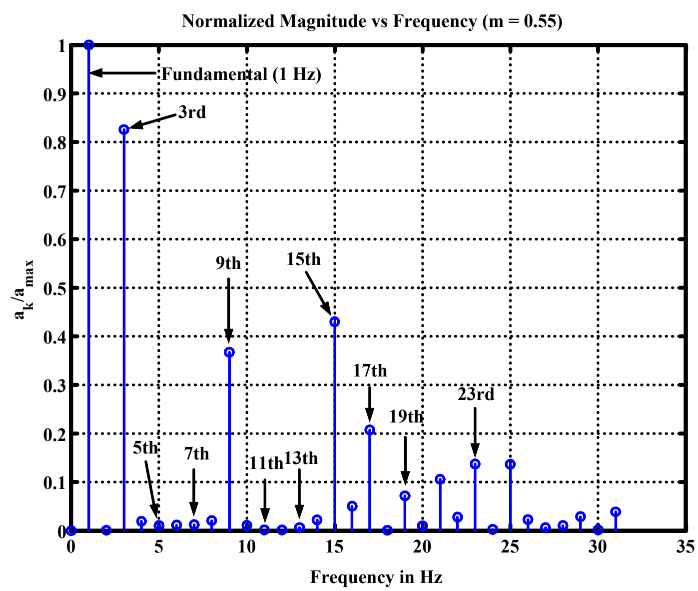

Fig. 14. Normalized FFT of the waveform in Figure 13.

VI. Conclusions

It has been shown how the complete solution to the harmonic elimination problem can be found using the theory of resultants from elimination theory. The solution is complete in the sense that any and all solutions were found. Experimental work was presented to corroborate the developed theory.

VII. Acknowledgements

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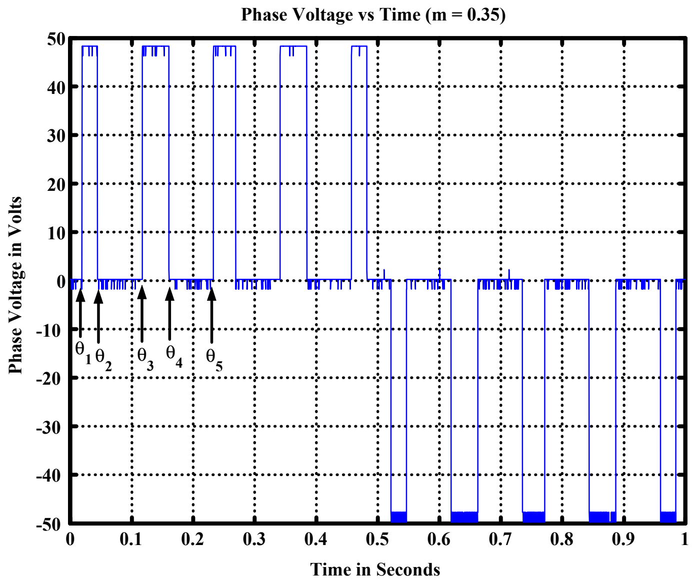

Fig. 15. Output waveform from a unipolar PWM scheme with $m = 0.35$.

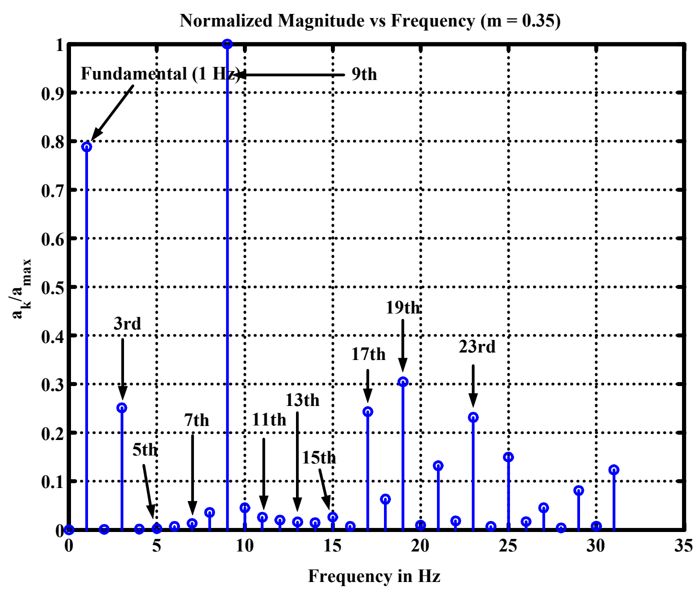

Fig. 16. Normalized FFT of the waveform in Figure 15.

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Appendix

I. Resultants

Given two polynomials $a(x_1, x_2)$ and $b(x_1, x_2)$ how does one find their common zeros? That is, the values (x_{10}, x_{20}) such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0.$$

Consider $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. There is always a polynomial $r(x_1)$ (called the *resultant polynomial*) such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1).$$

So if $a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0$ then $r(x_{10}) = 0$, that is, if (x_{10}, x_{20}) is a common zero of the pair $\{a(x_1, x_2), b(x_1, x_2)\}$, then the first coordinate x_{10} is a zero of $r(x_1) = 0$. To see how one obtains $r(x_1)$, let

$$\begin{aligned} a(x_1, x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1, x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1) \end{aligned}$$

Next, see if polynomials of the form

$$\begin{aligned} \alpha(x_1, x_2) &= \alpha_1(x_1)x_2 + \alpha_0(x_1) \\ \beta(x_1, x_2) &= \beta_2(x_1)x_2^2 + \beta_1(x_1)x_2 + \beta_0(x_1). \end{aligned}$$

can be found such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1). \qquad (10)$$

Equating powers of x_2, this equation may be rewritten in matrix form as

$$\begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & 0 & b_2(x_1) \end{bmatrix} \begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix on the left-hand side is called the *Sylvester* matrix and is denoted here by $S_{a,b}(x_1)$. The inverse of $S_{a,b}(x_1)$ has the form

$$S_{a,b}^{-1}(x_1) = \frac{1}{\det S_{a,b}(x_1)} \text{adj}\left(S_{a,b}(x_1)\right)$$

where $\text{adj}(S_{a,b}(x_1))$ is the adjoint matrix and is a 5×5 *polynomial* matrix in x_1. Solving for $\alpha_i(x_1), \beta_i(x_1)$ gives

$$\begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \frac{\text{adj} S_{a,b}(x_1)}{\det S_{a,b}(x_1)} \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Choosing $r(x_1) = \det S_{a,b}(x_1)$ guarantees that $\alpha_0(x_1), \alpha_1(x_1), \beta_0(x_1), \beta_1(x_1), \beta_2(x_1)$ are polynomials in x_1. That is, the *resultant polynomial* defined by $r(x_1) = \det S_{a,b}(x_1)$ is the polynomial required for (10).