Elimination of Harmonics in a Multilevel Converter with Non Equal DC Sources

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Abstract— **The problem of eliminating harmonics in a multilevel converter in which the separate DC sources vary is considered. That is, given a desired fundamental output voltage, the problem is to find the switching times (angles) that produce the fundamental while not generating specifically chosen harmonics. Assuming that the separate DC sources can be measured, a procedure is given to find *all* sets of switching angles for which the fundamental is produced while the 5^{th} and 7^{th} are eliminated. This is done by first converting the *transcendental* equations that specify the elimination of the harmonics into an equivalent set of *polynomial* equations. Then, using the mathematical theory of resultants, all solutions to this equivalent problem can be found. Experimental results are presented to validate the theory.**

Keywords— **Multilever Inverter, Harmonic Cancellation, Non Equal DC Sources.**

I. Introduction

Designs for heavy duty hybrid-electric vehicles (HEVs) that have large electric drives such as tractor trailers, transfer trucks, or military vehicles will require advanced power electronic inverters to meet the high power demands (> 100 kW) required of them. Development of large electric drive trains for these vehicles will result in increased fuel efficiency, lower emissions, and likely better vehicle performance (acceleration and braking).

One promising technology to interface battery packs in electric and hybrid electric vehicles are multilevel converters. Transformerless multilevel inverters are particularly suited for this application because of the high VA ratings possible with these inverters [1]. The multilevel voltage source inverter's unique structure allows it to reach high voltages with low harmonics without the use of transformers or series-connected, synchronized-switching devices. The general function of the multilevel inverter is to synthesize a desired voltage from several levels of dc voltages. For this reason, multilevel inverters can easily provide the high power required of a large electric traction drive.

For parallel-configured HEVs, a cascaded H-bridges inverter can be used to drive the traction motor from a set of batteries, ultracapacitors, or fuel cells. The use of a cascade inverter also allows the HEV drive to continue to operate even with the failure of one level of the inverter structure [2][3][4].

A key issue in designing an effective multilevel inverter is to ensure that the voltage total harmonic distortion (THD) is small enough. To do so requires both an (mathematical) algorithm to determine when the switching should be done so as to not produce harmonics and a fast real-time computing system to implement the strategy. Work was reported in [5] that presented a method to compute the switching angles for the H-bridges using the mathematical theory of resultants. In that work, a complete solution was presented for the three dc source case where it was assumed that the dc sources were all equal. However, in applications, this probably will not be the case even if the sources are nominally equal.

Here it is shown how the method in [5] can be extended for the non equal or varying dc source case. Specficially, the problem of eliminating harmonics in a multilevel converter in which the separate dc sources do not have equal voltage levels is considered. That is, given a desired fundamental output voltage, the problem is to find the switching times (angles) that produce the fundamental while not generating specifically chosen harmonics.

Assuming that the separate dc sources can be measured, a procedure is given to find *all* sets of switching angles for which the fundamental is produced while the 5^{th} and 7^{th} are eliminated. This is done by first converting the *transcendental* equations that specify the elimination of the harmonics into an equivalent set of *polynomial* equations. Then, using the mathematical theory of resultants, all solutions to this equivalent problem can be found. Experimental results are presented to validate the theory.

II. Cascaded H-bridges

The cascade multilevel inverter consists of a series of H-bridge (single-phase full-bridge) inverter units. As previously mentioned, the general function of this multilevel inverter is to synthesize a desired voltage from several separate dc sources (SDCSs), which may be obtained from batteries, fuel cells, or ultracapacitors in a HEV. Figure 1 shows a single-phase structure of a cascade inverter with SDCSs [1]. Each SDCS is connected to a single-phase full-bridge inverter. Each inverter level can generate three different voltage outputs, $+V_{dc}, 0$ and $-V_{dc}$ by connecting the dc source to the ac output side by different combinations of the four switches, S_1, S_2, S_3 and S_4. The ac output of each level's full-bridge inverter is connected in series such that the synthesized voltage waveform is the sum of all of the individual inverter outputs. The number of output phase voltage levels in a cascade mulitilevel inverter is then $2s+1$, where s is the number of dc sources. An example phase voltage waveform for an 11-level cascaded multilevel inverter with five SDSCs ($s = 5$) and five full bridges is shown in Figure 2. The output phase voltage is given by

$v_{an} = v_{a1} + v_{a2} + v_{a3} + v_{a4} + v_{a5}$.

With enough levels and an *appropriate* switching algorithm, the multilevel inverter results in an output voltage that is almost sinusoidal. For the 11-level example shown in Figure 2, the waveform has less than 5% THD with each of the active devices of the H-bridges active devices switching only at the fundamental frequency. Each H-bridge unit generates a quasi-square waveform by phase-shifting its positive and negative phase legs' switching timings. Each switching device always conducts for 180° (or $\frac{1}{2}$ cycle) regardless of the pulse width of the quasi-square wave so that this switching method results in equalizing the current stress in each active device.

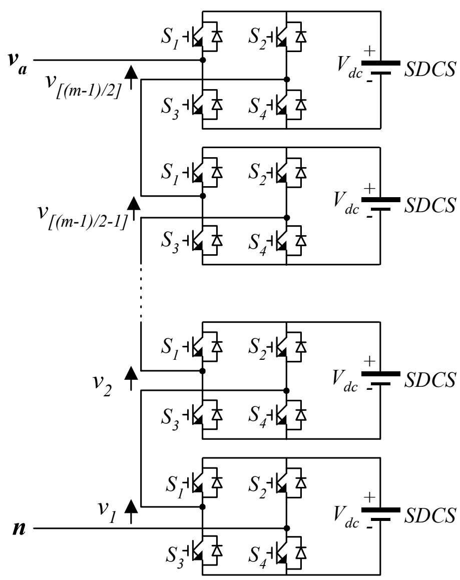


Fig. 1. Single-phase structure of a multilevel cascaded H-bridges inverter.

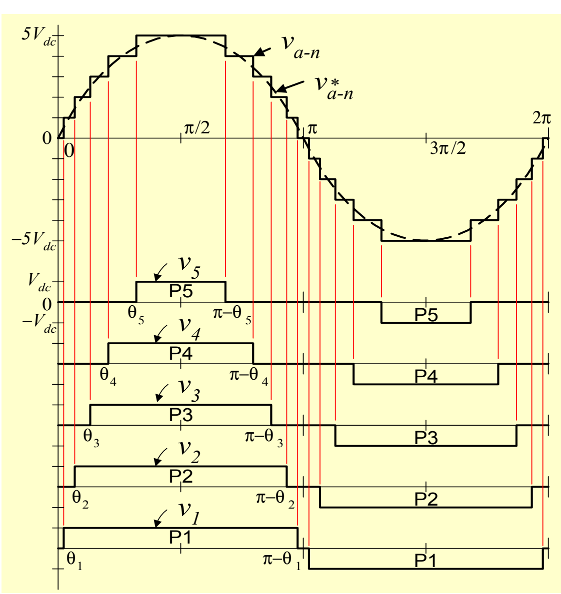


Fig. 2. Output waveform of an 11-level cascade multilevel inverter.

III. Switching Algorithm for the Multilevel Converter

The Fourier series expansion of the (stepped) output voltage waveform of the multilevel inverter as shown in Figure 2 is [3][2]

$$V(\omega t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4V_{dc}}{n\pi}\Big(V_1\cos(n\theta_1) + \cdots + V_s\cos(n\theta_s)\Big)\sin(n\omega t) \quad (1)$$

where s is the number of dc sources and the product V_iV_{dc} is the value of the i^{th} dc source (if all the dc sources have the same value V_{dc}, then $V_1 = V_2 = \cdots = V_s = 1$). The objective here is to choose the switching angles $0 \le \theta_1 < \theta_2 < \cdots < \theta_s \le \pi/2$ so as to make the first harmonic equal to the desired fundamental voltage V_f and specific higher harmonics of $V(\omega t)$ equal to zero. As the application of interest here is a three-phase power system, the triplen harmonics in each phase need not be canceled as they automatically cancel in the line-to-line voltages.

A three dc source case is now considered so that the switching angles are chosen so as to not generate the $5^{th}, 7^{th}$ order harmonics while achieving the desired fundamental voltage. The mathematical statement of these conditions is then

$$\begin{aligned} \frac{4V_{dc}}{\pi}\Big(V_1\cos(\theta_1) + V_2\cos(\theta_2) + V_3\cos(\theta_s)\Big) &= V_f \\ V_1\cos(5\theta_1) + V_2\cos(5\theta_2) + V_3\cos(5\theta_s) &= 0 \\ V_1\cos(7\theta_1) + V_2\cos(7\theta_2) + V_3\cos(7\theta_s) &= 0. \end{aligned} \quad (2)$$

This is a system of 3 transcendental equations in the unknowns $\theta_1, \theta_2, \theta_3$. One approach to solving this set of nonlinear transcendental equations (2) is to use an iterative method such as the Newton-Raphson method [6]. In this work, the method given in [5] is extended to find *all* solutions to (2). This methodology is based on the mathematical theory of resultants of polynomials which is a systematic procedure for finding the roots of systems of *polynomial* equations [7]. To use the method, equations (2) are first converted to a polynomial system by setting $x_1 = \cos(\theta_1), x_2 = \cos(\theta_2), x_3 = \cos(\theta_3)$ and using the trigonometric identities $\cos(5\theta) = 5\cos(\theta) - 20\cos^3(\theta) + 16\cos^5(\theta)$, $\cos(7\theta) = -7\cos(\theta) + 56\cos^3(\theta) - 112\cos^5(\theta) + 64\cos^7(\theta)$ to transform (2) into the equivalent conditions

$$\begin{aligned} p_1(x) &\triangleq V_1x_1 + V_2x_2 + V_3x_3 - m = 0 \\ p_5(x) &\triangleq \sum_{i=1}^{3} V_i\left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ p_7(x) &\triangleq \sum_{i=1}^{3} V_i\left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \end{aligned} \quad (3)$$

where $x = (x_1, x_2, x_3)$ and $m \triangleq V_f/(4V_{dc}/\pi)$. The modulation index is $m_a = m/s = V_1/(s4V_{dc}/\pi)$ (Each inverter has a dc source that is nominally equal to V_{dc} so that the maximum output voltage of the multilevel inverter is sV_{dc}. A square wave of amplitude sV_{dc} results in the maximum fundamental output possible of $V_{1\,\max} = 4sV_{dc}/\pi$ so $m_a \triangleq V_1/V_{1\,\max} = V_1/(s4V_{dc}/\pi) = m/s$)

This is now a set of three *polynomial* equations in the three unknowns x_1, x_2, x_3 (see also [8] where a polynomial system was used). Further, the solutions must satisfy $0 \leq x_3 < x_2 < x_1 \leq 1$.

Next, one substitutes $x_3 = \big(m - (V_1x_1 + V_2x_2)\big)/V_3$ into p_5, p_7 to get

$$p_5(x_1,x_2) = 5x_1 - 20x_1^3 + 16x_1^5 + 5x_2 - 20x_2^2 + 16x_2^5$$
$$+5\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right) - 20\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)^3$$
$$+16\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)^5$$

and

$$p_7(x_1,x_2) = -7x_1 + 56x_1^3 - 112x_1^5 + 64x_1^7 - 7x_2 + 56x_2^3$$
$$-112x_2^5 + 64x_2^7 - 7\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)$$
$$+56\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)^3 - 112\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)^5$$
$$+64\left(\frac{m-(V_1x_1+V_2x_2)}{V_3}\right)^7.$$

A. Elimination Using Resultants

In order to explain how one computes the zero sets of polynomial systems, a brief discussion of the procedure of solving such systems is now given. A systematic procedure to do this is known as *elimination theory* and uses the notion of *resultants* [9][10]. Briefly, one considers $a(x_1,x_2)$ and $b(x_1,x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. Then, for example, letting $a(x_1,x_2)$ and $b(x_1,x_2)$ have degrees 3 and 2, respectively in x_2, they may be written in the form

$$\begin{aligned} a(x_1,x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1,x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1). \end{aligned}$$

The $n \times n$ *Sylvester* matrix, where $n = \deg_{x_2}\{a(x_1,x_2)\} + \deg_{x_2}\{b(x_1,x_2)\} = 3 + 2 = 5$, is defined by

$$S_{a,b}(x_1) = \begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & b_3(x_1) & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & b_3(x_1) & b_2(x_1) \\ 0 & 0 & 0 & 0 & b_3(x_1) \end{bmatrix}.$$

The *resultant* polynomial is then defined by

$$r(x_1) = \mathrm{Res}\Big(a(x_1,x_2), b(x_1,x_2), x_2\Big) \triangleq \det S_{a,b}(x_1) \quad (4)$$

and is the result of solving $a(x_1,x_2) = 0$ and $b(x_1,x_2) = 0$ simultaneously for x_1, i.e., eliminating x_2. See [11][9][10][7] for an explanation of this fact.

B. The Switching Angle Solutions

The goal here is to find simultaneous solutions of $p_5(x_1,x_2) = 0, p_7(x_1,x_2) = 0$. For each fixed x_1, $p_5(x_1,x_2)$ can be viewed as a polynomial in x_2 whose coefficients are polynomials in x_1. For each fixed x_{10}, the pair of polynomials $p_5(x_1,x_2) = 0, p_7(x_1,x_2) = 0$ has a solution x_{20} if and only if their corresponding *resultant matrix* $S_{p_5,p_7}(x_{10})$ is singular. Here $\deg_{x_2}\{p_5(x_1,x_2)\} = 5$ and $\deg_{x_2}\{p_7(x_1,x_2)\} = 7$ so that the resultant matrix $S_{p_5,p_7}(x_1)$ is an element of $\Re^{12\times 12}[x_1]$ and its determinant $r_{5,7}(x_1) \triangleq \det S_{p_5,p_7}(x_1)$ is a *polynomial* in x_1.

The key point here is that for any (x_{10}, x_{20}) which is a simultaneous solution of $p_5(x_1,x_2) = 0, p_7(x_1,x_2) = 0$, it must be that $r_{5,7}(x_{10}) = 0$. Consequently, finding the roots of $r_{5,7}(x_1) = 0$ gives candidate values for x_1 to check for common zeros of $p_5(x_1,x_2) = 0, p_7(x_1,x_2) = 0$. The resultant polynomial $r_{5,7}(x_1)$ of the pair $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ was found with MATHEMATICA using the `Resultant` command and turned out to be a 35^{th} order polynomial. The explicit algorithm implemented to compute the switching angles is:

Algorithm for the 7 Level Case

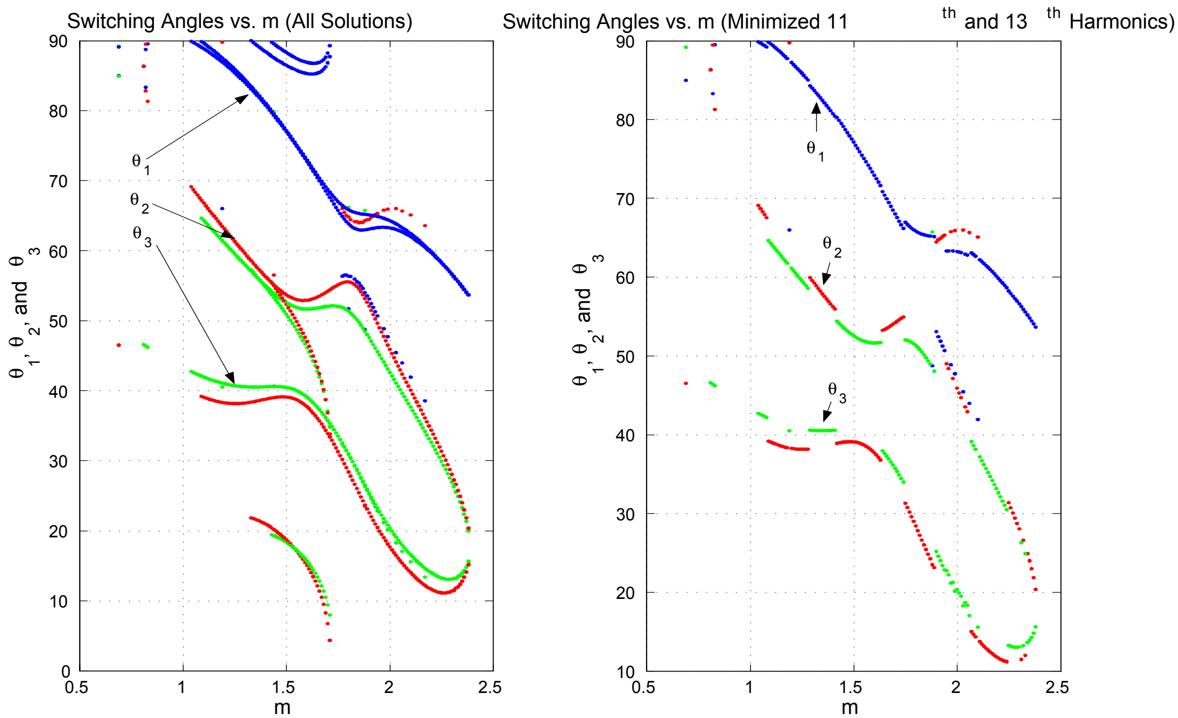
1. Given m and the measured values of V_1, V_2, V_3, find the roots of $r_{5,7}(x_1) = 0$.
2. Discard any roots that are less than zero, greater than 1 or that are complex. Denote the remaining roots as $\{x_{1i}\}$.
3. For each fixed zero x_{1i} in the set $\{x_{1i}\}$, substitute it into p_5 and solve for the roots of $p_5(x_{1i}, x_2) = 0$.
4. Discard any roots (in x_2) that are complex, less than zero or greater than one. Denote the pairs of remaining roots as $\{(x_{1j}, x_{2j})\}$.
5. Compute $m - x_{1j} - x_{2j}$ and discard any pair (x_{1j}, x_{2j}) that makes this quantity negative or greater than one. Denote the triples of remaining roots as $\{(x_{1k}, x_{2k}, x_{3k})\}$.
6. Discard any triple for which $x_{3k} < x_{2k} < x_{1k}$ does not hold. Denote the remaining triples as $\{(x_{1l}, x_{2l}, x_{3l})\}$.

The switching angles that are a solution to the three level system (2) are

$$\{(\theta_{1l}, \theta_{2l}, \theta_{3l})\} = \{\big(\cos^{-1}(x_{1l}), \cos^{-1}(x_{2l}), \cos^{-1}(x_{3l})\big)\}.$$

The results are plotted on the left side of Figure 3 for the case where dc source voltages are $V_1V_{dc} = 12.56$ Volts, $V_2V_{dc} = 10.19$ Volts and $V_3V_{dc} = 12.01$ Volts. The interest here is in a *symbolic* expression for the final resultant polynomial. That is, the final resultant polynomial is more precisely written as $r(x_1, m, V_{dc1}, V_{dc2}, V_{dc3})$ showing that not only is it a function of the indeterminate x_1, but also of the parameters $m, V_{dc1}, V_{dc2}, V_{dc3}$. This is the desired form because, for example, in a hybrid electric vehicle the batteries powering the vehicle will not usually be at the same voltage level and will vary with use. Consequently, the DC source voltages could be measured and the switching angles as in Figure 3 could be recomputed *online* to account for changes in the source voltages. This is because, starting with $r(x_1, m, V_{dc1}, V_{dc2}, V_{dc3})$, the calculation to compute the data of Figure 3 takes less than a second for any given modulation index.

If iterative numerical techniques are used, one is not guaranteed that the solution will converge (the initial guess has to be "close" to the solution or there may be no solution), nor that the particular solution obtained is the only



Left: All solution sets $\{\theta_1, \theta_2, \theta_3\}$ vs. m Right: The angles that generate the smallest 11^{th} and 13^{th} harmonics.

Fig. 3.

solution and, therefore, the best in any sense. This figure shows the switching angles $\theta_1, \theta_2, \theta_3$ vs. m for those values of m in which the system (2) has at least one solution set. The parameter m was incremented in steps of 0.01. Note that for m in the range from approximately 1.1 to 2.4, there are at least two different sets of solutions and sometimes three sets. The right side of Figure 3 is a plot of the set of switching angles chosen to give the smallest distortion generated by the 11^{th} and 13^{th} harmonics, i.e., the smallest value of $\sqrt{(p_{11}/11)^2 + (p_{13}/13)^2}$.

However, in an application such as electric vehicles, one needs to be concerned about another issue. Specifically, one would like the smallest switching angle $\theta_{\min}$ to correspond to the dc source (battery) with the highest voltage $V_{i_{\max}} V_{dc}$ so that the battery being drained the longest period of time is the one with the highest voltage. Of course, one would then want the next smallest switching angle to correspond to the next highest dc source voltage level, etc. If this is not the case, the voltages will continue to get further apart due to the fact that they are being discharged at different rates.

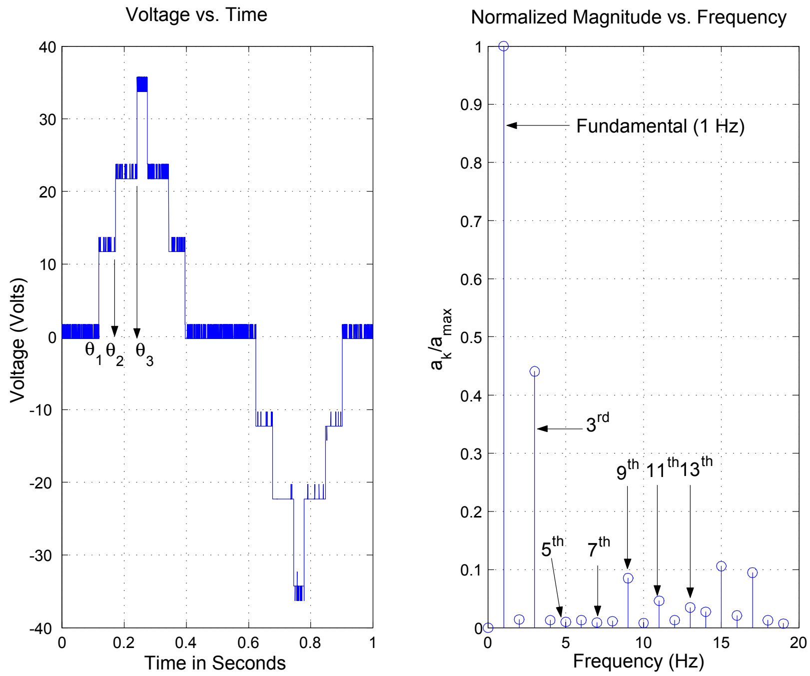
IV. Experimental Work

A prototype three-phase 11-level wye-connected cascaded inverter has been built using 100 V, 70 A MOSFETs as the switching devices. The gate driver boards and MOSFETs are shown in Figure 4. A battery bank of 15 SDCSs of 48 Volts dc (not shown) each feed the inverter configured with 5 SDCSs per phase [12]. In the experimental study here, this prototype system was configured to be an 7-level (3 SDCSs per phase).

The ribbon cable shown in the figure provides the communication link between the gate driver board and the real-time processor. In this work, a real-time computing platform [13] was used to interface the computer (which generates the logic signals) to this cable. This system allows one to implement the switching algorithm as a lookup table in Simulink which is then converted to C code using RTW (real-time workshop) from *Mathworks*. The software provides icons to interface the Simulink model to the digital I/O board and converts the C code into executables.

The time resolution (the precision for the time at which a switch is turned on or off) was chosen to be 1/1000 of an electrical cycle. For a 60 Hz frequency requirement, this comes to $(1/60)/1000 = 16.7$ microseconds. Note that while the computation of the lookup table of Figure 3 requires some offline computational effort, the real-time implementation is accomplished by putting the data (i.e., Figure 3) in a lookup table and therefore does not require high computational power for implementation.

In the first experiment, each dc source was nominally 12 Volts ($V_{dc} = 12$ V) while their actual values were $V_1 V_{dc} = 12.56$ Volts, $V_2 V_{dc} = 10.19$ Volts and $V_3 V_{dc} = 12.01$ Volts (i.e., $V_1 = 12.56/12 = 1.05, V_2 = 0.85, V_3 = 1.01$) corresponding to the values used in Figure 3. The left side of Figure 5 is a plot of the output waveform voltage with $m = 1.3$. (The spikes on the plot are due to the low bit resolution of the sampling scope and are not present on the actual scope display). The corresponding FFT of this signal is given on the right side of Figure 5. This shows the normalized magnitude of the 5^{th} and 7^{th} harmonics are both about 0.01 which corresponds well with the predicted



Left: Sampled output voltage with $m = 1.3$. Right: FFT of the sampled output voltage.

Fig. 5.

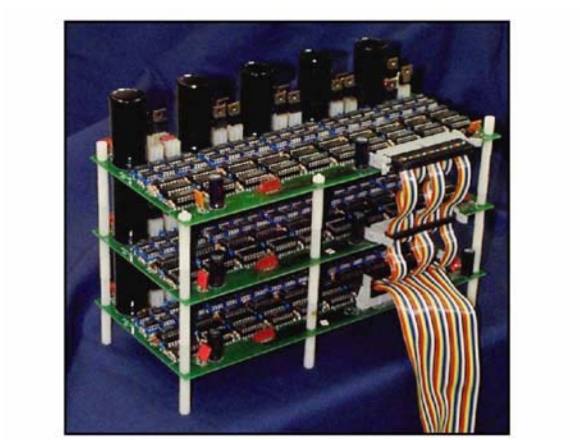


Fig. 4. Gate Driver Boards and MOSFETs for the Mulitlevel Inverter

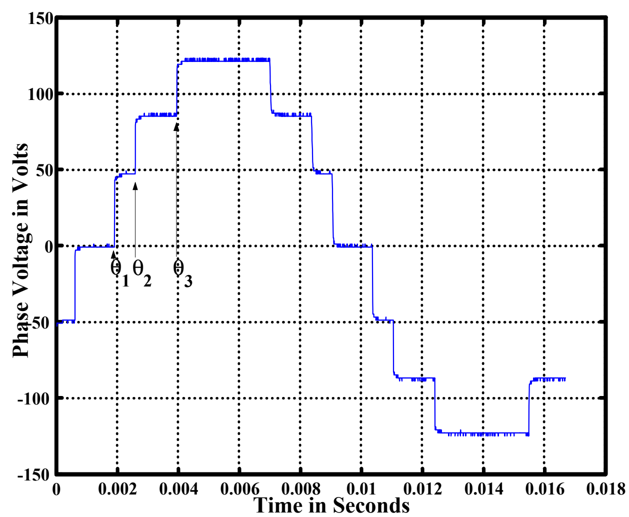


Fig. 6. Sampled output voltage with $m = 2.0$, $V_1 = 1, V_2 = 0.81, V_3 = 0.752$ and $f = 60$ Hz.

value of zero.

In the next experiment, the nominal value of the dc source voltages was 48 Volts ($V_{dc} = 48$ Volts) while the actual values were $V_1V_{dc} = 48.3$ Volts, $V_2V_{dc} = 38.9$ Volts and $V_3V_{dc} = 36.08$ Volts (i.e., $V_1 = 1, V_2 = 0.81, V_3 = 0.752$). Figure 6 is a plot of the output waveform voltage with $f = 60$ Hz and $m = 2.0$ and the corresponding FFT of this signal is given in Figure 7. This shows the normalized magnitude of the 5^{th} harmonic is zero and the 7^{th} harmonic is about 0.016 which corresponds well with the predicted value of zero.

In the final experiment presented here, the modulation index was changed to $m = 1$ while the dc source voltages were left at $V_1V_{dc} = 48.3$ Volts, $V_2V_{dc} = 38.9$ Volts and $V_3V_{dc} = 36.08$ Volts (i.e., $V_1 = 1, V_2 = 0.81, V_3 = 0.752$). Figure 8 is a plot of the output waveform voltage with $f = 60$ Hz and $m = 1.0$ and its corresponding FFT is given in Figure 9 which shows that the normalized magnitude of the 5^{th} and 7^{th} harmonics are zero.

V. Conclusions

It has been shown that elimnation theory and the notion of resultants can used to eliminate the lower order harmonices in a multilevel converter that has non equal dc sources. This method is expected to have widespread application as most multilevel converters do not have dc sources that are exactly equal. Future research includes

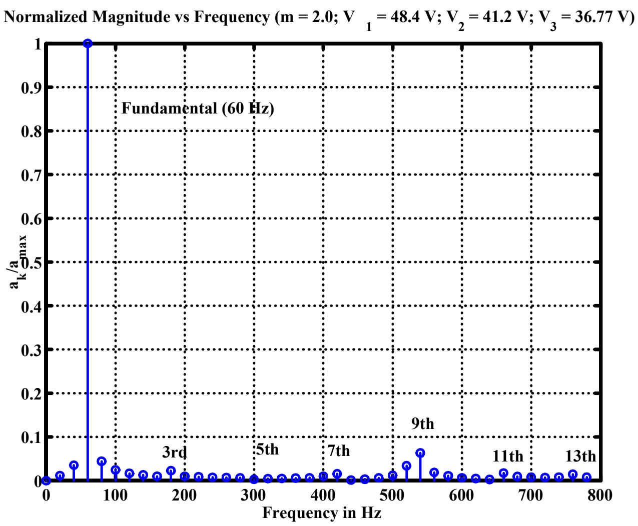


Fig. 7. Normalized FFT of the waveform in Figure 6

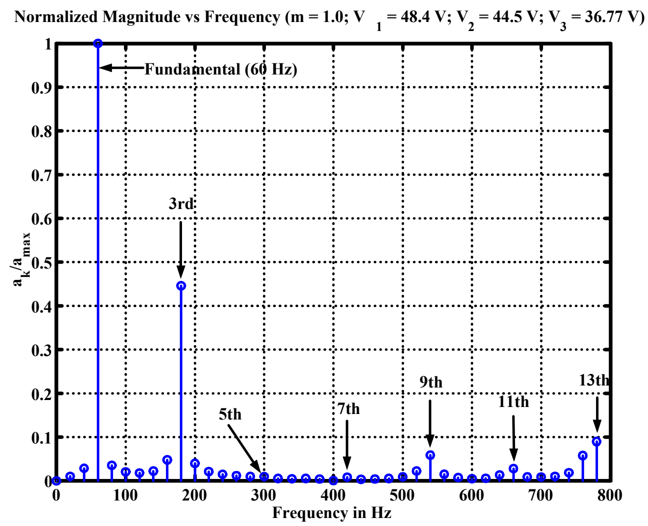


Fig. 9. Normalized FFT of the waveform in Figure 8

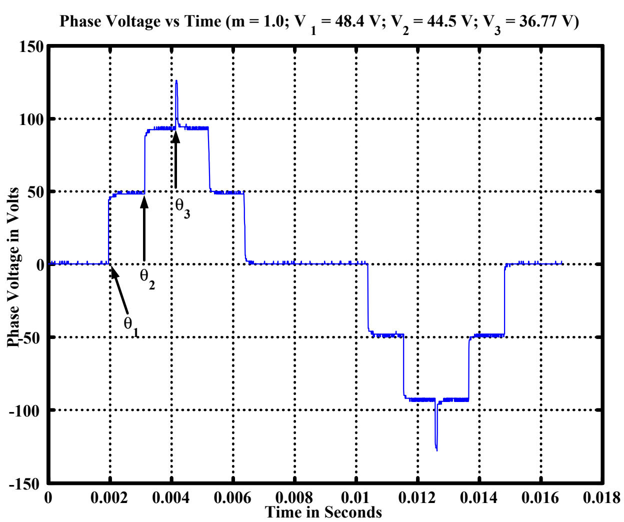


Fig. 8. Sampled output voltage with $m = 1.0$, $V_1 = 1, V_2 = 0.81, V_3 = 0.752$ and $f = 60$ Hz.

extending this to the case when there are more than three dc sources and methods to balance the dc sources against unequal discharge rates.

VI. Acknowledgements

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Appendix

I. Resultants

Given two polynomials $a(x_1, x_2)$ and $b(x_1, x_2)$ how does one find their common zeros? That is, the values (x_{10}, x_{20}) such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0.$$

Consider $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. There is always a polynomial $r(x_1)$ (called the *resultant polynomial*) such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1).$$

So if $a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0$ then $r(x_{10}) = 0$, that is, if (x_{10}, x_{20}) is a common zero of the pair $\{a(x_1, x_2), b(x_1, x_2)\}$, then the first coordinate x_{10} is a zero of $r(x_1) = 0$. To see how one obtains $r(x_1)$, let

$$\begin{aligned} a(x_1, x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1, x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1) \end{aligned}$$

Next, see if polynomials of the form

$$\begin{aligned} \alpha(x_1, x_2) &= \alpha_1(x_1)x_2 + \alpha_0(x_1) \\ \beta(x_1, x_2) &= \beta_2(x_1)x_2^2 + \beta_1(x_1)x_2 + \beta_0(x_1). \end{aligned}$$

can be found such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1). \tag{5}$$

Equating powers of x_2, this equation may be rewritten in matrix form as

$$\begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & 0 & b_2(x_1) \end{bmatrix} \begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix on the left-hand side is called the *Sylvester* matrix and is denoted here by $S_{a,b}(x_1)$. The inverse of $S_{a,b}(x_1)$ has the form

$$S_{a,b}^{-1}(x_1) = \frac{1}{\det S_{a,b}(x_1)} \operatorname{adj}\left(S_{a,b}(x_1)\right)$$

where $\operatorname{adj}(S_{a,b}(x_1))$ is the adjoint matrix and is a 5×5 *polynomial* matrix in x_1. Solving for $\alpha_i(x_1), \beta_i(x_1)$ gives

$$\begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \frac{\operatorname{adj} S_{a,b}(x_1)}{\det S_{a,b}(x_1)} \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Choosing $r(x_1) = \det S_{a,b}(x_1)$ guarantees that $\alpha_0(x_1), \alpha_1(x_1), \beta_0(x_1), \beta_1(x_1), \beta_2(x_1)$ are polynomials in x_1. That is, the *resultant polynomial* defined by $r(x_1) = \det S_{a,b}(x_1)$ is the polynomial required for (5).