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Observability of Speed in an Induction Motor from Stator Currents and Voltages

Mengwei Li, John Chiasson, Marc Bodson, and Leon Tolbert

Abstract— **This paper describes a new approach to estimating the speed of an induction motor from the measured terminal voltages and currents without the use of a speed/position sensor. The new observer uses a purely algebraic speed estimator to stabilize a dynamic speed estimator and it is shown that it has the potential to provide low speed (including zero speed) control of an induction motor under full rated load.**

Index Terms— **Sensorless Speed Observer, Induction Motor**

I. Introduction

Sensorless control of an induction motor refers the problem of controlling it without the use of a rotor position/speed sensor. Many different techniques have been proposed to estimate speed of an induction motor without a speed sensor. This area has a rather large literature and the reader is referred to [1][2][3][4][5][6][7][8][9][10] for an exposition of many of the existing approaches. The approach presented in this work is most closely related to the ideas described in [11][12][13][14][15]. In [11][12][13][14], observability is characterized as being able to reconstruct the unknown state variables as rational functions of the inputs, outputs, and their derivatives (See [12][13][14] for a more precise definition). We manage to obtain an algebraic expression for the rotor speed in terms of the machine inputs, machine outputs and their derivatives. In the systems theoretic approach considered in [15], the authors have shown that there are indistinguishable trajectories of the induction motor, i.e., pairs of different state trajectories with the same input/output behavior. That is, it is not possible to estimate the speed based on stator measurements for arbitrary trajectories [15]. A similar circumstance is shown here due to the fact that the "coefficients" of the algebraic expression for the speed all happen to be zero for some trajectories. We characterize a class of trajectories (or, modes of operation) from which the speed of the machine can be estimated from the stator currents and voltages. It is then shown how this speed estimate can be used in a field-oriented controller with the machine operating at low, or even zero, speed under full load.

M. Li, J. Chiasson, and L. M. Tolbert are with the ECE Department, University of Tennessee, Knoxville, TN 37996, mwl@utk.edu, chiasson@utk.edu, tolbert@utk.edu

M. Bodson is with the ECE Department, University of Utah, Salt Lake City, UT 84112, bodson@ee.utah.edu.

L. M. Tolbert is also with Oak Ridge National Laboratory, Oak Ridge TN. tolbertlm@ornl.gov

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II. Mathematical Model of the Induction Motor

A (two-phase equivalent) state-space mathematical model of the induction motor (see [16][17]) written in space vector form [9], by defining $\underline{i}_S \triangleq i_{Sa} + ji_{Sb}$, $\underline{\psi}_R \triangleq \psi_{Ra} + j\psi_{Rb}$, and $\underline{u}_S \triangleq u_{Sa} + ju_{Sb}$, is

$$\frac{d}{dt}\underline{i}_S = \frac{\beta}{T_R}\left(1 - jn_P\omega T_R\right)\underline{\psi}_R - \gamma\underline{i}_S + \frac{1}{\sigma L_S}\underline{u}_S \quad (1)$$

$$\frac{d}{dt}\underline{\psi}_R = -\frac{1}{T_R}\left(1 - jn_P\omega T_R\right)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S. \quad (2)$$

$$\frac{d\omega}{dt} = \frac{n_p M}{JL_R}\operatorname{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \frac{\tau_L}{J} \quad (3)$$

where, θ is the position of the rotor, $\omega = d\theta/dt$, n_p is the number of pole pairs, i_{Sa}, i_{Sb} are the (two phase equivalent) stator currents, and ψ_{Ra}, ψ_{Rb} are the (two phase equivalent) rotor flux linkages, R_S and R_R are the stator and rotor resistances, M is the mutual inductance, L_S and L_R are the stator and rotor inductances, J is the inertia of the rotor, and τ_L is the load torque. The symbols

$$T_R = \frac{L_R}{R_R} \qquad \sigma = 1 - \frac{M^2}{L_S L_R}$$

$$\beta = \frac{M}{\sigma L_S L_R} \qquad \gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}$$

have been used to simplify the expressions. T_R is referred to as the rotor time constant while σ is called the total leakage factor.

III. Algebraic Speed Observer

Differentiating (1) gives

$$\frac{d^2}{dt^2}\underline{i}_S = \frac{\beta}{T_R}\left(1 - jn_P\omega T_R\right)\frac{d}{dt}\underline{\psi}_R - jn_P\beta\underline{\psi}_R\frac{d\omega}{dt} - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S. \quad (4)$$

Using the complex-valued equations (1) and (2), one can eliminate $\underline{\psi}_R$ and $\frac{d}{dt}\underline{\psi}_R$ from (4) to obtain

$$\begin{aligned}\frac{d^2}{dt^2}\underline{i}_S = & -\frac{1}{T_R}\left(1 - jn_P\omega T_R\right)\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right) \\ & + \frac{\beta M}{T_R^2}\left(1 - jn_P\omega T_R\right)\underline{i}_S - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S \\ & - \frac{jn_P T_R}{1 - jn_P\omega T_R}\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right)\frac{d\omega}{dt}.\end{aligned} \quad (5)$$

Solving (5) for $d\omega/dt$ gives

$$\frac{d\omega}{dt} = -\frac{(1-jn_P\omega T_R)^2}{jn_P T_R^2} + \frac{1-jn_P\omega T_R}{jn_P T_R}\times \frac{\frac{\beta M}{T_R^2}(1-jn_P\omega T_R)\underline{i}_S - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S - \frac{d^2}{dt^2}\underline{i}_S}{\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S}. \tag{6}$$

If the signals are measured exactly and the dynamic model is correct, the right-hand side must be real. From (1) it is seen that the denominator in the last term of (6) is equal to $(\beta/T_R)(1-jn_P\omega T_R)\underline{\psi}_R$ and thus (6) is singular (i.e., the denominator in (6) is zero), if and only if $\left|\underline{\psi}_R\right| \equiv 0$.

Breaking down the right-hand side of (6) into its real and imaginary parts, the real part has the form

$$\frac{d\omega}{dt} = a_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega^2 + a_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega + a_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}). \tag{7}$$

The expressions for $a_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$, $a_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$, and $a_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$ are lengthy and therefore not explicitly presented here (Appendix VII-B gives their steady-state expressions). It is shown in Appendix VII-C that (7) is never stable in steady state.

On the other hand, the imaginary part of (6) has no derivatives in the speed leading to a 2^{nd} degree polynomial equation in ω of the form

$$q(\omega) \triangleq q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega^2 + q_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega + q_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}). \tag{8}$$

If ω is the speed of the motor, then $q(\omega)$ is zero. The expressions for $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$, $q_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$, and $q_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb})$ are lengthy and not explicitly presented here (Their steady-state expressions are given in Appendix VII-A.). There are two solutions to equation (8) and at least one of these two solutions must track the motor speed. This equation does not have any stability issue, but a procedure is required to determine which of the two solutions is correct. Further, there are situations when the speed cannot be determined by (8). For example, if u_{Sa} = constant and $u_{Sb}=0$, it turns out that $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) = q_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) = q_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \equiv 0$ and ω is not determinable from (8)[1]. On the other hand, if the machine is operated at zero speed ($\omega \equiv 0$) with a load on it, then $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \equiv 0$ and $q_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \neq 0$, and a unique solution is specified by (8) (see Appendix VII-A where this is proved in steady state). In fact, for low speed trajectories, consider equation (8) written in the form

$$(q_2\omega + q_1)\omega + q_0 = 0. \tag{9}$$

At low speeds, defined by $|q_2\omega| \ll |q_1|$, equation (9) reduces

$$q_1\omega + q_0 = 0$$

and ω is uniquely determined by $\omega = -q_0/q_1$. Appendix VII-B shows that, in steady state, $|q_2\omega| \ll |q_1|$ if $(T_R n_p \omega)^2 \ll 1$.

If $q_2(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \neq 0$, one determines the correct solution of (8) as follows: Differentiate equation (8) to obtain the new independent equation

$$(2q_2\omega + q_1)\frac{d\omega}{dt} + \dot{q}_2\omega^2 + \dot{q}_1\omega + \dot{q}_0 \equiv 0. \tag{10}$$

Next, $d\omega/dt$ is replaced by the right-hand side of equation (7) to obtain a new algebraic polynomial equation in ω given by

$$g(\omega) \triangleq 2q_2a_2\omega^3 + (2q_2a_1 + q_1a_2 + \dot{q}_2)\omega^2 + (2q_2a_0 + q_1a_1 + \dot{q}_1)\omega + q_1a_0 + \dot{q}_0. \tag{11}$$

$g(\omega)$ is a third-order polynomial equation in ω for which the speed of the motor is one of its zeros. Dividing[2] (11) by $q(\omega)$ from (8), the polynomial (11) has the form

$$g(\omega) = (2q_2a_2\omega + 2q_2a_1 - q_2q_1a_2 + \dot{q}_2)q(\omega) + r_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega + r_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}). \tag{12}$$

where

$$r_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \triangleq 2q_2^2a_0 - q_2q_1a_1 + q_2\dot{q}_1 - 2q_2q_0a_2 + q_1^2a_2 - q_1\dot{q}_2 \tag{13}$$

and

$$r_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) \triangleq q_2q_1a_0 + q_2\dot{q}_0 - 2q_2q_0a_1 + q_0q_1a_2 - q_0\dot{q}_2. \tag{14}$$

If ω is equal to the speed of the motor, then both $g(\omega)=0$ and $q(\omega)=0$, and one obtains

$$r(\omega) \triangleq r_1(u_{Sa},u_{Sb},i_{Sa},i_{Sb})\omega + r_0(u_{Sa},u_{Sb},i_{Sa},i_{Sb}) = 0. \tag{15}$$

This is now a first-order polynomial equation in ω with a unique solution as long as r_1 (the coefficient of ω) is nonzero (It is shown in Appendix VII-D that $r_1 \neq 0$ in steady state if $q_2 \neq 0$). The coefficients of r_1, r_0 contain 3^{rd} derivatives of the stator currents and 2^{nd} derivatives of the stator voltages and, therefore, noise is a concern. Rather than use this purely algebraic estimator, it is now shown how to combine it with the dynamic model to obtain a smoother (yet stable) speed estimator.

[1] An induction machine is not typically operated under these conditions. See [15] for more discussion of this issue.

[2] Given the polynomials $g(\omega), q(\omega)$ in ω with $\deg\{g(\omega)\} = n_g, \deg\{q(\omega)\} = n_q$, the Euclidean division algorithm ensures that there are polynomials $\gamma(\omega), r(\omega)$ such that $g(\omega) = \gamma(\omega)q(\omega) + r(\omega)$ and $\deg\{r(\omega)\} \leq \deg\{q(\omega)\} - 1 = n_q - 1$. Consequently, if ω_0 is a zero of both $g(\omega)$ and $q(\omega)$, then it must also be a zero of $r(\omega)$.

IV. Stable Dynamic Speed Observer

Dividing the right side of the differential equation model (7) by $q(\omega)$ $(q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0)$, one obtains

$$a_2\omega^2 + a_1\omega + a_0 = \gamma \times q(\omega, t) + \alpha\omega + \beta \quad (16)$$

where

$$\alpha \triangleq a_1 - a_2 q_1/q_2 \quad (17)$$

and

$$\beta \triangleq a_0 - a_2 q_0/q_2. \quad (18)$$

Then, as $q(\omega, t) \equiv 0$, equation (7) may be rewritten as

$$\frac{d\omega}{dt} = \alpha(t)\omega + \beta(t) \quad (19)$$

which is a *linear first-order time-varying* system. With

$$\Phi(t, t_0) \triangleq e^{\int_{t_0}^{t} \alpha(\tau)d\tau}$$

the fundamental solution of (19), the full solution is given by

$$\omega(t) = \Phi(t, t_0)\omega(0) + \int_{t_0}^{t} \Phi(t, \tau)\beta(\tau)d\tau.$$

Consequently, a sufficient condition for stability is that $\alpha(t) \leq -\kappa < 0$ for some $\kappa > 0$. It is shown in Appendix VII-C that $\alpha > 0$ in steady state, so the system is never stable in steady state.

For the case that $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0$, consider (19) to be the induction motor "model" and the solution ω of algebraic estimator (15) to be the "measurement". Then, let an observer be defined by

$$\frac{d\hat{\omega}}{dt} = \alpha(t)\hat{\omega} + \beta(t) + \ell(\omega - \hat{\omega}). \quad (20)$$

If $\ell - \alpha(t) > \kappa > 0$ for all t, then the estimator (20) is stable with a rate of decay of the error no less than κ. As this estimator is the result of integrating the signals $\alpha(t)$, $\beta(t)$, and ω (from (15)), it is a smoother estimate than the purely algebraic estimate.

In the case where $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0$, then the right side of equation (7) can be divided by $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0$ to obtain:

$$\frac{d\hat{\omega}}{dt} = c(t) + \ell(\omega - \hat{\omega}). \quad (21)$$

If $\ell > \kappa > 0$ for all t, then the equation (21) is stable with a rate of decay of the error no less than κ.

The estimate of speed proposed here is defined as the solution to the observer

$$\frac{d\hat{\omega}}{dt} \triangleq a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\hat{\omega}^2 + a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\hat{\omega} + a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) + \ell(\omega - \hat{\omega}) \quad (22)$$

where

$$\omega \triangleq \begin{cases} -q_0/q_1 & \text{if } |q_2\hat{\omega}| \leq 0.05\,|q_1| \quad \text{[See (8)]} \\ -r_0/r_1 & \text{if } |q_2\hat{\omega}| > 0.05\,|q_1| \quad \text{[See (15)].} \end{cases}$$

In Appendix VII-A it is shown that in steady state $q_1 \neq 0$ if $q_2 = 0$ while in Appendix VII-D it is shown that $r_1 \neq 0$ if $q_2 \neq 0$.

V. Simulation Results

As a first look at the viability of the observer (22), simulations were carried out to test it. Here, a three-phase (two-phase equivalent) induction motor model was simulated using SIMULINK with parameter values chosen to be

$$n_p = 2,\ R_S = 5.12 \text{ ohms},\ R_R = 2.23 \text{ ohms},$$
$$L_S = L_R = 0.2919 \text{ H},\ M = 0.2768 \text{ H},$$
$$J = 0.0021 \text{ k-gm}^2,\ \tau_{L\_rated} = 2.0337 \text{ N-m},$$
$$I_{\max} = 2.77 \text{ A},\ V_{\max} = 230 \text{ V}.$$

The induction motor model for the simulation is based on equations (1), (2), and (3). In the control scheme, the estimated speed is fed back to a current command field-oriented controller [9]. Figure 1 shows the simulation results of the motor speed and speed estimator with the motor under full load. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds, the brake is released and the machine is running on a low speed trajectory ($\omega_{\max} = 5\ rad/s$) with full load at the start. The estimated speed $\hat{\omega}$ is used in the field-oriented controller. In this simulation, the observer gain ℓ in equation (20) was chosen to be 1000.Figure 2 shows the simulation results of the motor speed and speed estimator with the motor under full load. From $t = 0$ to $t = 0.4$ seconds, a constant u_{Sa} is applied to the motor to build up the flux and the motor is considered to be held with a brake so that $\omega \equiv 0$. At $t = 0.4$ seconds, the brake is released and the machine is controlled to a zero speed trajectory ($\omega \equiv 0$) with full load at the start. $\hat{\omega}$ is used in the field-oriented controller. The observer gain ℓ in equation (20) was again chosen to be 1000.

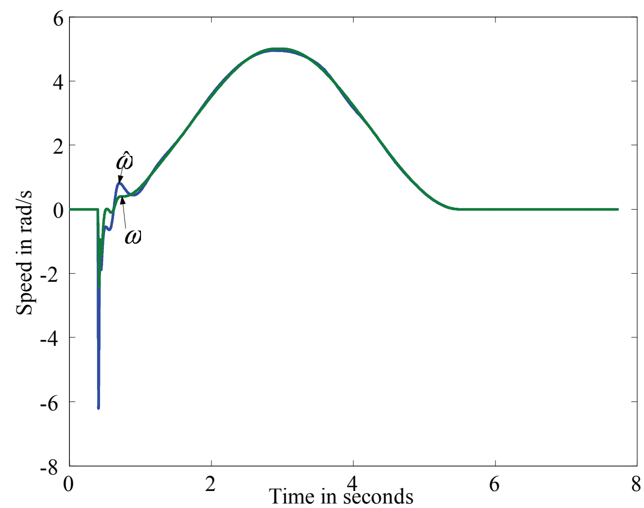

Fig. 1. ω and $\hat{\omega}$ with the motor tracking a low speed trajectory ($\omega_{\max} = 5\ rad/s$) with full load at the start.

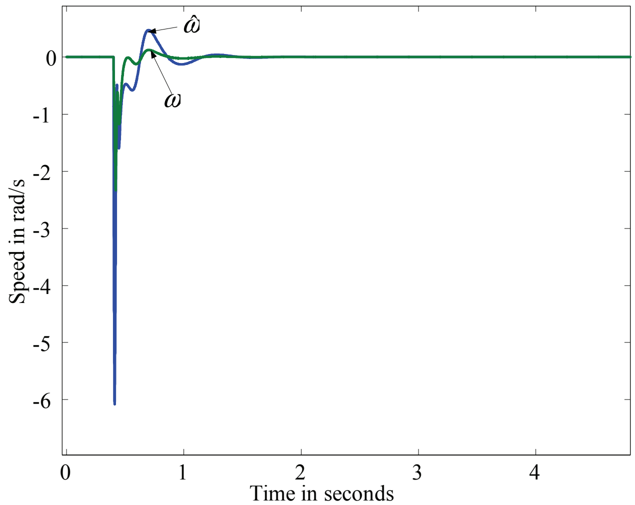

Fig. 2. ω and $\hat{\omega}$ with the motor tracking a zero speed trajectory ($\omega \equiv 0$) with full load at the start.

VI. Conclusions and Future Work

This paper introduced a new approach to speed sensorless control of an induction motor which entails using an algebraic estimate of the speed to stabilize a dynamic speed observer. The new observer does not require any sort of "slowly varying" speed assumption. The singularities of the observers were characterized under steady-state conditions. This sensorless speed controller shows potential for speed estimation at low speeds under full load. Future work will include the effect of parameter variation on the speed estimation as well as experimental results.

VII. Appendix: Steady-State Expressions

In the following, ω_S denotes the stator frequency and S denotes the normalized slip defined by $S \triangleq (\omega_S - n_p\omega)/\omega_S$. With $u_{Sa} + ju_{Sb} = \underline{U}_S e^{j\omega_S t}$ and $i_{Sa} + ji_{Sb} = \underline{I}_S e^{j\omega_S t}$, it is shown in [9] under steady-state conditions that the complex phasors $\underline{U}_S$ and $\underline{I}_S$ are related by ($S_p \triangleq R_R/(\sigma\omega_S L_R) = 1/(\sigma\omega_S T_R)$)

$$\begin{aligned}\underline{I}_S &= \frac{\underline{U}_S}{R_S + j\omega_S L_S\left(\dfrac{1+j\frac{S}{S_p}}{1+j\frac{S}{\sigma S_p}}\right)}\\ &= \frac{\underline{U}_S}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right) + j\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}.\end{aligned}$$

A. Steady-state expressions for q_2, q_1, and q_0

The steady-state expressions for q_2, q_1, and q_0 are now derived. These expressions are then used to show that $q_2 > 0$ for $\omega \neq 0$, $q_2 \equiv 0$ for $\omega = 0$, and $q_1 \neq 0$ if $q_2 \equiv 0$.

The explicit expression for q_2 is

$$\begin{aligned}q_2 \triangleq n_p^2 \times \Bigg(&\frac{1}{4}\sigma L_S T_R^2\left(\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)}{dt}\right)^2\\ &- T_R^2\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)}{dt}\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\\ &+ \frac{T_R^2}{\sigma L_S}\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}^2+u_{Sb}^2\right)\\ &+ \left(-\frac{\beta M}{T_R}+2\gamma\right)\frac{1}{4}\sigma L_S T_R^2\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)^2}{dt}\\ &+ \sigma L_S T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)^2\\ &+ 2T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right)\\ &\left(-\frac{\beta M}{T_R}+\gamma\right)\sigma L_S\gamma T_R^2\left(i_{Sa}^2+i_{Sb}^2\right)^2\\ &+ \left(\frac{\beta M}{T_R}-2\gamma\right)T_R^2\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right)\Bigg).\end{aligned}$$

In steady state, let (see [9])

$$\begin{aligned}u_{Sa} + ju_{Sb} &= \underline{U}_S e^{j\omega_S t}\\ i_{Sa} + ji_{Sb} &= \underline{I}_S e^{j\omega_S t}.\end{aligned}$$

The complex phasors $\underline{U}_S$ and $\underline{I}_S$ are related by

$$\underline{I}_S = \frac{\underline{U}_S}{R_S + j\omega_S L_S\left[\frac{1+j\frac{S}{S_p}}{1+j\frac{S}{\sigma S_p}}\right]}.$$

Here $S_p \triangleq \frac{R_R}{\sigma\omega_S L_R} = \frac{1}{\sigma\omega_S T_R}$ so that

$$\begin{aligned}\underline{I}_S &= \frac{\underline{U}_S}{R_S + j\omega_S L_S\left[\frac{1+jS\sigma\omega_S T_R}{1+jS\omega_S T_R}\right]}\\ &= \frac{\underline{U}_S}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right) + j\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}\end{aligned}$$

Further,

$$\begin{aligned}u_{Sa}i_{Sa} + u_{Sb}i_{Sb} &= \operatorname{Re}\left(\underline{U}_S\underline{I}_S^*\right)\\ u_{Sb}i_{Sa} - u_{Sa}i_{Sb} &= \operatorname{Im}\left(\underline{U}_S\underline{I}_S^*\right)\\ i_{Sa}^2 + i_{Sb}^2 &= \left|\underline{I}_S\right|^2\\ u_{Sa}^2 + u_{Sb}^2 &= \left|\underline{U}_S\right|^2.\end{aligned}$$

Thus, the 1^{st}, 2^{nd}, and 4^{th} terms of q_2 are all zero, i.e.,

$$n_p^2\frac{1}{4}\sigma L_S T_R^2\left(\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)}{dt}\right)^2 = 0$$

$$-n_p^2 T_R^2\frac{d\left(i_{Sa}^2+i_{Sb}^2\right)}{dt}\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right) = 0$$

$$n_p^2\left(-\beta M/T_R + 2\gamma\right)(1/4)\,\sigma L_S T_R^2 d\left(i_{Sa}^2+i_{Sb}^2\right)^2/dt = 0.$$

The 3^{rd} term of q_2 is given by

$$n_p^2 \frac{T_R^2}{\sigma L_S}\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}^2+u_{Sb}^2\right) = n_p^2 \frac{T_R^2}{\sigma L_S}\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2.$$

The 5^{th} term of q_2 is given by

$$n_p^2 \sigma L_S T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)^2 = n_p^2 \frac{\sigma L_S T_R^2 \omega_S^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.$$

The 6^{th} term of q_2 is

$$n_p^2 2T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt}-i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa}-u_{Sa}i_{Sb}\right) = n_p^2 \frac{-2T_R^2\omega_S\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2T_R^2\right)}{1+S^2\omega_S^2T_R^2}}{\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.$$

The 7^{th} term of q_2 is

$$n_p^2\left(-\frac{\beta M}{T_R}+\gamma\right)\sigma L_S\gamma T_R^2\left(i_{Sa}^2+i_{Sb}^2\right)^2 = n_p^2 \frac{\left(\frac{R_s^2}{\sigma L_s}+\frac{(1-\sigma)R_s}{\sigma T_R}\right)T_R^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}}.$$

The 8^{th} term of q_2 is

$$n_p^2\left(\frac{\beta M}{T_R}-2\gamma\right)T_R^2\left(i_{Sa}^2+i_{Sb}^2\right)\left(u_{Sa}i_{Sa}+u_{Sb}i_{Sb}\right) = n_p^2 \frac{-\left(\frac{2R_s}{\sigma L_s}+\frac{1-\sigma}{\sigma T_R}\right)T_R^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}} \times\left(R_S+\frac{(1-\sigma)S\omega_S^2L_ST_R}{1+S^2\omega_S^2T_R^2}\right).$$

Finally, substituting these steady-state expressions into the expression for q_2, one obtains

$$q_2 = \frac{n_p^2T_R^2\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{\omega_S^2L_S(1-\sigma)^2(1-S)}{\sigma\left(1+S^2\omega_S^2T_R^2\right)}. \quad (23)$$

With $\omega \neq 0$, it is seen that $q_2>0$ and $q_2=0$ if and only if $S=1$ (which is equivalent to $\omega=0$).

Similarly, it can be shown that the steady-state expression for q_1 is

$$q_1 = \frac{n_p\omega_S\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{L_S(1-\sigma)^2\left(1-\omega_S^2T_R^2(1-S)^2\right)}{\sigma\left(1+S^2\omega_S^2T_R^2\right)}. \quad (24)$$

If $\omega=0$, then $S=1$ and $q_1\neq 0$.

Finally, the steady-state expression for q_0 is

$$q_0 = \frac{-\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{\omega_S^2L_S(1-\sigma)^2(1-S)}{\sigma\left(1+S^2\omega_S^2T_R^2\right)}. \quad (25)$$

B. $(T_Rn_p\omega)^2 \ll 1 \Longrightarrow |q_2\omega| \ll |q_1|$

The purpose of this appendix is to show that in steady state, $|q_2\omega| \ll |q_1|$ if $(T_Rn_p\omega)^2 \ll 1$. In steady state,

$$|q_2\omega| = \frac{n_p\left|\omega_S\right|L_S(1-\sigma)^2\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{(T_Rn_p\omega)^2}{\sigma\left(1+S^2\omega_S^2T_R^2\right)}$$

and

$$|q_1| = \frac{n_p\left|\omega_S\right|L_S(1-\sigma)^2\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{\left|\left(1-(T_Rn_p\omega)^2\right)\right|}{\sigma\left(1+S^2\omega_S^2T_R^2\right)}.$$

Their ratio is then

$$\frac{|q_2\omega|}{|q_1|} = \left|\frac{(T_Rn_p\omega)^2}{1-(T_Rn_p\omega)^2}\right|$$

which shows that $(T_Rn_p\omega)^2 \ll 1 \Longrightarrow |q_2\omega| \ll |q_1|$.

C. *Steady-state expressions for* a_2, a_1, a_0, *and* α

The steady-state expressions for a_2, a_1, and a_0 are now given and used to show that the steady-state value for α is always positive. The steady-state expressions for a_2, a_1, a_0 are

$$a_2 = \frac{-n_p^2\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{\omega_S(1-\sigma)^2}{\sigma^2\left(1+S^2\omega_S^2T_R^2\right)}\times\frac{1}{den} \quad (26)$$

$$a_1 = \frac{n_p\left|\underline{U}_S\right|^4}{\left(\left(R_S+\frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2T_R^2}\right)^2+\frac{\omega_S^2L_S^2\left(1+\sigma S^2\omega_S^2T_R^2\right)^2}{\left(1+S^2\omega_S^2T_R^2\right)^2}\right)^2} \times\frac{2\omega_S^2(1-\sigma)^2(1-S)}{\sigma^2\left(1+S^2\omega_S^2T_R^2\right)}\times\frac{1}{den} \quad (27)$$

and

$$a_0 = \frac{-\left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\omega_S^3\left(1-\sigma\right)^2\left(1-S\right)^2}{\sigma^2\left(1+S^2\omega_S^2 T_R^2\right)} \times \frac{1}{den}. \quad (28)$$

where

$$den \triangleq \frac{n_p T_R\left|\underline{U}_S\right|^4\left(\frac{(1-\sigma)}{\sigma T_R}\frac{1+(S-1)S\omega_S^2 T_R^2}{1+S^2\omega_S^2 T_R^2}\right)^2}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} + \frac{n_p T_R\left|\underline{U}_S\right|^4\left(\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2} - \omega_S\right)^2}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.$$

Remark Recall that it was pointed out in Section III (following equation (6)) that $den = 0$ if and only if $\left|\underline{\psi}_R\right| \equiv 0$.

To compute the steady-state value of α, note that by (17),

$$\alpha = a_1 - a_2 q_1/q_2.$$

It is then easily seen that $a_1 > 0$, $a_2 q_1 < 0$, and $q_2 > 0$, so that in the steady state $\alpha > 0$. That is, the system (19) is never stable in steady state.

D. Steady-state expression for r_1

It is now shown that the steady-state value of $r_1\left(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}\right)$ in (13) is nonzero.

Substituting the steady-state values of q_2, q_1, q_0, a_2, a_1, and a_0 (noting that $\dot{q}_1 \equiv 0$ and $\dot{q}_2 \equiv 0$ in steady state) into (13) gives

$$r_1 = \frac{-\left|\underline{U}_S\right|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \times \frac{n_p^4\left(1-\sigma\right)^6\omega_S^3 L_S^2}{\sigma^4} \times \left(1 + T_R^2\omega_S^2\left(1-S\right)^2\right)^2 \times \frac{1}{den}$$

where den is given in Appendix VII-C. It is then seen that $r_1 \neq 0$ in steady state.

E. Steady-state speed

Substituting in the steady-state values of a_2, a_1, and a_0, it is seen that

$$a_1^2 - 4a_2 a_0 \equiv 0,$$

so that, interestingly, the steady-state value of the right-hand side of (7) may be rewritten as

$$a_2\omega^2 + a_1\omega + a_0 = a_2\left(\omega + \frac{a_1}{2a_2}\right)^2$$

where a_2 is nonzero by (26).

On the other hand, the steady-state solutions of (8) are

$$\omega_1 \triangleq \frac{-q_1 + \sqrt{q_1^2 - 4q_2 q_0}}{2q_2} = \omega$$

and

$$\omega_2 \triangleq \frac{-q_1 - \sqrt{q_1^2 - 4q_2 q_0}}{2q_2} = \frac{-1}{T_R^2 n_p^2 \omega}.$$

Interestingly the correct steady-state speed is found by choosing the + sign in the quadratic formula when computing the roots of (8). However, this is not necessarily true if the system is not in steady state.

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