A Differential-Algebraic Approach to Speed Estimation in an Induction Motor

Mengwei Li, John Chiasson, Marc Bodson, and Leon M. Tolbert

***Abstract*—This note considers a differential-algebraic approach to estimating the speed of an induction motor from the measured terminal voltages and currents. In particular, it is shown that the induction motor speed ω satisfies both a second- and a third-order polynomial equation whose coefficients depend on the stator voltages, stator currents, and their derivatives. It is shown that as long as the stator electrical frequency is nonzero, the speed is uniquely determined by these polynomials. The speed so determined is then used to stabilize a dynamic (Luenberger type) observer to obtain a smoother speed estimate. With full knowledge of the machine parameters and filtering of the sensor noise, simulations indicate that this estimator has the potential to provide low speed (including zero speed) control of an induction motor under full rated load.**

***Index Terms*—Differential-algebraic, induction motor, sensorless control, speed estimation.**

I. Introduction

Sensorless control refers to the problem of controlling an induction motor without the use of a rotor position/speed sensor. A disadvantage of existing, high-performance control algorithms for induction motors is that they require a shaft sensor to estimate the (unmeasured) rotor flux linkages for the field-oriented (vector control) algorithm. Here, a rotor speed/position sensor will be referred to as a *shaft sensor*. Other sensors, such as voltage and current sensors, are also used in a typical induction motor drive, but are far less vulnerable and are assumed present even in a "sensorless" drive (a truly sensorless algorithm is an open-loop control system). The shaft sensor issue is of utmost concern to industrial users as it represents a significant cost as well as a reliability issue. However, even if a shaft sensor is to be used, one can foresee the need for a control algorithm that is capable of tolerating the failure of the sensor. For example, it would be highly desirable to let an electric vehicle proceed to a garage for service after the shaft sensor has failed.

Many different techniques have been proposed to estimate speed of an induction motor without a shaft sensor. This area has a rather large literature, and the reader is referred to [1]–[12] for an exposition of many of the existing approaches. The approach presented in this work is most closely related to the ideas described in [13]–[18]. In [13]–[16], observability is characterized as being able to reconstruct the unknown state variables as rational functions of the inputs, outputs, and their derivatives (See [14]–[16] for a more precise definition). We manage to obtain an algebraic expression for the rotor speed in terms of the machine inputs, machine outputs and their derivatives. In the systems theoretic approach considered in [18], the authors have shown that there are indistinguishable trajectories of the induction motor, i.e., pairs of different state trajectories with the same input/output behavior. That is, it is not possible to estimate the speed based on stator measurements for arbitrary trajectories [18]. A similar circumstance is shown here due to the fact that the "coefficients" of the algebraic expression for the speed all happen to be zero for some trajectories. We characterize a class of trajectories (or, modes of operation) from which the speed of the machine can be estimated from the stator currents and voltages. It is then shown how this speed estimate can be used in a field-oriented controller with the machine operating at low, or even zero, speed under full load. A preliminary version of this work was presented in [19].

The starting point of the analysis is a space vector model of the induction motor given by (see, e.g., [20, p. 568])

$$\frac{d}{dt}\underline{i}_S = \frac{\beta}{T_R}(1 - jn_P\omega T_R)\underline{\psi}_R - \gamma\underline{i}_S + \frac{1}{\sigma L_S}\underline{u}_S \tag{1}$$

$$\frac{d}{dt}\underline{\psi}_R = -\frac{1}{T_R}(1 - jn_P\omega T_R)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S \tag{2}$$

$$\frac{d\omega}{dt} = \frac{n_p M}{J L_R}\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \frac{\tau_L}{J} \tag{3}$$

where $\underline{i}_S \triangleq i_{Sa} + j i_{Sb}$, $\underline{\psi}_R \triangleq \psi_{Ra} + j\psi_{Rb}$, $\underline{u}_S \triangleq u_{Sa} + j u_{Sb}$, θ is the position of the rotor, $\omega = d\theta/dt$, n_p is the number of pole pairs, i_{Sa}, i_{Sb} are the (two-phase equivalent) stator currents, and ψ_{Ra}, ψ_{Rb} are the (two-phase equivalent) rotor flux linkages, R_S and R_R are the stator and rotor resistances, M is the mutual inductance, L_S and L_R are the stator and rotor inductances, J is the inertia of the rotor, and τ_L is the load torque. The symbols $T_R = (L_R/R_R), \sigma = -(M^2/L_S L_R), \beta = (M/\sigma L_S L_R), \gamma = (R_S/\sigma L_S) + (1/\sigma L_S)(1/T_R)(M^2/L_R)$ has been used to simplify the expressions. T_R is referred to as the rotor time constant, and σ is called the total leakage factor.

II. Algebraic Speed Observer

Differentiating (1) gives

$$\frac{d^2}{dt^2}\underline{i}_S = \frac{\beta}{T_R}(1 - jn_P\omega T_R)\frac{d}{dt}\underline{\psi}_R - jn_P\beta\underline{\psi}_R\frac{d\omega}{dt} - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S. \tag{4}$$

Using the complex-valued (1) and (2), one can eliminate $\underline{\psi}_R$ and $(d/dt)\underline{\psi}_R$ from (4) to obtain

$$\begin{aligned}\frac{d^2}{dt^2}\underline{i}_S = &-\frac{1}{T_R}(1 - jn_P\omega T_R)\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right)\\ &+ \frac{\beta M}{T_R^2}(1 - jn_P\omega T_R)\underline{i}_S - \gamma\frac{d}{dt}\underline{i}_S + \frac{1}{\sigma L_S}\frac{d}{dt}\underline{u}_S\\ &- \frac{jn_P T_R}{1 - jn_P\omega T_R}\left(\frac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \frac{1}{\sigma L_S}\underline{u}_S\right)\frac{d\omega}{dt}.\end{aligned} \tag{5}$$

Solving (5) for $d\omega/dt$ gives (6), as shown at the bottom of the next page. If the signals are measured exactly and the dynamic model is correct, the right-hand side must be real. Note by (1) that $d\underline{i}_S/dt + \gamma\underline{i}_S - \underline{u}_S/(\sigma L_S) = (\beta/T_R)(1 - jn_P\omega T_R)\underline{\psi}_R$ so this formulation has the advantage that the right-hand side of (6) is singular if and only

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if $|\underline{\psi}_R| \equiv 0$. Breaking down the right-hand side of (6) into its real and imaginary parts, the real part has the form

$$\frac{d\omega}{dt} = a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}). \quad (7)$$

The expressions for $a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, $a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, and $a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are lengthy and therefore not explicitly presented here. Appendix 6.3 gives their steady-state expressions and further shows that (7) is never stable in steady state and thus cannot be used as an observer by just integrating it in real time. On the other hand, the imaginary part of the right-hand side of (6) is a second-degree polynomial in ω of the form

$$q(\omega) \triangleq q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega^2 + q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}). \quad (8)$$

If ω is the speed of the motor, then $q(\omega)$ is zero as the imaginary part of (6) is zero. The expressions for $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}), q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$, and $q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})$ are lengthy and not explicitly presented here. Their steady-state expressions are given in Appendix 6.1. There are two zeros of the polynomial (8), and at least one of these zeros is equal to the motor speed. There is no stability issue, but a procedure is required to determine which of the two zeros corresponds to the motor speed. Further, there are situations when the speed cannot be determined by (8). For example, if u_{Sa} = constant and $u_{Sb} = 0$, it turns out that $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = q_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \equiv 0$ and ω is not determinable from (8).[1] On the other hand, if the machine is operated at zero speed ($\omega \equiv 0$) with a load on it, then $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \equiv 0$ and $q_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0$, and a unique solution is specified by (8) (see Appendix 6.1 where this is proved in steady state). In fact, for low-speed trajectories, consider (8) written in the form $(q_2\omega + q_1)\omega + q_0 = 0$. At low speeds, defined by $|q_2\omega| \ll |q_1|$, this reduces to $q_1\omega + q_0 = 0$ and ω is uniquely determined by $\omega = -q_0/q_1$. Appendix 6.2 shows that, in steady state, $|q_2\omega| \ll |q_1|$ if $(T_R n_p \omega)^2 \ll 1$.

If $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0$, one determines the correct solution of (8) as follows: Differentiate (8) to obtain the new independent equation

$$(2q_2\omega + q_1)\frac{d\omega}{dt} + \dot{q}_2\omega^2 + \dot{q}_1\omega + \dot{q}_0 \equiv 0. \quad (9)$$

[1]An induction machine is not typically operated under these conditions. See [18] for more discussion of this issue.

Next, $d\omega/dt$ is replaced by the right-hand side of (7) to obtain the polynomial $g(\omega)$ defined by

$$g(\omega) \triangleq 2q_2a_2\omega^3 + (2q_2a_1 + q_1a_2 + \dot{q}_2)\omega^2 + (2q_2a_0 + q_1a_1 + \dot{q}_1)\omega + q_1a_0 + \dot{q}_0. \quad (10)$$

$g(\omega)$ is a third-order polynomial equation in ω for which the speed of the motor is one of its zeros. Dividing[2] $g(\omega)$ by $q(\omega)$ from (8), $g(\omega)$ may be rewritten in the form

$$g(\omega) = \frac{1}{q_2}((2q_2a_2\omega + 2q_2a_1 - q_1a_2 + \dot{q}_2)q(\omega) + r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + r_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})) \quad (11)$$

where

$$r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq 2q_2^2a_0 - q_2q_1a_1 + q_2\dot{q}_1 - 2q_2q_0a_2 + q_1^2a_2 - q_1\dot{q}_2 \quad (12)$$

and

$$r_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \triangleq q_2q_1a_0 + q_2\dot{q}_0 - 2q_2q_0a_1 + q_0q_1a_2 - q_0\dot{q}_2. \quad (13)$$

If ω is equal to the speed of the motor, then both $g(\omega) = 0$ and $q(\omega) = 0$, and one obtains

$$r(\omega) \triangleq r_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\omega + r_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) = 0. \quad (14)$$

This is now a first-order polynomial equation in ω with a unique solution as long as r_1 (the coefficient of ω) is nonzero (It is shown in Appendix 6.5 that $r_1 \neq 0$ in steady state if $q_2 \neq 0$). The coefficients of r_1, r_0 contain third derivatives of the stator currents and second derivatives of the stator voltages making noise a concern. Rather than use this purely algebraic estimator, it is now shown how to combine it with the dynamic model to obtain a smoother (yet stable) speed estimator. (See [21]–[24] for approaches to obtaining the derivatives of the measurements).

[2]Given the polynomials $g(\omega), q(\omega)$ in ω with $\deg\{g(\omega)\} = n_g, \deg\{q(\omega)\} = n_q$, the Euclidean division algorithm ensures that there are polynomials $c(\omega), r(\omega)$ such that $g(\omega) = c(\omega)q(\omega) + r(\omega)$ and $\deg\{r(\omega)\} \leq \deg\{q(\omega)\} - 1 = n_q - 1$. Consequently, if ω_0 is a zero of both $g(\omega)$ and $q(\omega)$, then it must also be a zero of $r(\omega)$.

$$\frac{d\omega}{dt} = -\frac{(1 - jn_P\omega T_R)^2}{jn_PT_R^2} + \frac{1 - jn_P\omega T_R}{jn_PT_R}\,\frac{\dfrac{\beta M}{T_R^2}(1 - jn_P\omega T_R)\underline{i}_S - \gamma\dfrac{d}{dt}\underline{i}_S + \dfrac{1}{\sigma L_S}\dfrac{d}{dt}\underline{u}_S - \dfrac{d^2}{dt^2}\underline{i}_S}{\dfrac{d}{dt}\underline{i}_S + \gamma\underline{i}_S - \dfrac{1}{\sigma L_S}\underline{u}_S}. \quad (6)$$

III. STABLE DYNAMIC SPEED OBSERVER

Dividing the right-hand side of (7) by $q(\omega)$ $(q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0)$, one obtains

$$a_2\omega^2 + a_1\omega + a_0 = c \times q(\omega) + \alpha\omega + \beta \quad (15)$$

for some c, where $\alpha \triangleq a_1 - a_2 q_1/q_2, \beta \triangleq a_0 - a_2 q_0/q_2$. Then, as $q(\omega) \equiv 0$, (7) may be rewritten as

$$\frac{d\omega}{dt} = \alpha(t)\omega + \beta(t) \quad (16)$$

which is a *linear first-order time-varying* system. With $\Phi(t, t_0) \triangleq e^{\int_{t_0}^{t} \alpha(\tau)d\tau}$ the fundamental solution of (16), the full solution is given by $\omega(t) = \Phi(t, t_0)\omega(0) + \int_{t_0}^{t} \Phi(t, \tau)\beta(\tau)d\tau$. Consequently, a sufficient condition for stability is that $\alpha(t) \leq -\kappa < 0$ for some $\kappa > 0$. It is shown in Appendix 6.3 that $\alpha > 0$ in steady state, so the system is never stable in steady state.

For the case that $q_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) \neq 0$, consider (16) to be the induction motor "model" and the solution ω of algebraic estimator (14) to be the "measurement." Define an observer by

$$\frac{d\hat{\omega}}{dt} = \alpha(t)\hat{\omega} + \beta(t) + \ell(\omega - \hat{\omega}). \quad (17)$$

If $\ell - \alpha(t) > \kappa > 0$ for all t, then the estimator (17) is stable with a rate of decay of the error no less than κ. As this estimator is the result of integrating the signals $\alpha(t)$, $\beta(t)$, and ω [from (14)], it is a smoother estimate than the purely algebraic estimate.

In the case where $q_2 = 0$, then the right side of (7) can be divided by the polynomial $q_1\omega + q_0$ to obtain

$$\frac{d\hat{\omega}}{dt} = c(t) + \ell(\omega - \hat{\omega}) \quad (18)$$

where $c(t)$ is a function of the stator currents, stator voltages and their derivatives. If $\ell > \kappa > 0$ for all t, then the (18) is stable with a rate of decay of the error no less than κ. The estimate of speed proposed here is defined as the solution to the observer

$$\frac{d\hat{\omega}}{dt} \triangleq a_2(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\hat{\omega}^2 + a_1(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb})\hat{\omega} + a_0(u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}) + \ell(\omega - \hat{\omega}) \quad (19)$$

where (see Appendices 6.1 and 6.5)

$$\omega \triangleq \begin{cases} -q_0/q_1, & \text{if } |q_2\hat{\omega}| \leq 0.05|q_1| \quad \text{[see (8)]} \\ -r_0/r_1, & \text{if } |q_2\hat{\omega}| > 0.05|q_1| \quad \text{[see (14)].} \end{cases}$$

IV. SIMULATION RESULTS

Here, a three-phase (two-phase equivalent) induction motor model was simulated using SIMULINK with parameter values $n_p = 2$, $R_S = 5.12$ ohms, $R_R = 2.23$ ohms, $L_S = L_R = 0.2919$ H, $M = 0.2768$ H, $J = 0.0021$ kg-m^2, $\tau_{L\_\text{rated}} = 2.0337$ N-m, $I_{\max} = 2.77$ A, $V_{\max} = 230$ V. In the simulation, a 20 kHz PWM inverter was included and white noise $(\sigma = 0.001 I_{\max})$ was added to the current measurements. Before sampling the currents, they were put through a low-pass analog filter (first-order, 1-kHz cutoff). After sampling, the current signals were quantized to simulate a 14 bit A/D and these quantized current measurements were filtered using a digital low-pass (third-order 500 Hz cutoff) Butterworth filter. The sample period was 1 μs and was found to be necessary to compute third-order derivatives of the currents and second-order derivatives of the voltages to estimate the speed. Though such a sample rate is not possible using the standard processor technology available in commercial electric drives, it may be possible in a few years. (For example, OPALRT has recently announced a new software product that allows one to convert Simulink files to run in real time on FPGA boards with sample rates of 1 μs [25]). It is standard practice to have DSPs (digital signal processors) in communication systems oversample at high rates in order to digitally filter received signals without aliasing. Further, if one uses a low-pass Butterworth filter, the derivatives of the filtered input signal (i.e., stator currents/voltages) are state variables in the state-space implementation of the filter, i.e., no differentiation is needed [24]. See also [23] for another approach to estimating derivatives without numerical differentiation.

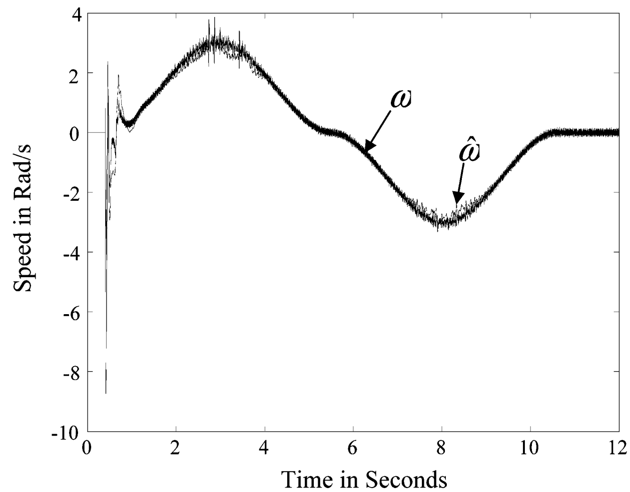

Fig. 1. ω and $\hat{\omega}$ in rad/s versus time in seconds with full load on the motor.

The interest here is in low-speed sensorless control of the machine with full load. The trajectory was chosen to have a maximum speed of 3 rad/s, to do a speed reversal, and to have zero speed at the end as shown in Fig. 1. Along with the stator currents, the estimated speed $\hat{\omega}$ is fed back to a current command field-oriented controller [8], [20]. Fig. 1 shows the simulation results of the motor speed and the stabilized speed estimator under full load. The full load is on the motor from $t = 0.4$ s to $t = 16$ s, that is, even during the zero speed part of the trajectory. From $t = 0$ to $t = 0.4$ s, a constant u_{Sa} is applied to the motor to build up the flux, and the motor is considered to be held with a (mechanical) brake so that $\omega \equiv 0$. Fig. 1 shows at $t = 0.4$ s the brake is released and the machine is running on a low-speed trajectory $(\omega_{\max} = 3\,\text{rad/s})$ with full load. In this simulation, the observer gain ℓ in (17) is chosen to be 1000. Fig. 2 shows the voltage u_{Sb} and current i_{Sb} corresponding to the trajectory in Fig. 1.

V. CONCLUSION

This note presented a characterization of the observability of the rotor speed of an induction motor based on input and output measurements (stator voltages and currents). This was done in terms of the speed being the solution to some polynomial equations whose coefficients were functions of the input–output measurements and their derivatives. The singularities of these algebraic equations (i.e., whether or not the leading coefficient is zero) were characterized under steady-state conditions. The new observer does not require any sort of "slowly varying" speed assumption and is stable. However, its computational requirements are quite high with respect to current technology.

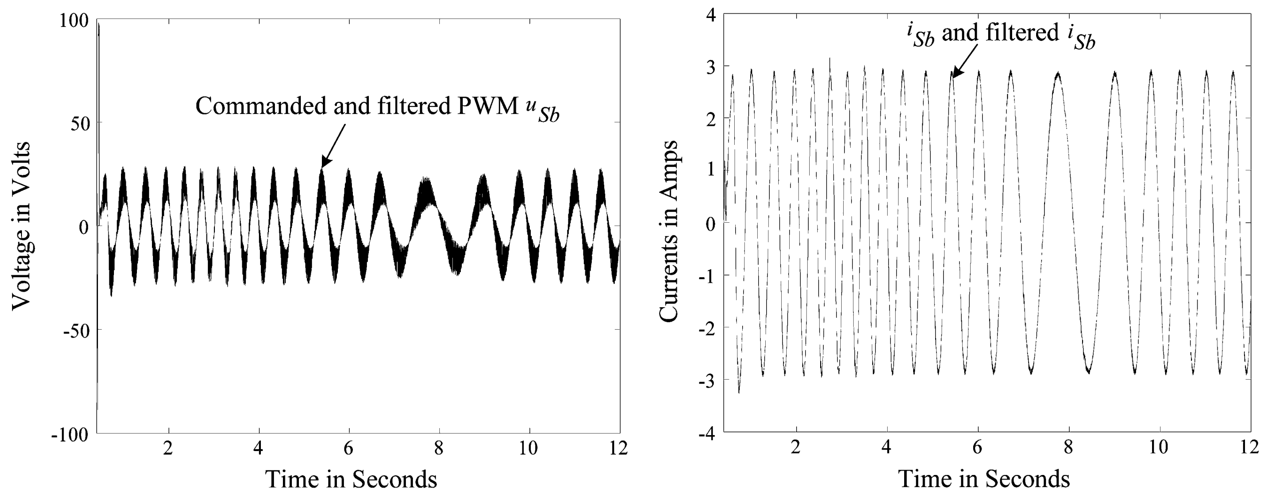

Fig. 2. (Left) u_{Sb} commanded and filtered. (Right) i_{Sb} measured and filtered.

Appendix
Steady-State Expressions

In the following, ω_S denotes the stator frequency and S denotes the normalized slip defined by $S \triangleq (\omega_S - n_p\omega)/\omega_S$. With $u_{Sa} + ju_{Sb} = \underline{U}_S e^{j\omega_S t}$ and $i_{Sa} + ji_{Sb} = \underline{I}_S e^{j\omega_S t}$, it is shown in [8] under steady-state conditions that the complex phasors $\underline{U}_S$ and $\underline{I}_S$ are related by $(S_p \triangleq (R_R/\sigma\omega_S L_R) = (1/\sigma\omega_S T_R))$

$$\underline{I}_S = \frac{\underline{U}_S}{R_S + j\omega_S L_S\left(1 + j\frac{S}{S_p}\right)/\left(1 + j\frac{S}{\sigma S_p}\right)} = \frac{\underline{U}_S}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right) + j\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}.$$

A. Steady-State Expressions for q_2, q_1, and q_0

The steady-state expressions for q_2, q_1, and q_0 are now derived. These expressions are then used to show that $q_2 > 0$ for $\omega \neq 0$, $q_2 \equiv 0$ for $\omega = 0$, and $q_1 \neq 0$ if $q_2 \equiv 0$. The explicit expression for q_2 is

$$\begin{aligned} q_2 \triangleq n_p^2 \times \Bigg(& \frac{1}{4}\sigma L_S T_R^2 \left(\frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\right)^2 \\ & - T_R^2 \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)}{dt}\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\ & + \frac{T_R^2}{\sigma L_S}\left(i_{Sa}^2 + i_{Sb}^2\right)\left(u_{Sa}^2 + u_{Sb}^2\right) \\ & + \left(-\frac{\beta M}{T_R} + 2\gamma\right)\frac{1}{4}\sigma L_S T_R^2 \frac{d\left(i_{Sa}^2 + i_{Sb}^2\right)^2}{dt} \\ & + \sigma L_S T_R^2 \left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_Sb}{dt}\right)^2 \\ & + 2T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) \\ & + \left(-\frac{\beta M}{T_R} + \gamma\right)\sigma L_S\gamma T_R^2\left(i_{Sa}^2 + i_{Sb}^2\right)^2 \\ & + \left(\frac{\beta M}{T_R} - 2\gamma\right)T_R^2\left(i_{Sa}^2 + i_{Sb}^2\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right)\Bigg). \end{aligned}$$

As the magnitude of the voltage and current phasors are constant in steady state, the first, second, and third terms of q_2 are all zero. The fourth term of q_2 is given by

$$n_p^2\frac{T_R^2}{\sigma L_S}\left(i_{Sa}^2 + i_{Sb}^2\right)\left(u_{Sa}^2 + u_{Sb}^2\right) = n_p^2\frac{T_R^2}{\sigma L_S}\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2.$$

The fifth term of q_2 is given by

$$n_p^2\sigma L_S T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)^2 = n_p^2\frac{\sigma L_S T_R^2\omega_S^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.$$

The sixth term of q_2 is

$$n_p^2 2T_R^2\left(i_{Sb}\frac{di_{Sa}}{dt} - i_{Sa}\frac{di_{Sb}}{dt}\right)\left(u_{Sb}i_{Sa} - u_{Sa}i_{Sb}\right) = n_p^2\frac{-2T_R^2\omega_S\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2\frac{\omega_S L_S\left(1+\sigma S^2\omega_S^2 T_R^2\right)}{1+S^2\omega_S^2 T_R^2}}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.$$

The seventh term of q_2 is

$$n_p^2\left(-\frac{\beta M}{T_R} + \gamma\right)\sigma L_S\gamma T_R^2\left(i_{Sa}^2 + i_{Sb}^2\right)^2 = n_p^2\frac{\left(\frac{R_s^2}{\sigma L_s} + \frac{(1-\sigma)R_s}{\sigma T_R}\right)T_R^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}}.$$

The eighth term of q_2 is

$$\begin{aligned} & n_p^2\left(\frac{\beta M}{T_R} - 2\gamma\right)T_R^2\left(i_{Sa}^2 + i_{Sb}^2\right)\left(u_{Sa}i_{Sa} + u_{Sb}i_{Sb}\right) \\ & = n_p^2\frac{-\left(\frac{2R_s}{\sigma L_s} + \frac{1-\sigma}{\sigma T_R}\right)T_R^2\left|\underline{I}_S\right|^2\left|\underline{U}_S\right|^2}{\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}} \\ & \quad \times\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1 + S^2\omega_S^2 T_R^2}\right). \end{aligned}$$

Finally, substituting these steady-state expressions into the expression for q_2, one obtains (20), as shown at the top of the next page. With $\omega \neq 0$, it is seen that $q_2 > 0$., while $q_2 = 0$ if and only if $S = 1$ (which is equivalent to $\omega = 0$). Similarly, it can be shown that the steady-state expression for q_1 is

$$q_1 = \frac{n_p\omega_S\left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times\frac{L_S(1-\sigma)^2\left(1 - \omega_S^2 T_R^2(1-S)^2\right)}{\sigma\left(1 + S^2\omega_S^2 T_R^2\right)}. \quad (21)$$

$$q_2 = \frac{n_p^2 T_R^2 \left|\underline{U}_S\right|^4}{\left(\left(R_S + (1-\sigma)\, S\omega_S^2 L_S T_R 1 + S^2\omega_S^2 T_R^2\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \frac{\omega_S^2 L_S\,(1-\sigma)^2\,(1-S)}{\sigma\,(1+S^2\omega_S^2 T_R^2)}. \tag{20}$$

If $\omega = 0$, then $S = 1$ and $q_1 \neq 0$. Finally, the steady-state expression for q_0 is

$$q_0 = \frac{-\left|\underline{U}_S\right|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\omega_S^2 L_S\,(1-\sigma)^2\,(1-S)}{\sigma\,(1+S^2\omega_S^2 T_R^2)}. \tag{22}$$

B. $(T_R n_p \omega)^2 \ll 1 \Longrightarrow |q_2\omega| \ll |q_1|$

To show that $|q_2\omega| \ll |q_1|$ if $(T_R n_p \omega)^2 \ll 1$, first note that $|q_2\omega|$ and $|q_1|$ are given by

$$|q_2\omega| = \frac{n_p|\omega_S|L_S(1-\sigma)^2|\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{(T_R n_p \omega)^2}{\sigma\,(1+S^2\omega_S^2 T_R^2)}$$

$$|q_1| = \frac{n_p|\omega_S|L_S(1-\sigma)^2|\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\left|\left(1-(T_R n_p \omega)^2\right)\right|}{\sigma\,(1+S^2\omega_S^2 T_R^2)}.$$

Their ratio is then $|q_2\omega|/|q_1| = (T_R n_p \omega)^2/|1-(T_R n_p \omega)^2|$ so that $(T_R n_p \omega)^2 \ll 1 \Longrightarrow |q_2\omega| \ll |q_1|$.

C. Steady-State Expressions for a_2, a_1, a_0, and α

The steady-state expressions for a_2, a_1, and a_0 are now given and used to show that the steady-state value for α is always positive. The steady-state expressions for a_2, a_1, a_0 are

$$a_2 = \frac{-n_p^2|\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\omega_S(1-\sigma)^2}{\sigma^2\,(1+S^2\omega_S^2 T_R^2)} \frac{1}{\mathrm{den}} \tag{23}$$

$$a_1 = \frac{n_p|\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{2\omega_S^2\,(1-\sigma)^2\,(1-S)}{\sigma^2\,(1+S^2\omega_S^2 T_R^2)} \frac{1}{\mathrm{den}} \tag{24}$$

and

$$a_0 = \frac{-|\underline{U}_S|^4}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2} \times \frac{\omega_S^3\,(1-\sigma)^2\,(1-S)^2}{\sigma^2\,(1+S^2\omega_S^2 T_R^2)} \frac{1}{\mathrm{den}} \tag{25}$$

where

$$\mathrm{den} = n_p T_R|\underline{U}_S|^4 \times \frac{\left(\frac{(1-\sigma)}{\sigma T_R}\frac{1+S^2\omega_S^2 T_R^2 - S\omega_S^2 T_R^2}{1+S^2\omega_S^2 T_R^2}\right)^2 + \left(\frac{(1-\sigma)}{\sigma}\frac{\omega_S}{1+S^2\omega_S^2 T_R^2}\right)^2}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^2}.$$

To compute the steady-state value of α, recall that $\alpha = a_1 - a_2 q_1/q_2$ [see just following (15)]. It is then easily seen that $a_1 > 0, a_2 q_1 < 0$, and $q_2 > 0$, so that in the steady state $\alpha > 0$.

D. Steady-State Speed

Substituting the steady-state values of a_2, a_1, and a_0, it is seen that $a_1^2 - 4a_2 a_0 \equiv 0$, so that the steady-state value of the right-hand side of (7) may be rewritten as $a_2\omega^2 + a_1\omega + a_0 = a_2(\omega + a_1/(2a_2))^2$ where a_2 is nonzero by (23). On the other hand, the steady-state solutions of (8) are

$$\omega_1 \triangleq \left(-q_1 + \sqrt{q_1^2 - 4q_2 q_0}\right)/(2q_2) = \omega$$

$$\omega_2 \triangleq \left(-q_1 - \sqrt{q_1^2 - 4q_2 q_0}\right)/(2q_2) = -1/\left(T_R^2 n_p^2 \omega\right).$$

E. Steady-State Expression for r_1

It is now shown that the steady-state value of r_1 in (12) is nonzero. Substituting the steady-state values of q_2, q_1, q_0, a_2, a_1, and a_0 (noting that $\dot{q}_1 \equiv 0$ and $\dot{q}_2 \equiv 0$ in steady state) into (12) gives

$$r_1 = \frac{-|\underline{U}_S|^{12}}{\left(\left(R_S + \frac{(1-\sigma)S\omega_S^2 L_S T_R}{1+S^2\omega_S^2 T_R^2}\right)^2 + \frac{\omega_S^2 L_S^2\left(1+\sigma S^2\omega_S^2 T_R^2\right)^2}{\left(1+S^2\omega_S^2 T_R^2\right)^2}\right)^6} \times \left(\frac{1}{1+S^2\omega_S^2 T_R^2}\right)^3 \frac{n_p^4\,(1-\sigma)^6\,\omega_S^3 L_S^2}{\sigma^4} \times \left(1 + T_R^2\omega_S^2(1-S)^2\right)^2 \frac{1}{\mathrm{den}}$$

where den is given in Appendix 6.3. It is then seen that $r_1 \neq 0$ in steady state.

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Global Practical Stabilization of Planar Linear Systems in the Presence of Actuator Saturation and Input Additive Disturbance

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***Abstract*—In this note, we revisit the problem of global practical stabilization for planar linear systems subject to actuator saturation and input additive disturbances. A parameterized linear state feedback law is designed such that, by tuning the value of the parameter, all trajectories of the closed-loop system converge to an arbitrarily small neighborhood of the origin in a finite time and remain in there.**

***Index Terms*—Actuator saturation, disturbance rejection, practical stabilization.**

I. Introduction

Disturbances are common in control systems. There is vast literature that addresses the problem of disturbance rejection for linear systems subject to actuator saturation[3]–[13]. On this topic, two lines of research have been pursued. In the first line, disturbances are assumed to be in the $\mathcal{L}_p$ space and controllers are constructed to result in $\mathcal{L}_p$ state trajectories, possibly with a small $\mathcal{L}_p$ gain from the disturbance to the state[1], [2], [6], [8], [9]. In particular, it is established that $\mathcal{L}_p$ gain from the disturbance to the state can be made arbitrarily small by linear state feedback if the $\mathcal{L}_p$ disturbances are also bounded in magnitude[8]. Such boundedness assumption on the disturbances can be removed if nonlinear feedback control is allowed [9].

The other line of research focuses on disturbances that are magnitude bounded and may be persistent, such as constant disturbances and sinusoidal disturbances. The objective here is to design controllers such that the closed-loop trajectories enter and remain in an *a-priori* given arbitrarily small neighborhood of the origin in a finite time. Such a design objective is usually called practical stabilization. If all trajectories are required of the aforementioned behavior, the design objective is called global practical stabilization. If only trajectories starting from a prespecified, but arbitrarily large, bounded set of the state space, then the design objective is called semiglobal practical stabilization. Two pieces of work along this line are Lin [10] and Saberi *et al.* [13]. In particular, [13] established that semiglobal practical stabilization for a linear system subject to actuator saturation and input additive disturbances can be achieved as long as the open loop system is not exponentially unstable. For the same class of systems, Lin[10] constructed nonlinear feedback laws that achieve global practical stabilization.

In this note, we revisit the problem of practical stabilization for planar linear systems subject to actuator saturation and input additive disturbances

$$\dot{x} = Ax + B\sigma(u + d) \tag{1}$$

where $x \in \mathbf{R}^2$ is the state, $u \in \mathbf{R}$ is the input, $d \in \mathbf{R}$ is the disturbance, and σ is a standard saturation function with unity saturation level, i.e.,

$$\sigma(u) = \mathrm{sign}(u) \min(1, |u|).$$

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