it is evident in the derivative plots, but not in the nonlinearities, that the cubic splines converge around two different solutions.

V. Conclusion

An algorithm employing a separable least squares Levenberg–Marquardt optimization was developed for the identification of Wiener systems with IIR linear dynamics and arbitrary, but fixed, degree spline nonlinearities with variable knot points. Monte Carlo simulations were used to evaluate the algorithm with both cubic and linear spline nonlinearities.

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A Nonlinear Least-Squares Approach for Identification of the Induction Motor Parameters

Kaiyu Wang, John Chiasson, Marc Bodson, and Leon M. Tolbert

***Abstract*—A nonlinear least-squares method is presented for the identification of the induction motor parameters. A major difficulty with the induction motor is that the rotor state variables are not available measurements so that the system identification model cannot be made linear in the parameters without overparametrizing the model. Previous work in the literature has avoided this issue by making simplifying assumptions such as a "slowly varying speed." Here, no such simplifying assumptions are made. The problem is formulated as a nonlinear least-squares identification problem and uses elimination theory (resultants) to compute the parameter vector that minimizes the residual error. The only requirement is that the system must be sufficiently excited. The method is suitable for online operation to continuously update the parameter values. Experimental results are presented.**

***Index Terms*—Induction motor, least-squares identification, parameter identification, resultants.**

I. Introduction

The induction motor parameters are the mutual inductance M, the stator inductance L_S, the rotor inductance L_R, the stator resistance R_S, the rotor resistance R_R, the inertia of the rotor J, and the load torque τ_L. Standard methods for the estimation of the induction motor parameters include the locked rotor test, the no-load test, and the standstill frequency response test. However, these approaches cannot be used online, that is, during normal operation of the machine, which is a disadvantage because some of the parameters vary during operation. For example, field-oriented control requires knowledge of the rotor time constant $T_R = L_R/R_R$ in order to estimate the rotor flux linkages and R_R varies significantly due to ohmic heating.

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K. Wang and J. Chiasson are with the Electrical and Computer Engineering Department, The University of Tennessee, Knoxville, TN 37996 USA (e-mail: wkaiyu@utk.edu; chiasson@utk.edu).

M. Bodson is with the Electrical and Computer Engineering Department, The University of Utah, Salt Lake City, UT 84112 USA (e-mail: bodson@ece.utah.edu).

L. M. Tolbert is with the Electrical and Computer Engineering Department, The University of Tennessee, Knoxville, TN 37996 USA, and also with Oak Ridge National Laboratory, Knoxville, TN 37932 USA (e-mail: tolbert@utk.edu; tolbertlm@ornl.gov).

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The work presented here is an approach to identify the induction motor parameters based on a nonlinear least-squares criteria. As the rotor state variables are not available measurements, the system identification model cannot be made linear in the parameters without overparameterizing the model [1], [2]. Previous work in the literature has avoided this issue by making simplifying assumptions such as a "slowly varying speed." Specifically, the proposed method improves upon the linear least-squares approach formulated in [1] and [2]. In [1] and [2], the approach was limited in that the acceleration was required to be small and that an iterative method used to solve for the parameter vector was not guaranteed to converge, nor necessarily produce the minimum residual error. Here, no such simplifying assumptions are made. The problem is formulated as a nonlinear system identification problem and uses elimination theory (resultants) to compute the parameter vector that minimizes the residual error with the only assumption being that the system is sufficiently excited. This method is suitable for *online* implementation so that during regular operation of the machine, the stator currents and voltages along with the rotor speed can be used to continuously update the parameters of the machine. Experimental results are presented to demonstrate the validity of the approach.

A combined parameter identification and velocity estimation problem is discussed in [3]–[5]. In contrast, we do not consider the velocity estimation problem, as the velocity is assumed to be known. Other related work includes [6]–[8]. For background on various approaches to machine parameter estimation, see [9] and [10].

II. Induction Motor Model

The work here is based on standard models of induction machines available in the literature [11]. These models neglect parasitic effects such as hysteresis, eddy currents, and magnetic saturation. The particular model formulation used here is the state–space model of the system given by (cf. [12] and [13])

$$\begin{aligned}
\frac{di_{Sa}}{dt} &= \frac{\beta}{T_R}\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + \frac{1}{\sigma L_S}u_{Sa}\\
\frac{di_{Sb}}{dt} &= \frac{\beta}{T_R}\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + \frac{1}{\sigma L_S}u_{Sb}\\
\frac{d\psi_{Ra}}{dt} &= -\frac{1}{T_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M}{T_R}i_{Sa}\\
\frac{d\psi_{Rb}}{dt} &= -\frac{1}{T_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M}{T_R}i_{Sb}\\
\frac{d\omega}{dt} &= \frac{n_pM}{JL_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - f\omega - \frac{\tau_L}{J}
\end{aligned} \tag{1}$$

where $\omega = d\theta/dt$ with θ the position of the rotor, n_p is the number of pole pairs, i_{Sa}, i_{Sb} are the (two-phase equivalent) stator currents, and ψ_{Ra}, ψ_{Rb} are the (two-phase equivalent) rotor flux linkages.

As stated in the introduction, the (unknown) parameters of the model are the five electrical parameters, R_S and R_R (the stator and rotor resistances), M (the mutual inductance), L_S and L_R (the stator and rotor inductances), and the mechanical parameters, J (the inertia of the rotor), f (viscous friction coefficient), and τ_L (the load torque). The symbols

$$T_R = \frac{L_R}{R_R} \quad \sigma = -\frac{M^2}{L_SL_R}$$
$$\beta = \frac{M}{\sigma L_S L_R} \quad \gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}$$

have been used to simplify the expressions. T_R is referred to as the rotor time constant while σ is called the total leakage factor.

This model is then transformed into a coordinate system attached to the rotor. For example, the current variables are transformed according to

$$\begin{bmatrix} i_{Sx} \\ i_{Sy}\end{bmatrix} = \begin{bmatrix}\cos(n_p\theta) & \sin(n_p\theta)\\ -\sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix} i_{Sa} \\ i_{Sb}\end{bmatrix}. \tag{2}$$

The transformation simply projects the vectors in the (a,b) frame onto the axes of the moving coordinate frame. An advantage of this transformation is that the signals in the moving frame (i.e., the (x,y) frame) typically vary slower than those in the (a,b) frame (they vary at the slip frequency rather than at the stator frequency). At the same time, the transformation does not depend on any unknown parameter, in contrast to the field-oriented dq transformation. The stator voltages and the rotor flux linkages are transformed in the same manner as the currents resulting in the following model (see [1] and [2]):

$$\frac{di_{Sx}}{dt} = \frac{\beta}{T_R}\psi_{Rx} + \beta n_p\omega\psi_{Ry} - \gamma i_{Sx} + n_p\omega i_{Sy} + \frac{u_{Sx}}{\sigma L_S} \tag{3}$$
$$\frac{di_{Sy}}{dt} = \frac{\beta}{T_R}\psi_{Ry} - n_p\beta\omega\psi_{Rx} - \gamma i_{Sy} - n_p\omega i_{Sx} + \frac{u_{Sy}}{\sigma L_S} \tag{4}$$
$$\frac{d\psi_{Rx}}{dt} = \frac{M}{T_R}i_{Sx} - \frac{1}{T_R}\psi_{Rx} \tag{5}$$
$$\frac{d\psi_{Ry}}{dt} = \frac{M}{T_R}i_{Sy} - \frac{1}{T_R}\psi_{Ry} \tag{6}$$
$$\frac{d\omega}{dt} = \frac{n_pM}{JL_R}(i_{Sy}\psi_{Rx} - i_{Sx}\psi_{Ry}) - f\omega - \frac{\tau_L}{J}. \tag{7}$$

III. Linear Overparameterized Model

Measurements of the stator currents i_{Sa}, i_{Sb} and voltages u_{Sa}, u_{Sb} as well as the position θ of the rotor are assumed to be available (velocity may then be reconstructed from position measurements). However, the rotor flux linkages ψ_{Rx}, ψ_{Ry} are not assumed to be measured. Standard methods for parameter estimation are based on equalities where known signals depend *linearly* on unknown parameters. However, the induction motor model described above does not fit in this category unless the rotor flux linkages are measured.

To proceed, a new set of independent equations are found by differentiating (3) and (4) to obtain

$$\begin{aligned}
\frac{1}{\sigma L_s}\frac{du_{Sx}}{dt} &= \frac{d^2i_{Sx}}{dt^2} + \gamma\frac{di_{Sx}}{dt} - \frac{\beta}{T_R}\frac{d\psi_{Rx}}{dt} - n_p\beta\omega\frac{d\psi_{Ry}}{dt}\\
&\quad - n_p\beta\psi_{Ry}\frac{d\omega}{dt} - n_p\omega\frac{di_{Sy}}{dt} - n_pi_{Sy}\frac{d\omega}{dt}
\end{aligned} \tag{8}$$
$$\begin{aligned}
\frac{1}{\sigma L_s}\frac{du_{Sy}}{dt} &= \frac{d^2i_{Sy}}{dt^2} + \gamma\frac{di_{Sy}}{dt} - \frac{\beta}{T_R}\frac{d\psi_{Ry}}{dt} + n_p\beta\omega\frac{d\psi_{Rx}}{dt}\\
&\quad + n_p\beta\psi_{Rx}\frac{d\omega}{dt} + n_p\omega\frac{di_{Sx}}{dt} + n_pi_{Sx}\frac{d\omega}{dt}.
\end{aligned} \tag{9}$$

Next, (3)–(6) are solved for $\psi_{Rx}, \psi_{Ry}, d\psi_{Rx}/dt, d\psi_{Ry}/dt$ and substituted into (8) and (9) to obtain

$$\begin{aligned}
0 = &-\frac{d^2i_{Sx}}{dt^2} + \frac{di_{Sy}}{dt}n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sx}}{dt}\\
&- i_{Sx}\left(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}\right) + i_{Sy}n_p\omega\left(\frac{1}{T_R} + \frac{\beta M}{T_R}\right)\\
&+ \frac{u_{Sx}}{\sigma L_S T_R} + n_p\frac{d\omega}{dt}i_{Sy} - n_p\frac{d\omega}{dt}\frac{1}{\sigma L_S\left(1 + n_p^2\omega^2T_R^2\right)}\\
&\times\Big(-\sigma L_S T_R\frac{di_{Sy}}{dt} - \gamma i_{Sy}\sigma L_S T_R - i_{Sx}n_p\omega\sigma L_S T_R\\
&-\frac{di_{Sx}}{dt}n_p\omega\sigma L_S T_R^2 - \gamma i_{Sx}n_p\omega\sigma L_S T_R^2\\
&+ i_{Sy}n_p^2\omega^2\sigma L_S T_R^2 + n_p\omega T_R^2 u_{Sx} + T_R u_{Sy}\Big)
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
0 = & -\frac{d^2 i_{Sy}}{dt^2} - \frac{di_{Sx}}{dt} n_p \omega + \frac{1}{\sigma L_S} \frac{du_{Sy}}{dt} - \left(\gamma + \frac{1}{T_R}\right) \frac{di_{Sy}}{dt} \\
& - i_{Sy} \left(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}\right) - i_{Sx} n_p \omega \left(\frac{1}{T_R} + \frac{\beta M}{T_R}\right) \\
& + \frac{u_{Sy}}{\sigma L_S T_R} - n_p \frac{d\omega}{dt} i_{Sx} + n_p \frac{d\omega}{dt} \frac{1}{\sigma L_S \left(1 + n_p^2 \omega^2 T_R^2\right)} \\
& \times \Bigg(-\sigma L_S T_R \frac{di_{Sx}}{dt} - \gamma i_{Sx} \sigma L_S T_R + i_{Sy} n_p \omega \sigma L_S T_R \\
& + \frac{di_{Sy}}{dt} n_p \omega \sigma L_S T_R^2 + \gamma i_{Sy} n_p \omega \sigma L_S T_R^2 \\
& + i_{Sx} n_p^2 \omega^2 \sigma L_S T_R^2 - n_p \omega T_R^2 u_{Sy} + T_R u_{Sx} \Bigg). \qquad (11)
\end{aligned}
$$

The set of (10) and (11) may be rewritten in regressor form as

$$y(t) = W(t)K \qquad (12)$$

where $K \in \mathbb{R}^{15}, W \in \mathbb{R}^{2\times 15}$, and $y \in \mathbb{R}^2$ are given by

$$
K \triangleq \begin{bmatrix} \gamma & \beta M & \frac{1}{\sigma L_S} & \frac{\beta M}{T_R^2} & \frac{1}{T_R} & \frac{\gamma}{T_R} & \frac{\beta M}{T_R} & T_R & \gamma T_R & \beta M T_R & T_R^2 & \gamma T_R^2 & \frac{T_R^2}{\sigma L_S} & \frac{1}{\sigma L_S T_R} & \frac{T_R}{\sigma L_S} \end{bmatrix}^T
$$

$$
W(t) \triangleq \begin{bmatrix}
-\frac{di_{Sx}}{dt} & n_p^2\omega^2 i_{Sx} & \frac{du_{Sx}}{dt} & i_{Sx} & n_p\omega i_{Sy} - \frac{di_{Sx}}{dt} & -i_{Sx} & n_p\omega i_{Sy} & -n_p^2\omega^2\frac{di_{Sx}}{dt} + n_p^3\omega^3 i_{Sy} + \frac{d\omega}{dt}\left(n_p di_{Sy}/dt + n_p^2\omega i_{Sx}\right) & n_p i_{Sy}\frac{d\omega}{dt} - n_p^2\omega^2 i_{Sx} & n_p^3\omega^3 i_{Sy} & n_p^2\left(\omega\frac{di_{Sx}}{dt}\frac{d\omega}{dt} - \omega^2\frac{d^2 i_{Sx}}{dt^2}\right) + n_p^3\omega^3\frac{di_{Sy}}{dt} & n_p^2\left(\omega i_{Sx}\frac{d\omega}{dt} - \omega^2\frac{di_{Sx}}{dt}\right) & n_p^2\left(\omega^2\frac{du_{Sx}}{dt} - u_{Sx}\omega\frac{d\omega}{dt}\right) & u_{Sx} & n_p^2\omega^2 u_{Sx} - n_p u_{Sy}\frac{d\omega}{dt} \\
-\frac{di_{Sy}}{dt} & n_p^2\omega^2 i_{Sy} & \frac{du_{Sy}}{dt} & i_{Sy} & -n_p\omega i_{Sx} - \frac{di_{Sy}}{dt} & -i_{Sy} & -n_p\omega i_{Sx} & -n_p^2\omega^2\frac{di_{Sy}}{dt} - n_p^3\omega^3 i_{Sx} + \frac{d\omega}{dt}\left(-n_p di_{Sx}/dt + n_p^2\omega i_{Sy}\right) & -n_p i_{Sx}\frac{d\omega}{dt} - n_p^2\omega^2 i_{Sy} & -n_p^3\omega^3 i_{Sx} & n_p^2\left(\omega\frac{di_{Sy}}{dt}\frac{d\omega}{dt} - \omega^2\frac{d^2 i_{Sy}}{dt^2}\right) - n_p^3\omega^3\frac{di_{Sx}}{dt} & n_p^2\left(\omega i_{Sy}\frac{d\omega}{dt} - \omega^2\frac{di_{Sy}}{dt}\right) & n_p^2\left(\omega^2\frac{du_{Sy}}{dt} - u_{Sy}\omega\frac{d\omega}{dt}\right) & u_{Sy} & n_p^2\omega^2 u_{Sy} + n_p u_{Sx}\frac{d\omega}{dt}
\end{bmatrix}
$$

and

$$
y(t) \triangleq \begin{bmatrix} \frac{d^2 i_{Sx}}{dt^2} - n_p i_{Sy}\frac{d\omega}{dt} - n_p\omega\frac{di_{Sy}}{dt} \\ \frac{d^2 i_{Sy}}{dt^2} + n_p i_{Sx}\frac{d\omega}{dt} + n_p\omega\frac{di_{Sx}}{dt} \end{bmatrix}.
$$

Though system (12) is linear in the parameters, it is overparameterized, resulting in poor numerical conditioning if standard least-squares techniques are used. Specifically

$$
\begin{aligned}
&K_1 = K_6 K_8 \quad K_2 = K_4 K_8^2 \quad K_3 = K_8 K_{14} \quad K_5 = 1/K_8 \\
&K_7 = K_4 K_8 \quad K_9 = K_6 K_8^2 \quad K_{10} = K_4 K_8^3 \quad K_{11} = K_8^2 \\
&K_{12} = K_6 K_8^3 \quad K_{13} = K_{14} K_8^3 \quad K_{15} = K_{14} K_8^2 \qquad (13)
\end{aligned}
$$

so that only the four parameters K_4, K_6, K_8, K_{14} are independent. Also, not all five electrical parameters R_S, L_S, R_R, L_R, and M can be retrieved from the K_i's. The four parameters K_4, K_6, K_8, K_{14} determine only the four independent parameters R_S, L_S, σ, and T_R by

$$
\begin{aligned}
R_S &= \frac{K_6 - K_4}{K_{14}} \quad T_R = K_8 \\
L_S &= \frac{1 + K_4 K_8^2}{K_{14} K_8} \quad \sigma = \frac{1}{1 + K_4 K_8^2}. \qquad (14)
\end{aligned}
$$

As $T_R = L_R/R_R$ and $\sigma = 1 - M^2/(L_S L_R)$, only L_R/R_R and M^2/L_R can be obtained, and not M, L_R, and R_R independently. This situation is inherent to the identification problem when rotor flux linkages are unknown and is not specific to the method. If the rotor flux linkages are not measured, machines with different R_R, L_R, and M, but identical L_R/R_R and M^2/L_R will have the same input/output (i.e., voltage to current and speed) characteristics. However, machines with different K_i parameters, yet satisfying the nonlinear relationships (13), will be distinguishable. For a related discussion of this issue, see [14] where parameter identification is performed using torque-speed and stator current-speed characteristics.

IV. Nonlinear Least-Squares Identification

Due to modeling errors, noise, etc., (12) will not hold for a single parameter vector K and all data values. Instead, the problem is reformulated as a least-squares minimization, that is, one minimizes

$$E^2(K) = \sum_{n=1}^{N} |y(n) - W(n)K|^2 \qquad (15)$$

subject to the constraints (13). On physical grounds, the parameters K_4, K_6, K_8, K_{14} are constrained to

$$0 < K_i < \infty, \qquad \text{for } i = 4, 6, 8, 14. \qquad (16)$$

Also, based on physical grounds, the squared error $E^2(K)$ will be minimized in the interior of this region. Define $E^2(K_p)$, as shown in (17), at the bottom of the page, where where

$$
\begin{aligned}
&K_p \triangleq [K_4 \quad K_6 \quad K_8 \quad K_{14}]^T \\
&R_y \triangleq \sum_{n=1}^{N} y^T(n) y(n) \quad R_{Wy} \triangleq \sum_{n=1}^{N} W^T(n) y(n) \\
&R_W \triangleq \sum_{n=1}^{N} W^T(n) W(n).
\end{aligned}
$$

For any chosen value of $K_p, E^2(K_p)$ is referred to as the *residual error*. The desire here is to find the value of K_p that minimizes the residual error. As just explained, the minimum of (17) must occur in

$$
\begin{aligned}
E^2(K_p) &\triangleq \sum_{n=1}^{N} |y(n) - W(n)K|^2 \Big|_{\substack{K_1 = K_6 K_8 \\ K_2 = K_4 K_8^2 \\ \vdots}} \\
&= R_y - 2R_{Wy}^T K \Big|_{\substack{K_1 = K_6 K_8 \\ K_2 = K_4 K_8^2 \\ \vdots}} + \left(K^T R_W K\right) \Big|_{\substack{K_1 = K_6 K_8 \\ K_2 = K_4 K_8^2 \\ \vdots}} \qquad (17)
\end{aligned}
$$

the interior of the region and, therefore, at an extremum point. This then entails solving the four equations

$$r_1(K_p) \triangleq \frac{\partial E^2(K_p)}{\partial K_4} = 0 \tag{18}$$

$$r_2(K_p) \triangleq \frac{\partial E^2(K_p)}{\partial K_6} = 0 \tag{19}$$

$$r_3(K_p) \triangleq \frac{\partial E^2(K_p)}{\partial K_8} = 0 \tag{20}$$

$$r_4(K_p) \triangleq \frac{\partial E^2(K_p)}{\partial K_{14}} = 0. \tag{21}$$

The partial derivatives in (18)–(21) are *rational* functions in the parameters K_4, K_6, K_8, K_{14}. Defining

$$p_1(K_p) \triangleq K_8 r_1(K_p) = K_8 \frac{\partial E^2(K_p)}{\partial K_4} \tag{22}$$

$$p_2(K_p) \triangleq K_8 r_2(K_p) = K_8 \frac{\partial E^2(K_p)}{\partial K_6} \tag{23}$$

$$p_3(K_p) \triangleq K_8^3 r_3(K_p) = K_8^3 \frac{\partial E^2(K_p)}{\partial K_8} \tag{24}$$

$$p_4(K_p) \triangleq K_8 r_4(K_p) = K_8 \frac{\partial E^2(K_p)}{\partial K_{14}} \tag{25}$$

results in the $p_i(K_p)$ being *polynomials* in the parameters K_4, K_6, K_8, K_{14} and having the same positive zero set (i.e., the same roots satisfying $K_i > 0$) as the system (18)–(21). The degrees of the polynomials p_i are given in the following table.

	deg K_4	deg K_6	deg K_8	deg K_{14}
$p_1(K_p)$	1	1	7	1
$p_2(K_p)$	1	1	7	1
$p_3(K_p)$	2	2	8	2
$p_4(K_p)$	1	1	7	1

All possible solutions to this set may be found using elimination theory, as is now summarized.

A. Solving Systems of Polynomial Equations [15], [16]

The question at hand is "Given two polynomial equations $a(K_1, K_2) = 0$ and $b(K_1, K_2) = 0$, how does one solve them simultaneously to eliminate (say) K_2?" A systematic procedure to do this is known as *elimination theory* and uses the notion of *resultants*. Briefly, one considers $a(K_1, K_2)$ and $b(K_1, K_2)$ as polynomials in K_2 whose coefficients are polynomials in K_1. Then, for example, letting $a(K_1, K_2)$ and $b(K_1, K_2)$ have degrees 3 and 2, respectively, in K_2, they may be written in the form

$$\begin{aligned} a(K_1, K_2) &= a_3(K_1)K_2^3 + a_2(K_1)K_2^2 + a_1(K_1)K_2 \\ &\quad + a_0(K_1) \\ b(K_1, K_2) &= b_2(K_1)K_2^2 + b_1(K_1)K_2 + b_0(K_1). \end{aligned}$$

The $n \times n$ *Sylvester* matrix, where $n = \deg_{K_2}\{a(K_1, K_2)\} + \deg_{K_2}\{b(K_1, K_2)\} = 3 + 2 = 5$, is defined by

$$S_{a,b}(K_1) \triangleq \begin{bmatrix} a_0(K_1) & 0 & b_0(K_1) & 0 & 0 \\ a_1(K_1) & a_0(K_1) & b_1(K_1) & b_0(K_1) & 0 \\ a_2(K_1) & a_1(K_1) & b_2(K_1) & b_1(K_1) & b_0(K_1) \\ a_3(K_1) & a_2(K_1) & 0 & b_2(K_1) & b_1(K_1) \\ 0 & a_3(K_1) & 0 & 0 & b_2(K_1) \end{bmatrix}.$$

The *resultant polynomial* is then defined by

$$r(K_1) = \mathrm{Res}(a(K_1, K_2), b(K_1, K_2), K_2) \triangleq \det S_{a,b}(K_1) \tag{26}$$

and is the result of *eliminating* the variable K_2 from $a(K_1, K_2)$ and $b(K_1, K_2)$. In fact, the following is true.

Theorem 1: [15], [16] Any solution (K_1^0, K_2^0) of $a(K_1, K_2) = 0$ and $b(K_1, K_2) = 0$ must have $r(K_1^0) = 0$.

Though the converse of this theorem is not necessarily true, the solutions of $r(K_1) = 0$ are the *only* possible candidates for the first coordinate (partial solutions) of the common zeros of $a(K_1, K_2)$ and $b(K_1, K_2)$, and the number of solutions is finite. Whether or not such a partial solution extends to a full solution is easily determined by back solving and checking the solution.

B. Solving the Polynomial Equations (22)–(25)

Using the polynomials (22)–(25) and the computer algebra software program MATHEMATICA [17], the variable K_4 is eliminated first to obtain three polynomials in three unknowns as

$$\begin{aligned} r_{p1p2}(K_6, K_8, K_{14}) &\triangleq \mathrm{Res}(p_1, p_2, K_4) \\ r_{p1p3}(K_6, K_8, K_{14}) &\triangleq \mathrm{Res}(p_1, p_3, K_4) \\ r_{p1p4}(K_6, K_8, K_{14}) &\triangleq \mathrm{Res}(p_1, p_4, K_4) \end{aligned}$$

where

	deg K_6	deg K_8	deg K_{14}
r_{p1p2}	1	14	1
r_{p1p3}	2	22	2
r_{p1p4}	1	14	1

Next K_6 is eliminated to obtain two polynomials in two unknowns as

$$\begin{aligned} r_{p1p2p3}(K_8, K_{14}) &\triangleq \mathrm{Res}(r_{p1p2}, r_{p1p3}, K_6) \\ r_{p1p2p4}(K_8, K_{14}) &\triangleq \mathrm{Res}(r_{p1p2}, r_{p1p4}, K_6) \end{aligned}$$

where

	deg K_8	deg K_{14}
r_{p1p2p3}	50	2
r_{p1p2p4}	28	1

Finally, K_{14} is eliminated to obtain a single polynomial in K_8 as

$$r(K_8) \triangleq \mathrm{Res}(r_{p1p2p3}(K_8, K_{14}), r_{p1p2p4}(K_8, K_{14}), K_{14})$$

where

$$\deg_{K_8}\{r(K_8)\} = 104.$$

- The parameter K_8 was chosen as the variable *not* eliminated because its degree was the highest at each step, meaning it would have a larger (in dimension) Sylvester matrix than using any other variable.
- The positive roots of $r(K_8) = 0$ are found, which are then substituted into $r_{p1p2p3} = 0$ (or $r_{p1p2p4} = 0$), which in turn are solved to obtain the partial solutions (K_8, K_{14}).
- The partial solutions (K_8, K_{14}) are then substituted into $r_{p1p2} = 0$ (or $r_{p1p3} = 0$ or $r_{p1p4} = 0$), which are solved to obtain the partial solutions (K_6, K_8, K_{14}) so that they in turn may be substituted into $p_1 = 0$ (or $p_2 = 0$ or $p_3 = 0$ or $p_4 = 0$), which are solved to obtain the solutions (K_4, K_6, K_8, K_{14}).

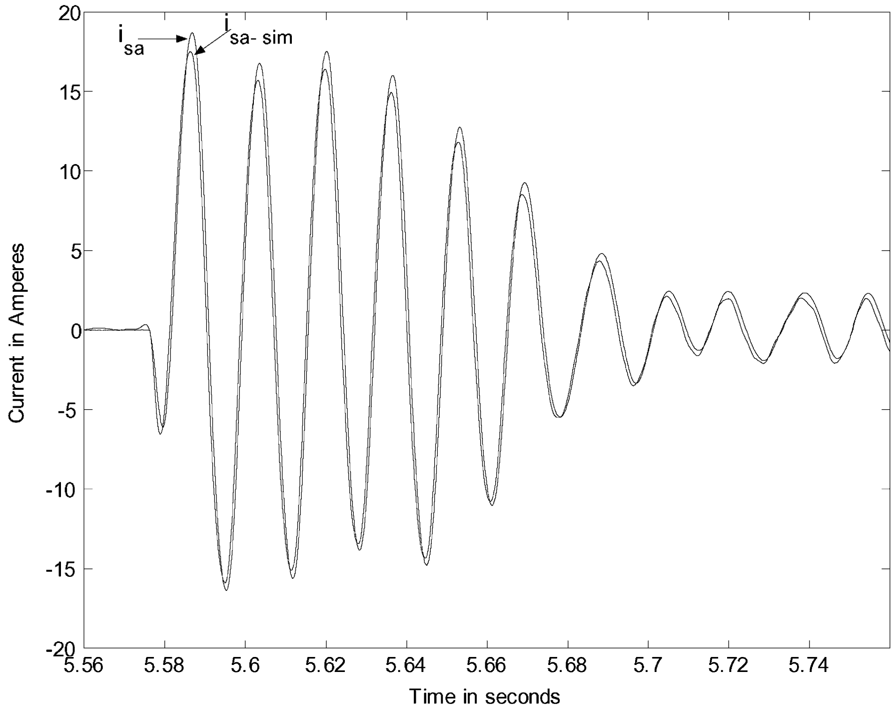

Fig. 1. Phase $\boldsymbol{a}$ current $\boldsymbol{i}_{Sa}$ and its simulated response $\boldsymbol{i}_{Sa\_\mathrm{sim}}$.

- These solutions are then checked to see which ones satisfy the complete system of polynomials equations (22)–(25) and those that do constitute the *candidate* solutions for the minimization. Based on physical considerations, the set of candidate solutions is non empty.
- From the set of candidate solutions, the one that gives the smallest squared error is chosen.

C. Testing for Sufficient Richness

After finding the solution that gives the minimal value for $E^2(K_p)$, one needs to know if the solution is well defined. For example, in the *linear* least-squares problem, there is a unique well defined solution provided that the regressor matrix R_W is nonsingular (or, in practical terms, its condition number is not too large). In the nonlinear case here, a Taylor series expansion is done about the computed minimum point $K_p^* = [K_4^*, K_6^*, K_8^*, K_{14}^*]^T$ to obtain $(i, j = 4, 6, 8, 14)$

$$E^2(K_p) = E^2(K_p^*) + \frac{1}{2}[K_p - K_p^*]^T \frac{\partial^2 E^2(K_p^*)}{\partial K_i \partial K_j}[K_p - K_p^*] + \cdots. \tag{27}$$

One then checks that the Hessian matrix $\partial^2 E^2(K_p^*)/\partial K_i \partial K_j$ is positive definite to ensure that the data is sufficiently rich (sufficient excitation) and checks if its condition number is small enough to ensure numerical reliability of the computations.

D. Mechanical Parameters

Once the electrical parameters have been found, the two mechanical parameters J, f $(\tau_L = -f\omega)$ can be found using a linear least-squares algorithm. To do so, (8) and (9) are solved for $M\psi_{Rx}/L_R, M\psi_{Ry}/L_R$ resulting in

$$\begin{bmatrix} M\psi_{Rx}/L_R \\ M\psi_{Ry}/L_R \end{bmatrix} = \frac{\sigma L_S}{(1/T_R)^2 + n_p^2\omega^2} \begin{bmatrix} 1/T_R & -\omega \\ \omega & 1/T_R \end{bmatrix} \times \begin{bmatrix} di_{Sx}/dt - u_{Sx}/(\sigma L_S) + \gamma i_{Sx} - n_p\omega i_{Sy} \\ di_{Sy}/dt - u_{Sy}/(\sigma L_S) + \gamma i_{Sy} + n_p\omega i_{Sx} \end{bmatrix}. \tag{28}$$

Noting that

$$\gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R} = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}(1-\sigma)L_S \tag{29}$$

then quantities on the right-hand side of (28) are all known once the electrical parameters have been computed. With $K_{16} \triangleq n_p/J, K_{17} \triangleq f/J$, (7) may be rewritten as

$$\frac{d\omega}{dt} = \begin{bmatrix} \frac{M\psi_{Rx}}{L_R} i_{Sy} - \frac{M\psi_{Ry}}{L_R} i_{Sx} & -\omega \end{bmatrix} \begin{bmatrix} K_{16} \\ K_{17} \end{bmatrix}$$

so that the standard approach for solving linear least-squares problems is directly applicable. Then

$$J \triangleq n_p/K_{16}, f \triangleq n_p K_{17}/K_{16}. \tag{30}$$

V. Experimental Results

A three-phase, 0.5 hp, 1735 rpm ($n_p = 2$ pole-pair) induction machine was used for the experiments. A 4096 pulse/rev optical encoder was attached to the motor for position measurements. The motor was connected to a three-phase, 60 Hz, 230 V source through a switch with no load on the machine. When the switch was closed, the stator currents and voltages along with the rotor position were sampled at 4 kHz. Filtered differentiation (using digital filters) was used for calculating the acceleration and the derivatives of the voltages and currents. Specifically, the signals were filtered with a lowpass digital Butterworth filter followed by reconstruction of the derivatives using $dx(t)/dt = (x(t) - x(t-T))/T$ where T is the sampling interval. The voltages and currents were put through a 3-to-2 transformation to obtain the two-phase equivalent voltages and currents $u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}$.

The sampled two-phase equivalent current i_{Sa} and its simulated response i_{Sa_sim} are shown in Fig. 1 (The simulated current will be discussed later). The phase b current i_{Sb} is similar, but shifted by $\pi/(2n_p)$. The calculated speed ω (from the position measurements) and the simulated speed ω_{sim} are shown in Fig. 2 (The simulated speed ω_{sim} will be discussed later). Using the data $\{u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}, \theta\}$ collected

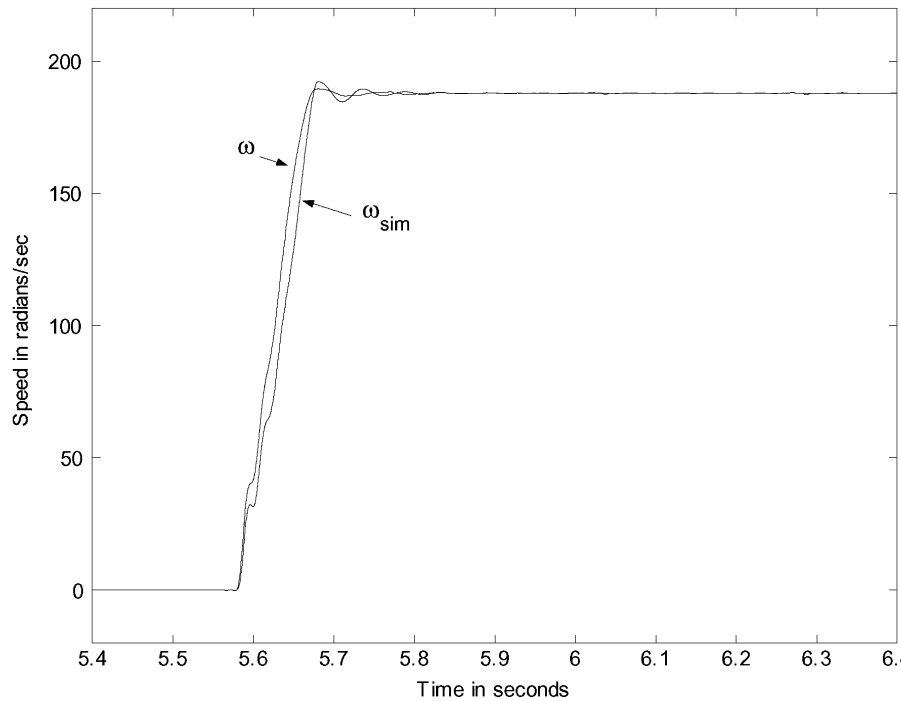

Fig. 2. Calculated speed $\boldsymbol{\omega}$ and simulated speed $\boldsymbol{\omega}_{\mathrm{sim}}$.

between 5.57 and 5.8 s, the quantities u_{Sx}, u_{Sy}, du_{Sx}/dt, du_{Sy}/dt, $i_{Sx}, i_{Sy} di_{Sx}/dt, di_{Sy}/dt, d^2 i_{Sx}/dt^2, d^2 i_{Sy}/dt^2, \omega = d\theta/dt, d\omega/dt$ were calculated and the regressor matrices R_W, R_y, and R_{Wy} were computed. The procedure explained in Section IV-B was then carried out to compute K_4, K_6, K_8, K_{14}. In this experiment, there was only one extremum point that had *positive* values for all the K_i.

The table that follows presents the parameter values determined using the nonlinear least-squares methodology along with their corresponding parametric error indexes. The *parametric error index* is the maximum amount the parameter can be changed until it results in a 25% increase in the residual error (see [2]).

Parameter	Estimated Value	Parametric Error
K_4	519.7	185.8
K_6	1848.3	796.4
K_8	0.1311	0.0103
K_{14}	259.5	59.4

The *residual error index* is defined to be $E^2(K_p^*)/R_y$ and satisfies $0 \leq E^2(K_p^*)/R_y \leq 1$ (see [2]). It was calculated to be 13.43%. The motor's electrical parameters are computed using (14) to obtain

$$R_S = 5.12\ \mathrm{Ohms} \quad T_R = 0.1311\ \mathrm{s} \tag{31}$$

$$L_S = 0.2919\ \mathrm{H} \quad \sigma = 0.1007. \tag{32}$$

By way of comparison, the stator resistance was measured using an Ohmmeter giving the value of 4.9 Ohms, and a no load test was also run to compute the value of L_S resulting in 0.33 H.

The Hessian matrix for the identification of the parameters K_4, K_6, K_8, K_{14} was calculated at the minimum point according to (27) resulting in

$$\left\{\frac{\partial^2 E^2(K_p^*)}{\partial K_i \partial K_j}\right\} = \begin{bmatrix} 0.0574 & 0.1943 & -0.0034 & -0.7655 \\ 0.1943 & 2.584 & 10.17 & -44.35 \\ -0.0034 & 10.17 & 631.4 & 193.8 \\ -0.7655 & -44.35 & 193.8 & 3012 \end{bmatrix}$$

which is positive definite and has a condition number of 8.24×10^4. The large value of the condition number is related to the nature of the system dynamics and not the methodology. Specifically, the voltage drops $R_S i_{Sx}$ and $R_S i_{Sy}$ are, respectively, much smaller compared to the other terms in the current (3) and (4), respectively, during normal operation of the machine. In other words, the effect of the ohmic voltage drops on the system response is small making it hard to identify R_S. Consequently, the identified value of R_S is more sensitive to the data than the other parameters and the condition number is large because of this. To make this more evident, consider four cases in which one of the four parameters R_S, T_R, σ, L_S is assumed to be known (using the value identified in this work). The following table gives the computed condition number of the 3×3 Hessian matrix for each of these cases.

Known parameter	Condition number
R_S	4.81×10^3
L_S	6.19×10^4
σ	6.65×10^4
T_R	5.92×10^4

Note the order of magnitude reduction in the condition number if R_S is the known parameter. Again, the ohmic voltage drop is much smaller than the other voltage drops in the current dynamics making it hard to identify R_S.

Using the electrical parameters, the rotor flux linkages $(M/L_R)\psi_{Rx}$ and $(M/L_R)\psi_{Ry}$ were reconstructed and used to identify the mechanical parameters. The following table gives the estimated values and the parametric error indexes.

Parameter	Estimated Value	Parametric Error
K_{16}	952.38	126.92
K_{17}	0.5714	0.1528

The residual error index was calculated to be 18.6%. The 2×2 regressor matrix R_W for these two parameters had a condition number of 1.060×10^3. The corresponding values for the motor parameters J and f are then computed using (30) to obtain

$$J = n_p / K_{16} = 0.0021 \text{ kgm}^2 \tag{33}$$

$$f = n_p K_{17} / K_{16} = 0.0012 \text{ Nm/(rad/s)}. \tag{34}$$

A. Simulation of the Experimental Motor

Another useful way to evaluate the identified parameters (31)–(32) and (33)–(34) is to simulate the motor using these values with the measured voltages as input. One then compares the simulation's output (stator currents) with the measured outputs. To proceed in this manner, recall that only R_S, T_R, L_S, and σ can be identified. However, this is all that is needed for the simulation. Specifically, defining

$$\phi_{Ra} \triangleq \frac{M}{L_R}\psi_{Ra} \quad \phi_{Rb} \triangleq \frac{M}{L_R}\psi_{Rb}$$

the model (1) may be rewritten as

$$\begin{aligned}
\frac{di_{Sa}}{dt} &= \frac{1}{\sigma L_S T_R}\phi_{Ra} + \frac{1}{\sigma L_S} n_p \omega \phi_{Rb} - \gamma i_{Sa} + \frac{1}{\sigma L_S} u_{Sa} \\
\frac{di_{Sb}}{dt} &= \frac{1}{\sigma L_S T_R}\phi_{Rb} - \frac{1}{\sigma L_S} n_p \omega \phi_{Ra} - \gamma i_{Sb} + \frac{1}{\sigma L_S} u_{Sb} \\
\frac{d\phi_{Ra}}{dt} &= -\frac{1}{T_R}\phi_{Ra} - n_p \omega \phi_{Rb} + \frac{1}{T_R}\frac{M^2}{L_R} i_{Sa} \\
\frac{d\phi_{Rb}}{dt} &= -\frac{1}{T_R}\phi_{Rb} + n_p \omega \phi_{Ra} + \frac{1}{T_R}\frac{M^2}{L_R} i_{Sb} \\
\frac{d\omega}{dt} &= \frac{n_p}{J}\left(i_{Sb}\phi_{Ra} - i_{Sa}\phi_{Rb}\right) - f\omega - \frac{\tau_L}{J}
\end{aligned} \tag{35}$$

where

$$\frac{M^2}{L_R} = (1-\sigma)L_S \quad \gamma = \frac{R_S}{\sigma L_S} + \frac{1}{\sigma L_S}\frac{1}{T_R}\frac{M^2}{L_R}.$$

Model (35) uses only parameters that can be estimated. The collected experimental voltages were then used as input to a simulation of the model (35) using the parameter values from (31)–(32) and (33)–(34). The resulting phase a current i_{Sa_sim} from the simulation is shown in Fig. 1 and matches well with the actual measured current i_{Sa}. Similarly, the resulting speed ω_{sim} from the simulation is shown in Fig. 2 where it is seen that the simulated speed is somewhat more oscillatory than the measured speed ω.

Remark 2: As explained previously, M is not identifiable by the aforementioned method. However, it can be (and often is) estimated by assuming that $L_R = L_S$ (rotor and stator leakages are equal) so that $M = \sqrt{L_S L_R(1-\sigma)} = L_S\sqrt{1-\sigma}$. For the above experimental motor, this gives $M = 0.2768$. Under this same assumption, an estimate for the rotor resistance is given by $R_R = L_R/T_R = L_S/T_R = 2.23$ Ohms.

VI. Conclusion

A method for estimating the parameters of an induction machine was presented. The problem was addressed using a nonlinear least-squares formulation and solved using elimination theory. An important advantage of the procedure is that it can be used *online*, i.e., during regular operation of the machine. The parameter values can be continuously updated assuming sufficient excitation of the machine.

The experiment in this note is done with voltages from the main line. If the machine is fed with an inverter as is typical in industrial applications, the commanded voltage would be used for computations rather than the measured PWM voltage and the inverter noise would not be an issue [2].

The technique proposed here is relevant for other systems and not just for induction machine parameter identification. The first issue of concern in applying this methodology to other systems is that the parameters used to develop a linear overparameterized model be *rationally* related. The next issue is the symbolic computation of the Sylvester matrices to compute the resultant polynomials. As the degrees of the polynomials to be solved increase, the dimensions of the corresponding Sylvester matrices increase and, therefore, the *symbolic* computation of their determinants becomes more intensive. The particular application considered in this work did not experience such computational difficulty. Interestingly, the recent work of [18] and [19] is promising for the efficient symbolic computation of the determinants of large Sylvester matrices. Finally, one must be able to accurately compute the roots of a large degree polynomial.

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