Elimination of Harmonics in a Multilevel Converter Using the Theory of Symmetric Polynomials and Resultants

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***Abstract*—A method is presented to compute the switching angles in a multilevel converter so as to produce the required fundamental voltage while at the same time not generate higher order harmonics. Previous work has shown that the transcendental equations characterizing the harmonic content can be converted to polynomial equations which are then solved using the method of resultants from elimination theory. A difficulty with this approach is that when there are several dc sources, the degrees of the polynomials are quite large making the computational burden of their resultant polynomials (as required by elimination theory) quite high. Here, it is shown that the theory of symmetric polynomials can be exploited to reduce the degree of the polynomial equations that must be solved which in turn greatly reduces the computational burden. In contrast to results reported in the literature that use iterative numerical techniques to solve these equations, the approach here produces all possible solutions.**

***Index Terms*—Multilevel inverter, resultants, symmetric polynomials.**

I. INTRODUCTION

A MULTILEVEL inverter is a power electronic device built to synthesize a desired ac voltage from several levels of dc voltages. For example, the output of solar cells are dc voltages, and if this energy is to be fed into an ac power grid, a power electronic interface is required. A multilevel inverter is ideal for connecting such distributed dc energy sources (solar cells, fuel cells, the rectified output of wind turbines) to an existing ac power grid. Transformerless multilevel inverters are uniquely suited for these applications because of the high power ratings possible with these inverters [1]. The devices in a multilevel inverter have a much lower dV/dt per switching, and they operate at high efficiencies because they can switch at a much lower frequency than pulseiwdth modulation (PWM)-controlled inverters. Three, four, and five level rectifier-inverter drive systems that have used some form of multilevel PWM as a means to control the switching of the rectifier and inverter sections have been investigated in [2]–[6]. Here, a fundamental frequency switching scheme (rather than PWM) is considered because, as just mentioned, this results in significantly lower switching losses.

A key issue in the fundamental switching scheme is to determine the switching angles (times) so as to produce the fundamental voltage and not generate specific higher order harmonics. Here, an harmonic elimination technique is presented that allows one to control a multilevel inverter in such a way that it is an efficient low total harmonic distortion (THD) inverter that can be used to interface distributed dc energy sources to a main ac grid or as an interface to a traction drive powered by fuel cells, batteries or ultracapacitors. Harmonic elimination techniques go back to the work of Hoft and Patel [7], [8], and the recent book by Holmes and Lipo [9] documents the current state of the art of such techniques.

Previous work in [10]–[14] has shown that the transcendental equations characterizing the harmonic content can be converted into *polynomial equations* which are then solved using the method of *resultants* from *elimination theory* [15], [16]. However, if there are several dc sources, the degrees of the polynomials in these equations are large. As a result, one reaches the limitations of the capability of contemporary computer algebra software tools (e.g., MATHEMATICA or MAPLE) to solve the system of polynomial equations using elimination theory (by computing the resultant polynomial of the system). A major distinction between the work in [10]–[14] and the work presented here is that here it is shown how the theory of *symmetric polynomials* [17] can be exploited to reduce the degree of the polynomial equations that must be solved so that they are well within the capability of existing computer algebra software tools. As in [12], the approach here produces all possible solutions in contrast to iterative numerical techniques that have been used to solve the harmonic elimination equations [18]. Furthermore, in the experiments reported here, an induction motor load is connected to the three-phase multilevel inverter, and the current as well as the voltage waveforms are collected for analysis. The fast Fourier transforms (FFTs) of these waveforms show that their harmonic content is close to the theoretically predicted value. A preliminary (oral) presentation of this work was given at [19] (see also [20] and [21]).

II. CASCADED H-BRIDGES

A cascade multilevel inverter consists of a series of H-bridge (single-phase full-bridge) inverter units. The general function of this multilevel inverter is to synthesize a desired voltage from

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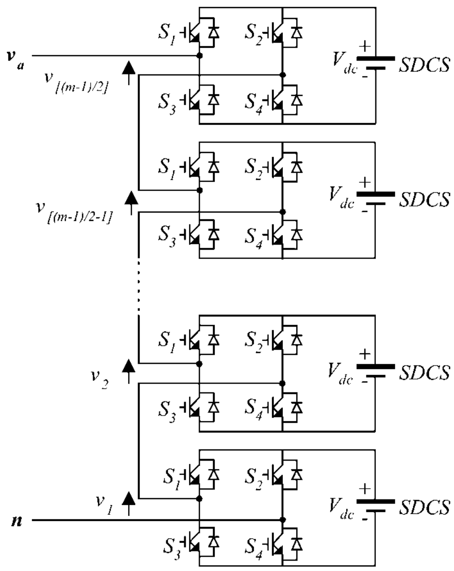

Fig. 1. Single-phase structure of a multilevel cascaded H-bridges inverter.

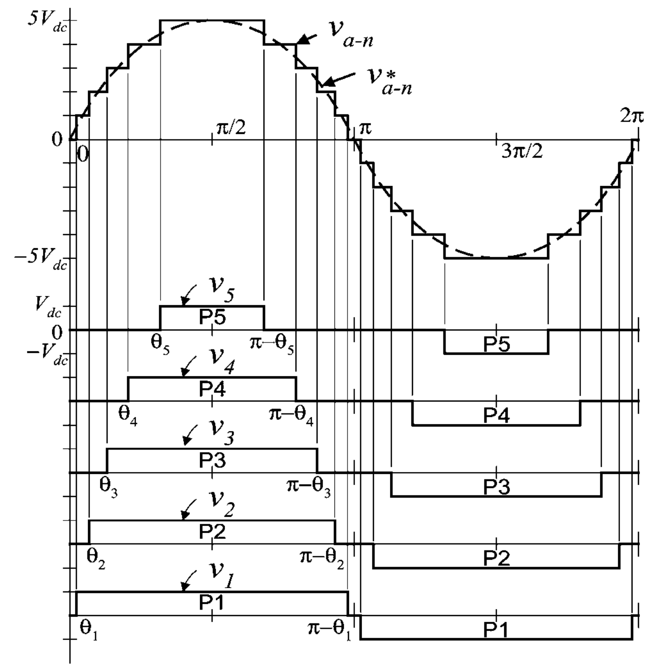

Fig. 2. Output waveform of an 11-level cascade multilevel inverter.

several separate dc sources (SDCSs), which may be obtained from solar cells, fuel cells, batteries, ultracapacitors, etc. Fig. 1 shows a single-phase structure of a cascade inverter with SDCSs [1]. Each SDCS is connected to a single-phase full-bridge inverter. Each inverter level can generate three different voltage outputs $+V_{dc}$, 0 and $-V_{dc}$ by connecting the dc source to the ac output side by different combinations of the four switches, S_1, S_2, S_3 and S_4. The ac output of each level's full-bridge inverter is connected in series such that the synthesized voltage waveform is the sum of all of the individual inverter outputs. The number of output phase voltage levels in a cascade mulitilevel inverter is then $2s+1$, where s is the number of dc sources. An example phase voltage waveform for an 11-level cascaded multilevel inverter with five SDCSs $(s=5)$ and five full bridges is shown in Fig. 2. The output phase voltage is given by $v_{an}=v_1+v_2+v_3+v_4+v_5$. With enough levels and an *appropriate* switching algorithm, the multilevel inverter results in an output voltage that is almost sinusoidal.

III. Mathematical Model of Switching for the Multilevel Converter

Following the development in [12] (see also [22]–[24]), the Fourier series expansion of the (staircase) output voltage waveform of the multilevel inverter as shown in Fig. 2 is

$$V(\omega t)=\frac{4V_{dc}}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\times\left(\cos(n\theta_1)+\cos(n\theta_2)+\cdots+\cos(n\theta_s)\right)\sin(n\omega t) \quad (1)$$

where s is the number of dc sources. Ideally, given a desired fundamental voltage V_1, one wants to determine the switching angles $\theta_1,\ldots,\theta_s$ so that (1) becomes $V(\omega t)=V_1\sin(\omega t)$. In practice, one is left with trying to do this approximately. The goal here is to choose the switching angles $0\le\theta_1<\theta_2<\cdots<\theta_s\le\pi/2$ so as to make the first harmonic equal to the desired fundamental voltage V_1 and specific higher harmonics of $V(\omega t)$ equal to zero. As the application of interest here is a three-phase system, the triplen harmonics in each phase need not be canceled as they automatically cancel in the line-to-line voltages. Specifically, in the case of $s=5$ dc sources, the desire is to cancel the fifth, seventh, 11th, and 13th-order harmonics as they dominate the THD. The mathematical statement of these conditions is then

$$\begin{aligned}\frac{4V_{dc}}{\pi}\left(\cos(\theta_1)+\cos(\theta_2)+\cdots+\cos(\theta_5)\right)&=V_1\\ \cos(5\theta_1)+\cos(5\theta_2)+\cdots+\cos(5\theta_5)&=0\\ \cos(7\theta_1)+\cos(7\theta_2)+\cdots+\cos(7\theta_5)&=0\\ \cos(11\theta_1)+\cos(11\theta_2)+\cdots+\cos(11\theta_5)&=0\\ \cos(13\theta_1)+\cos(13\theta_2)+\cdots+\cos(13\theta_5)&=0.\end{aligned} \quad (2)$$

This is a system of five transcendental equations in the five unknowns θ_1, θ_2, θ_3, θ_4, θ_5. The question here is "When does the set of (2) have a solution?". One approach to solving this set of nonlinear transcendental (2) is to use an iterative method such as the Newton–Raphson method [22]–[25]. In contrast to iterative methods, here a new approach is presented that produces all possible solutions and requires significantly less computational effort than the approach in [12]. To proceed let $s=5$, and, as in [12], define $x_i=\cos(\theta_i)$ for $i=1,\ldots,5$. Using the trigonometric identities

$$\begin{aligned}\cos(5\theta)&=5\cos(\theta)-20\cos^3(\theta)+16\cos^5(\theta)\\ \cos(7\theta)&=-7\cos(\theta)+56\cos^3(\theta)-112\cos^5(\theta)\\ &\quad+64\cos^7(\theta)\\ \cos(11\theta)&=-11\cos(\theta)+220\cos^3(\theta)-1232\cos^5(\theta)\\ &\quad+2816\cos^7(\theta)-2816\cos^9(\theta)+1024\cos^{11}(\theta)\\ \cos(13\theta)&=13\cos(\theta)-364\cos^3(\theta)+2912\cos^5(\theta)\\ &\quad-9984\cos^7(\theta)+16\,640\cos^9(\theta)\\ &\quad-13\,312\cos^{11}(\theta)+4096\cos^{13}(\theta)\end{aligned}$$

the conditions (2) become

$$\begin{aligned}
p_1(x) &\triangleq x_1+x_2+x_3+x_4+x_5-m=0\\
p_5(x) &\triangleq \sum_{i=1}^{5}\left(5x_i-20x_i^3+16x_i^5\right)=0\\
p_7(x) &\triangleq \sum_{i=1}^{5}\left(-7x_i+56x_i^3-112x_i^5+64x_i^7\right)=0\\
p_{11}(x) &\triangleq \sum_{i=1}^{5}\left(-11x_i+220x_i^3-1232x_i^5+2816x_i^7\right.\\
&\qquad \left.-2816x_i^9+1024x_i^{11}\right)=0\\
p_{13}(x) &\triangleq \sum_{i=1}^{5}\left(13x_i-364x_i^3+2912x_i^5-9984x_i^7\right.\\
&\qquad \left.+16\,640x_i^9-13\,312x_i^{11}+4096x_i^{13}\right)=0
\end{aligned} \tag{3}$$

where $x=(x_1,x_2,x_3,x_4,x_5)$ and $m\triangleq V_1/(4V_{dc}/\pi)$. The modulation index is $m_a=m/s=V_1/(s4V_{dc}/\pi)$ (Each inverter has a dc source of V_{dc} so that the maximum output voltage of the multilevel inverter is sV_{dc}. A square wave of amplitude sV_{dc} results in the maximum fundamental output possible of $V_{1\max}=4sV_{dc}/\pi$ so $m_a\triangleq V_1/V_{1\max}=V_1/(s4V_{dc}/\pi)=m/s$).

This is a set of five equations in the five unknowns x_1,x_2,x_3,x_4,x_5. Further, the solutions must satisfy $0\le x_5<\cdots<x_2<x_1\le 1$. This development has resulted in a set of polynomial equations rather than trigonometric equations. In [10]–[12], the authors considered the three-dc source case (seven levels) and solved the corresponding system of three equations in three unknowns using elimination theory by computing the resultant polynomial of the system (In [26], polynomial systems were also used, but solved by an iterative method). It turns out this procedure can be used for the four-dc source case (nine levels), but requires several hours of computation on a Pentium III. However, when one goes to five dc sources (11 levels), the computations using contemporary computer algebra software tools, e.g., the `Resultant` command in MATHEMATICA [27]) on a Pentium III (512 Mb RAM) appear to reach their limit (i.e., the authors were unable to get a solution before the computer gave out memory error messages). This computational complexity is because the degrees of the polynomials are large which in turn requires the *symbolic* computation of the determinant of large $n\times n$ matrices. Here, a new approach to solving the system (3) is presented which greatly reduces the computational burden. This is done by taking into account the symmetry of the polynomials making up the system (3). Specifically, the *theory of symmetric polynomials* [15], [28] is exploited to obtain a new set of relatively low-degree polynomials whose resultants can easily be computed using existing computer algebra software tools. Further, in contrast to results reported in the literature that use iterative numerical techniques to solve these type of equations [18], the approach here produces all possible solutions.

IV. SOLVING POLYNOMIAL EQUATIONS

For the purpose of exposition, the three source (7 level) multilevel inverter will be used to illustrate the approach. The conditions are then

$$\begin{aligned}
p_1(x) &\triangleq x_1+x_2+x_3-m=0,\quad m\triangleq \frac{V_1}{\frac{4V_{dc}}{\pi}}\\
p_5(x) &\triangleq \sum_{i=1}^{3}\left(5x_i-20x_i^3+16x_i^5\right)=0\\
p_7(x) &\triangleq \sum_{i=1}^{3}\left(-7x_i+56x_i^3-112x_i^5+64x_i^7\right)=0.
\end{aligned} \tag{4}$$

Eliminating x_3 by substituting $x_3=m-(x_1+x_2)$ into p_5,p_7 gives

$$\begin{aligned}
p_5(x_1,x_2) =&\, 5x_1-20x_1^3+16x_1^5+5x_2-20x_2^2+16x_2^5\\
&+5(m-x_1-x_2)-20(m-x_1-x_2)^3\\
&+16(m-x_1-x_2)^5\\
p_7(x_1,x_2) =&-7x_1+56x_1^3-112x_1^5+64x_1^7-7x_2+56x_2^3\\
&-112x_2^5+64x_2^7-7(m-x_1-x_2)\\
&+56(m-x_1-x_2)^3-112(m-x_1-x_2)^5\\
&+64(m-x_1-x_2)^7
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\deg_{x_1}\{p_5(x_1,x_2)\} &= 4 \quad \deg_{x_2}\{p_5(x_1,x_2)\}=4\\
\deg_{x_1}\{p_7(x_1,x_2)\} &= 6 \quad \deg_{x_2}\{p_7(x_1,x_2)\}=6.
\end{aligned} \tag{6}$$

A. Elimination Using Resultants

In order to explain the computational issues with finding the zero sets of polynomial systems, a brief discussion of the procedure to solve such systems is now given. The question at hand is "Given two polynomial equations $a(x_1,x_2)=0$ and $b(x_1,x_2)=0$, how does one solve them simultaneously to eliminate (say) x_2?" A systematic procedure to do this is known as *elimination theory* and uses the notion of *resultants* [15], [16]. Briefly, one considers $a(x_1,x_2)$ and $b(x_1,x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. Then, for example, letting $a(x_1,x_2)$ and $b(x_1,x_2)$ have degrees 3 and 2, respectively, in x_2, they may be written in the form

$$\begin{aligned}
a(x_1,x_2) &= a_3(x_1)x_2^3+a_2(x_1)x_2^2+a_1(x_1)x_2+a_0(x_1)\\
b(x_1,x_2) &= b_2(x_1)x_2^2+b_1(x_1)x_2+b_0(x_1).
\end{aligned}$$

The $n\times n$ *Sylvester* matrix, where $n=\deg_{x_2}\{a(x_1,x_2)\}+\deg_{x_2}\{b(x_1,x_2)\}=3+2=5$, is defined by

$$S_{a,b}(x_1)\triangleq\begin{bmatrix}
a_0(x_1) & 0 & b_0(x_1) & 0 & 0\\
a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0\\
a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1)\\
a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1)\\
0 & a_3(x_1) & 0 & 0 & b_2(x_1)
\end{bmatrix}.$$

The *resultant* polynomial is then defined by

$$r(x_1) = \mathrm{Res}\,(a(x_1,x_2), b(x_1,x_2), x_2) \triangleq \det S_{a,b}(x_1) \quad (7)$$

and any solution (x_{10}, x_{20}) of $a(x_1,x_2)=0$ and $b(x_1,x_2)=0$ must have $r(x_{10})=0$ [15]–[17].

Regarding the converse of this result, that is, does $r(x_{10}) \triangleq \det S_{a,b}(x_{10}) = 0$ imply that there exists an x_{20} such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0?.$$

Not necessarily. However, the answer is yes if either of the leading coefficients in x_2 of $a(x_1,x_2)$, $b(x_1,x_2)$ are not zero at x_{10}, i.e., $a_3(x_{10}) \neq 0$ *or* $b_2(x_{10}) \neq 0$ (see [15]–[17] for a detailed explanation). Further, the finite number of solutions of $r(x_1)=0$ are the *only* possible candidates for the first coordinate (partial solutions) of the common zeros of $a(x_1,x_2)$ and $b(x_1,x_2)$. Whether or not a partial solution extends to a full solution is simply determined by back solving and checking the solution.

B. Symmetric Polynomials

Consider once again the system of polynomial (5). In [12] (see also [10] and [11]), the authors computed the resultant polynomial of the pair $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ to obtain the solutions to (4). This involved setting up a 10×10 Sylvester matrix ($10 = \deg_{x_2}\{p_5(x_1,x_2)\} + \deg_{x_2}\{p_7(x_1,x_2)\}$) and then computing its determinant to obtain the resultant polynomial $r(x_1)$ whose degree is 22. However, as one adds more dc sources to the multilevel inverter, the degrees of the polynomials go up rapidly. For example, in the case of four dc sources, the final step of the method requires computing (symbolically) the determinant of a 27×27 Sylvester matrix to obtain a resultant polynomial of degree 221. In the case of five sources, using this method, the authors were only able to get the system of five polynomial equations in five unknowns to reduce to three equations in three unknowns. The computation to get it down to two equations in two unknowns requires the symbolic computation of the determinant of a 33×33 Sylvester matrix. This was attempted on a PC Pentium III (512 Mb RAM), but after several hours of computation, the computer complained of low memory and failed to produce an answer. To get around this difficulty, a new approach is developed here which exploits the fact that the polynomials in (3) are symmetric.

The polynomials $p_1(x)$, $p_5(x)$, $p_7(x)$ in (4) are *symmetric polynomials* [28], [29], that is

$$p_i(x_1,x_2,x_3) = p_i(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \text{ for all } i = 1,5,7$$

and any permutation. $\pi(\cdot)$[1] Define the *elementary symmetric functions* (polynomials) s_1, s_2, s_3 as

$$\begin{aligned} s_1 &\triangleq x_1 + x_2 + x_3 \\ s_2 &\triangleq x_1x_2 + x_1x_3 + x_2x_3 \\ s_3 &\triangleq x_1x_2x_3. \end{aligned} \quad (8)$$

[1]That is, $p_i(x_1,x_2,x_3) = p_i(x_2,x_1,x_3) = p_i(x_3,x_2,x_1)$, etc.

A basic theorem of symmetric polynomials is that they can be rewritten in terms of the elementary symmetric functions [28], [29] (this is easy to do using the `SymmetricReduction` command in MATHEMATICA [27]). In the case at hand, it follows that with $s = (s_1, s_2, s_3)$ and using (8), the polynomials (4) become

$$\begin{aligned} p_1(s) &= s_1 - m \\ p_5(s) &= 5s_1 - 20s_1^3 + 16s_1^5 + 60s_1s_2 - 80s_1^3s_2 + 80s_1s_2^2 \\ &\quad - 60s_3 + 80s_1^2s_3 - 80s_2s_3 \\ p_7(s) &= -7s_1 + 56s_1^3 - 112s_1^5 + 64s_1^7 - 168s_1s_2 + 560s_1^3s_2 \\ &\quad - 448s_1^5s_2 - 560s_1s_2^2 + 896s_1^3s_2^2 - 448s_1s_2^3 \\ &\quad + 168s_3 - 560s_1^2s_3 + 448s_1^4s_3 + 560s_2s_3 \\ &\quad - 1344s_1^2s_2s_3 + 448s_2^2s_3 + 448s_1s_3^2. \end{aligned} \quad (9)$$

One uses $p_1(s) = s_1 - m = 0$ to eliminate s_1 so that

$$\begin{aligned} q_5(s_2,s_3) &\triangleq p_5(m,s_2,s_3) = 5m - 20m^3 + 16m^5 \\ &\quad + 60ms_2 - 80m^3s_2 + 80ms_2^2 \\ &\quad - 60s_3 + 80m^2s_3 - 80s_2s_3 \\ q_7(s_2,s_3) &\triangleq p_7(m,s_2,s_3) = -7m + 56m^3 \\ &\quad - 112m^5 + 64m^7 - 168ms_2 \\ &\quad + 560m^3s_2 - 448m^5s_2 - 560ms_2^2 \\ &\quad + 896m^3s_2^2 - 448ms_2^3 + 168s_3 \\ &\quad - 560m^2s_3 + 448m^4s_3 + 560s_2s_3 \\ &\quad - 1344m^2s_2s_3 + 448s_2^2s_3 + 448ms_3^2 \end{aligned}$$

where

$$\begin{aligned} \deg_{s_2}\{q_5(s_2,s_3)\} &= 2, \quad \deg_{s_3}\{q_5(s_2,s_3)\} = 1 \\ \deg_{s_2}\{q_7(s_2,s_3)\} &= 3, \quad \deg_{s_3}\{q_7(s_2,s_3)\} = 2. \end{aligned}$$

The key point here is that degrees of these polynomials in s_2, s_3 are much less than the degrees of $p_5(x_1,x_2)$, $p_7(x_1,x_2)$ in x_1, x_2 as shown in (6). In particular, the Sylvester matrix of the pair $\{q_5(s_2,s_3), q_7(s_2,s_3)\}$ is 3×3 (if the variable s_3 is eliminated) rather than being 10×10 in the case of $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ in (5). Eliminating s_3, the resultant polynomial $r_{q_5,q_7}(s_2)$ is given by

$$\begin{aligned} r_{q_5,q_7}(s_2) &\triangleq \mathrm{Res}\,(q_5(s_2,s_3), q_7(s_2,s_3), s_3) \\ &= -16m \times \Big(-1575 + 9800m^2 - 24\,080m^4 \\ &\quad + 28\,160m^6 - 15\,360m^8 + 3072m^{10} \\ &\quad - 10\,500s_2 + 56\,000m^2s_2 \\ &\quad - 103\,040m^4s_2 + 78\,080m^6s_2 \\ &\quad - 20\,480m^8s_2 - 19\,600s_2^2 \\ &\quad + 89\,600m^2s_2^2 - 116\,480m^4s_2^2 \\ &\quad + 46\,080m^6s_2^2 - 11\,200s_2^3 \\ &\quad + 44\,800m^2s_2^3 - 35\,840m^4s_2^3\Big) \end{aligned}$$

which is only of degree 3 in s_2. For each m, one would solve $r_{q_5,q_7}(s_2) = 0$ for the roots $\{s_{2i}\}_{i=1,..3}$. These roots are then used to solve $q_5(s_{2i}, s_3) = 0$ for the root s_{3i} resulting in the set of 3-tuples $\{(s_1, s_2, s_3) \in C^3 \mid (s_1, s_2, s_3) = (m, s_{2i}, s_{3i})_{i=1,..,3}\}$ as the only possible solutions to (9).

C. Solving the Symmetric Polynomials

For each solution triple (s_1, s_2, s_3), the corresponding values of (x_1, x_2, x_3) are required to obtain the switching angles. Consequently, the system of polynomial equations (8) must be solved for the x_i. To do so, one simply uses the resultant method to solve the system of polynomials

$$\begin{aligned} f_1(x_1,x_2,x_3) &= s_1 - (x_1 + x_2 + x_3) = 0 \\ f_2(x_1,x_2,x_3) &= s_2 - (x_1x_2 + x_1x_3 + x_2x_3) = 0 \\ f_3(x_1,x_2,x_3) &= s_3 - x_1x_2x_3 = 0. \end{aligned}$$

That is, one computes

$$\begin{aligned} r_1(x_2,x_3) &= \mathrm{Res}\,(f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3), x_1) \\ &= -s_2 + s_1x_2 - x_2^2 + s_1x_3 - x_2x_3 - x_3^2 \\ r_2(x_2,x_3) &= \mathrm{Res}\,(f_1(x_1,x_2,x_3), f_3(x_1,x_2,x_3), x_1) \\ &= -s_3 + s_1x_2x_3 - x_2^2x_3 - x_2x_3^2 \end{aligned}$$

so that

$$\begin{aligned} r(x_3) &= \mathrm{Res}(r_1(x_2,x_3), r_2(x_2,x_3), x_2) \\ &= \left(s_3 - s_2x_3 + s_1x_3^2 - x_3^3\right)^2. \end{aligned} \tag{10}$$

The procedure is to substitute the solutions of (9) into (10) and solve for the roots $\{x_{3i}\}$. For each x_{3i}, one then solves $r_1(x_2, x_{3i})$ for the roots x_{2j}. Finally, one solves $f_1(x_1, x_{2j}, x_{3i}) = 0$ for x_{1j} to obtain the triples $\{(x_1,x_2,x_3) = (x_{1j}, x_{2j}, x_{3i}), i = 1,2,3, j = 1,2\}$ as the only possible solutions to (4). This finite set of possible solutions can then be checked as to which are solutions of (4) satisfying $0 \le x_3 \le x_2 \le x_1 \le 1$.

V. Five-dc Source Case

In this section, the five-dc source case is summarized. The polynomials $p_1(x), p_5(x), p_7(x), p_{11}(x), p_{13}(x)$ in (3) are *symmetric polynomials* [28], [29], and the *elementary symmetric functions* (polynomials) s_1, s_2, s_3, s_4, s_5 are defined as

$$\begin{aligned} s_1 &\triangleq x_1 + x_2 + x_3 + x_4 + x_5 \\ s_2 &\triangleq x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_5 \\ &\quad + x_3x_4 + x_3x_5 + x_4x_5 \\ s_3 &\triangleq x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 \\ &\quad + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 \\ s_4 &\triangleq x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 \\ &\quad + x_2x_3x_4x_5 \\ s_5 &\triangleq x_1x_2x_3x_4x_5. \end{aligned}$$

Rewriting the polynomials $p_i(x)$ in terms of the elementary symmetric polynomials gives

$$p_1(s) = s_1 - m = 0 \tag{11}$$

$$\begin{aligned} p_5(s) &= 5s_1 - 20s_1^3 + 16s_1^5 + 60s_1s_2 - 80s_1^3s_2 + 80s_1s_2^2 \\ &\quad - 60s_3 + 80s_1^2s_3 - 80s_2s_3 - 80s_1s_4 + 80s_5 \\ &= 0 \end{aligned} \tag{12}$$

$$\begin{aligned} p_7(s) &= -7s_1 + 56s_1^3 - 112s_1^5 + 64s_1^7 \\ &\quad - 168s_1s_2 + 560s_1^3s_2 - 448s_1^5s_2 - 560s_1s_2^2 \\ &\quad + 896s_1^3s_2^2 - 448s_1s_2^3 + 168s_3 - 560s_1^2s_3 \\ &\quad + 448s_1^4s_3 + 560s_2s_3 - 1344s_1^2s_2s_3 + 448s_2^2s_3 \\ &\quad + 448s_1s_3^2 + 560s_1s_4 - 448s_1^3s_4 + 896s_1s_2s_4 \\ &\quad - 448s_3s_4 - 560s_5 + 448s_1^2s_5 - 448s_2s_5 \\ &= 0 \end{aligned} \tag{13}$$

$$p_{11}(s) = -11s_1 + 220s_1^3 - 1232s_1^5 + \cdots = 0 \tag{14}$$

$$p_{13}(s) = 13s_1 - 364s_1^3 + 2912s_1^5 \ldots = 0 \tag{15}$$

where the complete expressions for $p_{11}(s)$ and $p_{13}(s)$ are rather long and their exact expressions are not needed for the explanation here. One uses $p_1(s) = s_1 - m = 0$ to eliminate s_1 so that

$$\begin{aligned} q_5(s_2,s_3,s_4,s_5) &\triangleq p_5(m,s_2,s_3,s_4,s_5) \\ q_7(s_2,s_3,s_4,s_5) &\triangleq p_7(m,s_2,s_3,s_4,s_5) \\ q_{11}(s_2,s_3,s_4,s_5) &\triangleq p_{11}(m,s_2,s_3,s_4,s_5) \\ q_{13}(s_2,s_3,s_4,s_5) &\triangleq p_{13}(m,s_2,s_3,s_4,s_5) \end{aligned}$$

where

	deg s_2	deg s_3	deg s_4	deg s_5
$q_5(s)$	2	1	1	1
$q_7(s)$	3	2	1	1
$q_{11}(s)$	5	3	2	2
$q_{13}(s)$	6	4	3	2

The key point here is that the maximum degrees of each of these polynomials in s_2, s_3, s_4, s_5 are much less than the maximum degrees of $p_1(x)$, $p_5(x)$, $p_7(x)$, $p_{11}(x)$, $p_{13}(x)$ in x_1, x_2, x_3, x_4, x_5 as seen by comparing with their values given in the table below.

	degree in x_1, x_2, x_3, x_4, x_5
$p_5(x_1,x_2,x_3,x_4,x_5)$	5
$p_7(x_1,x_2,x_3,x_4,x_5)$	7
$p_{11}(x_1,x_2,x_3,x_4,x_5)$	11
$p_{13}(x_1,x_2,x_3,x_4,x_5)$	13

Consequently, the computational burden of finding the resultant polynomials (i.e., the determinants of the Sylvester matrices) is greatly reduced. Also, as each of the $q_i(s)$'s has its maximum degree in s_2, the overall computational burden is further reduced by choosing this as the variable that is *not* eliminated. Proceeding, the indeterminate s_5 is eliminated first by computing

$$\begin{aligned} r_{q_5,q_7}(s_2,s_3,s_4) &= \mathrm{Res}\,(q_5(s_2,s_3,s_4,s_5) \\ &\qquad q_7(s_2,s_3,s_4,s_5), s_5) \\ r_{q_5,q_{11}}(s_2,s_3,s_4) &= \mathrm{Res}\,(q_5(s_2,s_3,s_4,s_5) \\ &\qquad q_{11}(s_2,s_3,s_4,s_5), s_5) \\ r_{q_5,q_{13}}(s_2,s_3,s_4) &= \mathrm{Res}\,(q_5(s_2,s_3,s_4,s_5) \\ &\qquad q_{13}(s_2,s_3,s_4,s_5), s_5) \end{aligned} \tag{16}$$

where

	$\deg s_2$	$\deg s_3$	$\deg s_4$
$r_{q_5,q_7}(s_2,s_3,s_4)$	2	2	1
$r_{q_5,q_{11}}(s_2,s_3,s_4)$	4	3	2
$r_{q_5,q_{13}}(s_2,s_3,s_4)$	5	4	2

Eliminating s_4 from these three polynomials gives the two polynomials

$$\begin{aligned} r_1(s_2,s_3) &\triangleq \text{Res}\,(r_{q_5,q_7}(s_2,s_3,s_4), \\ &\qquad r_{q_5,q_{11}}(s_2,s_3,s_4), s_4) \\ r_2(s_2,s_3) &\triangleq \text{Res}\,(r_{q_5,q_7}(s_2,s_3,s_4), \\ &\qquad r_{q_5,q_{13}}(s_2,s_3,s_4), s_4) \end{aligned} \tag{17}$$

where

	$\deg s_2$	$\deg s_3$
$r_1(s_2,s_3)$	6	4
$r_2(s_2,s_3)$	7	3

Finally, eliminating s_3 from $r_1(s_2,s_3)$ and $r_2(s_2,s_3)$, one obtains the resultant polynomial

$$\begin{aligned} r(s_2) &\triangleq \text{Res}\,(r_1(s_2,s_3), r_2(s_2,s_3), s_3) \\ &= Cm^{12}\left(5-20m^2+16m^4\right) \\ &\quad \times \left(-35+140m^2-140m^4+32m^6-35s_2\right. \\ &\qquad \left. +140m^2s_2-112m^4s_2\right)^4 g(s_2) \end{aligned}$$

where C is a constant and $g(s_2)$ is a polynomial of degree 9. One then back solves these equations for the five tuples (s_1,s_2,s_3,s_4,s_5) that are solutions to the system of polynomial equations (11)–(15).

To obtain the corresponding values of (x_1,x_2,x_3,x_4,x_5) for each of the solutions (s_1,s_2,s_3,s_4,s_5), elimination theory is again used to solve the system of polynomial equations as shown in Section IV-C.

Remark: Rather than using resultants, one could compute the Gröbner basis of $polys \triangleq \{q_5(s_2,s_3,s_4,s_5), q_7(s_2,s_3,s_4,s_5,), q_{11}(s_2,s_3,s_4,s_5), q_{13}(s_2,s_3,s_4,s_5)\}$ to find the solutions. However, it was found that the computation of this basis is slow (Using the command `GroebnerBasis[polys, {s2, s3, s4, s5}]` MATHEMATICA was unable to compute the answer after running more than 9 hours on a 1.2 MHz, Pentium III with 0.5 G of RAM.). On the other hand, the computation using resultants was less than a minute. This may be due to the fact that some of the intermediate resultant expressions (see (16), (17)) factor so that the computation of the resultant polynomial is simplified by working with these factors individually rather than the whole expression.

VI. Computational Results

Using the fundamental switching scheme of Fig. 2, the solutions of (2) were computed using the method described above. These solutions are plotted in Fig. 3 versus the parameter m. As the plots show, for m in the intervals $[2.21, 3.66]$ and $[3.74, 4.23]$ as well as $m = 1.88$, 1.89, the output waveform can have the desired fundamental with the 5th, 7th, 11th, 13th harmonics absent. Further, in the subinterval $[2.53, 2.9]$ two sets of solutions exist while in the subinterval $[3.05, 3.29]$, there are three sets of

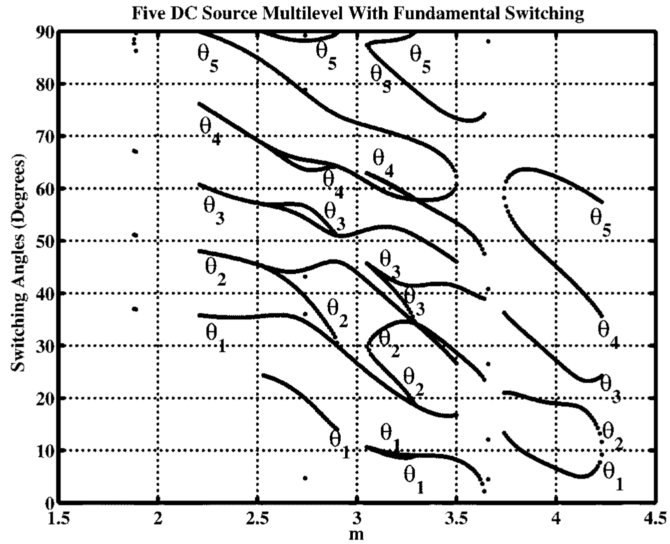

Fig. 3. Switching angles versus m for the five-dc source multilevel converter ($m_a = m/s$ with $s = 5$).

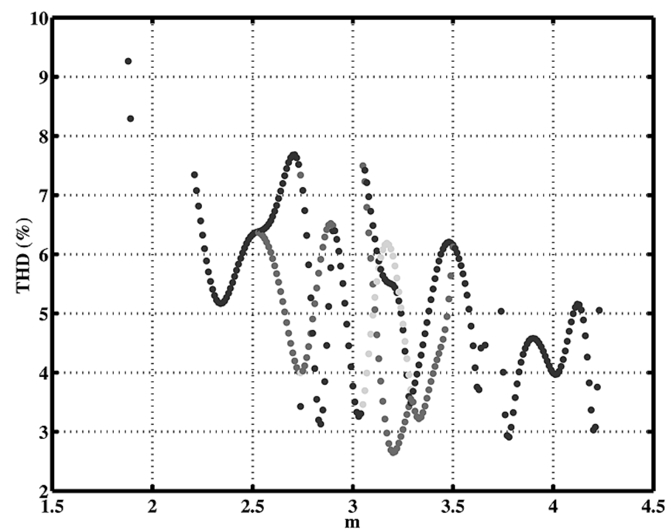

Fig. 4. THD versus m for each solution set ($m_a = m/s$ with $s = 5$).

solutions. In the case of multiple solution sets, one would typically choose the set that gives the lowest THD. In those intervals for which no solutions exist, one must use a different switching scheme (see [14] and [30] for a discussion on such possibilities). The corresponding THD was computed out to the 31st according to

$$\text{THD} = 100 \times \sqrt{\frac{V_5^2+V_7^2+V_{11}^2+V_{13}^2+V_{17}^2+\cdots+V_{31}^2}{V_1^2}}$$

where $V_n = (4V_{dc}/n\pi)(\cos(n\theta_1)+\cos(n\theta_2)+\cdots+\cos(n\theta_s))$ is the amplitude of the nth harmonic term of (1). The THD versus m is plotted in Fig. 4 for each of the solution sets shown in Fig. 3. As this figure shows, one can choose a particular solution for the switching angles such that the THD is 6.5% or less for $2.25 \le m \le 4.23$ $(0.45 \le m_a \le 0.846)$. For those values of m for which multiple solution sets exist, an appropriate choice is the one that results in the lowest THD. A look at Fig. 4 shows that this difference in THD can be as much as 3.5%, which is significant.

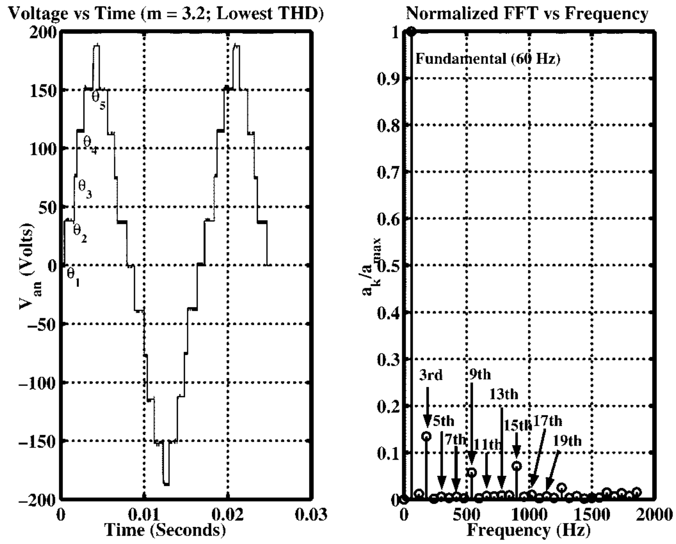

Fig. 5. Phase a output voltage waveform ($m = 3.2$) using the solutions set with the lowest THD and its normalized FFT.

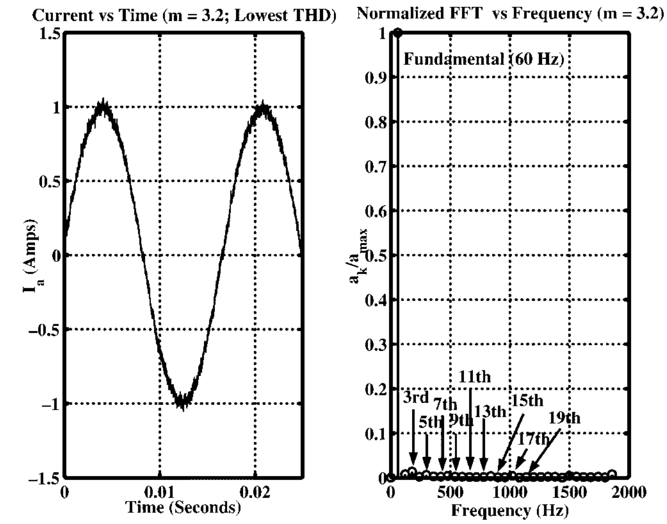

Fig. 6. Phase a current corresponding to the voltage in Fig. 5 and its normalized FFT.

VII. Experimental Results

The same experimental setup described in [12] was used for this work. It is a three-phase 11-level (five dc sources) wye-connected cascaded inverter using 100-V 70-A MOSFETs as the switching devices [31]. A battery bank of 15 SDCS of 36-V dc (not shown) each feed the inverter (five SDCS per phase). In this work, the RT-LAB real-time computing platform from *Opal-RT-Technologies Inc*. [32] was used to interface the computer to the multilevel inverter. This system allows one to implement the switching algorithm as a lookup table in SIMULINK which is then converted to C code using RTW (real-time workshop) from *Mathworks*. The RT-LAB software provides icons to interface the SIMULINK model to the digital I/O board and converts the C code into executables. The step size for the real time implementation was 32 μs.

Note that while the calculations for the lookup table of Fig. 3 requires some offline computational effort, the real-time implementation is accomplished by putting the data of Fig. 3 in a lookup table and therefore does not require high computational power for implementation.

The multilevel converter was attached to a three-phase induction motor with the following nameplate data: 1/3 hp, rated current 1.5 A, 1725 rpm, 208 V (RMS line-to-line at 60 Hz). In the experiment, $m = 3.2$ was chosen to produce a fundamental voltage of $V_1 = m(4V_{\mathrm{dc}}/\pi) = 3.2(4 \times 36/\pi) = 146.7$ V along with $f = 60$ Hz. As can be seen in Fig. 4, there are three different solution sets for $m = 3.2$. The solution set that gave the smallest THD ($= 2.65\%$ see Fig. 4) was used. Fig. 5 shows the phase a voltage and its corresponding FFT showing that the fifth, seventh, 11th, and 13th are absent from the waveform as predicted. The THD of the line-line voltage was computed using the data in Fig. 5 and was found to be 2.8%, comparing favorably with the value of 2.65% predicted in Fig. 4. Fig. 6 contains a plot of both the phase a current and its corresponding FFT showing that the harmonic content of the current is less than the voltage due to the filtering by the motor's inductance. The THD of this current waveform was computed using the FFT data and was found to be 1.9%.

VIII. Conclusion

A procedure to eliminate harmonics in a multilevel inverter has been given which exploits the properties of the transcendental equations that define the harmonic content of the converter output. Specifically, it was shown that one can transform the transcendental equations into symmetric polynomials which are then further transformed into another set of polynomials in terms of the elementary symmetric functions. This formulation resulted in a drastic reduction in the degrees of the polynomials that characterize the solution. Consequently, the computation of solutions of this final set of polynomial equations could be carried out using elimination theory (resultants) as the required symbolic computations were well within the capabilities of contemporary computer algebra software tools. This methodology resulted in the complete characterization of the solutions to the harmonic elimination problem. Experiments were performed, and the data presented corresponded well with the predicted results. Though not presented here, the authors have been successful solving for the angles for the case of seven dc sources. However, as one increases the number of dc sources, the degrees of polynomials representing the harmonic elimination equations increase as well and, thus, the dimension of Sylvester matrices. Even with the use of the symmetric polynomials, one will eventually run into computational difficulty with the symbolic computation of the determinants of the Sylvester matrices. Recent work of [33], [34] is promising for the efficient symbolic computation of these determinants.

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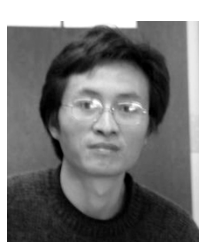

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