wdy ECE 300 AND 301 FALL 2005

# An Introduction To Using Laplace Transforms In Circuit Analysis

Consider

$$i(t) = C\frac{dV}{dt} \qquad (\text{capacitor})$$

Taking the Laplace transform of this expression gives,

$$\mathcal{L}[i(t)] = \mathcal{L}\left[C\frac{dV}{dt}\right]$$

or

$$I(s) = C[sV(s) - V(0)]$$

We assume here that $V(0) = 0$, so

$$I(s) = sC\,V(s)$$

We can write this as

$$\frac{V(s)}{I(s)} = \frac{1}{sC} \qquad (1)$$

We recall from DC circuits that

$$\frac{V}{I} = R \qquad (\text{resistance})$$

We define, from Eq. 1,

$$\frac{V(s)}{I(s)} = Z_c(s) = \frac{1}{sC} \qquad (2)$$

We call $Z(s)$, in general, complex impedance. Recall that $s$ is a complex variable and that

$$s = \sigma + j\omega \qquad (3)$$

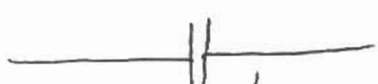
Later, when we tackle AC circuit, $\sigma = 0$. and we write

$$Z_C(s)\Big|_{s=j\omega} = Z_c(j\omega) = \frac{1}{sC}\Big|_{s=j\omega} = \frac{1}{j\omega C}$$

that is,

$$Z_c(j\omega) = \frac{1}{j\omega C}$$

Going back to Laplace, when we see a capacitor in a circuit we replace it with impedance, $\frac{1}{sC}$



$\frac{1}{sC}$

We will see how we use this later, Now consider

$$v(t) = L\frac{di}{dt} \quad (\text{inductor}) \qquad (4)$$

We take the Laplace Transform of (4)

$$V(s) = L\left[s\, I(s) - i(0)\right] \qquad (5)$$

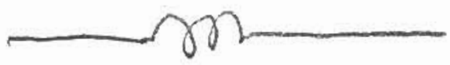
We assume $i(0)=0$. This leads to

$$\frac{V(s)}{I(s)} = sL$$

or

$$Z_L(s) = sL \qquad (6)$$

So when we have an inductor we associate $sL$ with it as the complex impedance.



$sL$

## Application

Consider the following circuit.

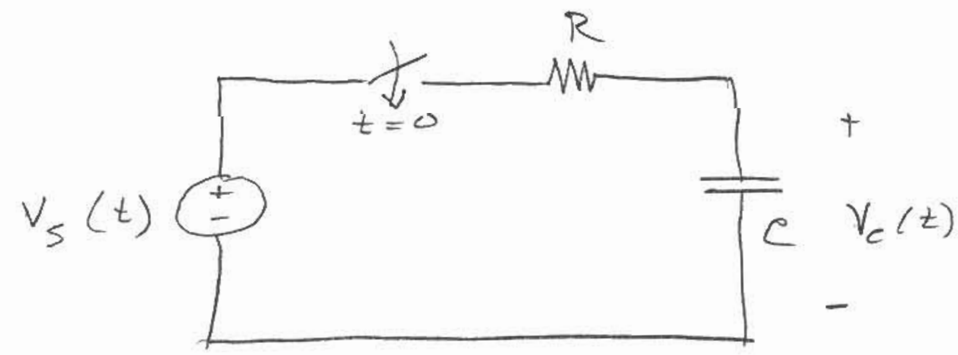



Figure 1: RC circuit, time domain.

We have studied this circuit earlier.

We recall that when $V_s(t) = V_s$ (DC voltage) and $V_c(0) = 0$,

$$V_c(t) = V_s - V_s e^{-\frac{t}{RC}} \qquad (7)$$

If we consider the circuit of Figure 1 in the Laplace sense, we have the circuit of Figure 2.

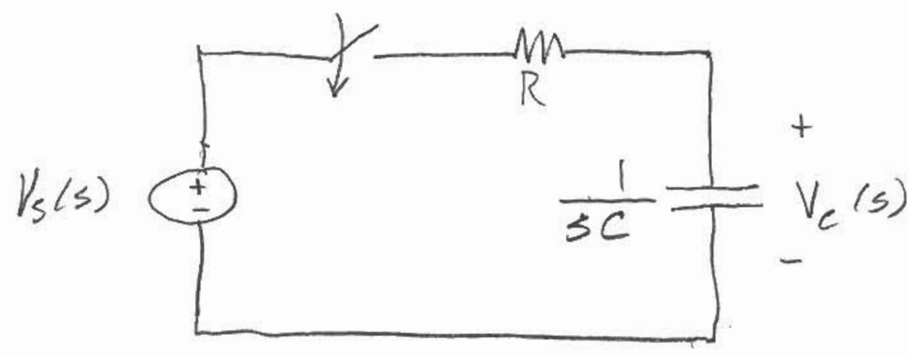



Figure 2: RC circuit, complex domain.

We apply voltage division in exactly the same form we used for DC circuits. This gives

$$V_c(s) = \frac{V_s(s) \times \frac{1}{sC}}{R + \frac{1}{sC}} \qquad (8)$$

or

$$V_c(s) = \frac{V_s(s)}{1 + sRC} \qquad (9)$$

If

$$V_s(t) = V_s$$

then

taking the Laplace gives,

$$V_s(s) = \frac{V_s}{s} \qquad (10)$$

where $V_s$ is some constant (a DC voltage)

substituting Eq (10) in Eq (9) and dividing the numerator and denominator by RC gives

$$V_c(s) = \frac{V_c/RC}{s\left(s + \frac{1}{RC}\right)} \qquad (11)$$

We now take the inverse Laplace of Eq (11) to find $V_c(t)$.

$$V_c(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}} \qquad (12)$$

Using partial fraction expansion and Heavyside we have

$$\left(s+\frac{1}{RC}\right) \times \frac{V_c/RC}{s\left(s+\frac{1}{RC}\right)}\Bigg|_{s=-\frac{1}{RC}} = \frac{A\left(s+\frac{1}{RC}\right)}{s}\Bigg|_{s=-\frac{1}{RC}} + B$$

This gives

$$B = \left.\frac{V_c/RC}{s}\right|_{s=-\frac{1}{RC}} = -\frac{V_c/RC}{1/RC}$$

$$\boxed{B = -V_c}$$

Similarly,

$$A = \left.\frac{V_c/RC}{s+\frac{1}{RC}}\right|_{s=0} = V_c$$

Placing this back in Eq(12) gives

$$V_c(s) = \frac{V_c}{s} - \frac{V_c}{s+\frac{1}{RC}} \qquad (13)$$

Taking the inverse Laplace transform of Eq (12) gives

$$\mathcal{L}^{-1}[V_c(s)] = \mathcal{L}^{-1}\left[\frac{V_c}{s}\right] - \mathcal{L}^{-1}\left[\frac{V_c}{s+1/RC}\right]$$

or

$$\boxed{V_c(t) = V_c - V_c\, e^{-\frac{t}{RC}}} \qquad (14)$$

This is the same expression obtain by using the differential equation approach. See Eq. (7).

Generally, using Laplace transforms for analysis of R2C transient circuits simplifies the task as compared to writing out differential equations in the time domain. For a good introductory level presentation on using Laplace for RC, RL and RLC circuits see;

Mersereau, R.M. & Jackson, J.R.; Circuit Analysis. A Systems Approach [Chapter 6]; Prentice-Hall Publishing Co. ISBN 0-13-093224-8, 2006.

We also recall that the differential equation for the circuit of Figure 1 is

$$\frac{dV_c(t)}{dt} + \frac{V_c(t)}{RC} = \frac{V_s(t)}{RC} \qquad (15)$$

Assume zero I.C. for $V_c(0)$. Taking the Laplace of Eq (15) gives

$$s V_c(s) + \frac{V_c(s)}{RC} = \frac{V_s(s)}{RC} \qquad (16)$$

We use that $V_s(t) = V_s$ and

$$\mathcal{L}[V_s(t)] = V_s(s) = \frac{V_s}{s} \qquad (17)$$

Equation (16) becomes,

$$\left(s + \frac{1}{RC}\right)V_c(s) = \frac{V_s}{s\,RC}$$

or

$$V_c(s) = \frac{V_s/RC}{s\left(s+\frac{1}{RC}\right)} \qquad (18)$$

This is identical to Equation (11) so we know that taking the inverse Laplace gives Eq (14). QED.

Example 1:

Find $i(t)$ for the following circuit using Laplace Transforms. Assume $i(0) = 0$

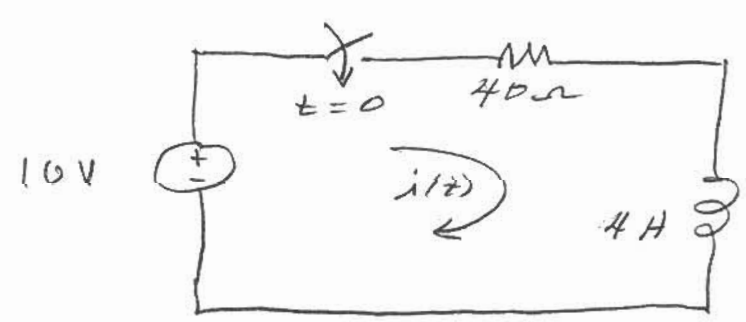



Figure 3; RL circuit, Example 1, time domain.

We convert the circuit to the Laplace form as shown in Figure 4.

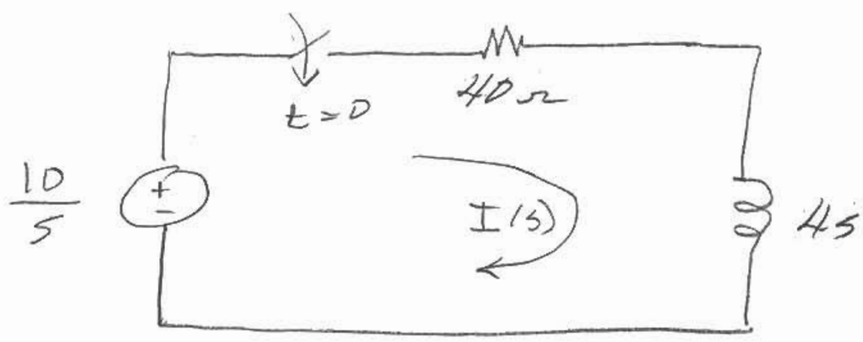



Figure 4: RL circuit, Example 1, Laplace Form.

Solving for I(s) gives,

$$I(s) = \frac{10/s}{4s + 40}$$

or

$$I(s) = \frac{10/4}{s(s+10)}$$

$$I(s) = \frac{2.5}{s(s+10)} \qquad (19)$$

Taking the inverse Laplace gives

$$\boxed{i(t) = 2.5 - 2.5e^{-10t}} \qquad (20$$

This presentation does not really show the power of using Laplace Transforms in circuits and more particular in systems in general. Hopefully, the notes will stimulate you to look further into the subject.